

On crystal sets

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Abstract

We are interested in *2-crystal sets* and *protocrystal sets* in which every difference between distinct elements occurs zero or an even number of times. We show that several infinite families of such sets exist. We also give non-existence theorems for infinite families. We find conditions to limit the computer search space for such sets. We note that search for *2-crystal sets* $(n; k_1, k_2)$, $k = k_1 + k_2$ even, in a set of size n , immediately cuts the search space for two circulant weighing matrices with periodic autocorrelation function zero from 3^{2n} to 2^{2n-k} . We show that $2-(2n; 4, 1)$, for n odd, can only exist when $7|n$ and conjecture that $2-(2n; q^2, 1)$ crystal sets will only exist when $q^2 + q + 1$ is a prime and $(q^2 + q + 1)|n$.

1 Introduction

Two sequences with elements $0, \pm 1$, very small periodic or non-periodic autocorrelation function, and small cross correlation function are of considerable interest in signal processing. Two such sequences with zero periodic or non-periodic autocorrelation function are also used to form weighing matrices.

This paper concentrates on searching for the zeros of such sequences; this is called *crystallization* and the zero positions form *crystal sets*. This paper gives conditions on crystal sets and gives algorithms for their construction preparatory to searching for weighing matrices.

2 Definitions and Preliminaries

2.1 Protocrystal Sets and Crystal Sets

Difference sets [1] and *supplementary difference sets* (sds) [8, 9] and their applications have been extensively studied in the past.

We now study two more relaxed sets, *protocrystal sets* and *crystal sets*, which can sometimes be used to form difference sets and supplementary difference sets.

Definition 1 Let K be a subset of size k , written as $(n; k; \mu)$ protocystal set, of a set of n elements, V . Then K will be called a *protocystal set* if in the totality (multiset), written as Λ , of all the differences between all distinct elements in the subset, K has an even number of even elements, $|\Lambda| = \mu$. Since $\mu = k(k - 1)$, we will omit μ and write $(n; k)$ PCset.

Lemma 1 *If n is odd, the number of elements of Λ which are even equals the number of elements which are odd; that is, $\frac{k(k-1)}{2}$. If n is even, the number of odd elements in Λ is even and the number of even elements is even, but they may not be equal.*

Proof. If n is odd and the protocystal set has k elements, then the differences $(a_i - b_i) \pmod n$ and $(b_i - a_i) \pmod n$ both occur in Λ . Hence each difference d and $n - d$ occurs; one is even and the other is odd, so the number of even and odd elements in Λ is the same. The total number of elements in Λ is $k(k - 1)$; hence in this case the number of even elements is $\frac{k(k-1)}{2}$.

However, if n is even, $(a_i - b_i) \pmod n$ even (or odd) implies $(b_i - a_i) \pmod n$ even (or odd, respectively). Hence the number of even elements in Λ is even and the number of odd elements is also even, but they may not equal each other. \square

Corollary 1 *Suppose n is odd. We write Λ_i for the number of elements in Λ for $k \equiv 0, 1, 2, \text{ or } 3 \pmod 4$ respectively. Then we see Λ_0 and Λ_1 have an even number of even elements; but Λ_2 and Λ_3 have an odd number of even elements.*

Hence crystal sets can be made only by having two sets of size k_i , $i = 0$ and/or $1 \pmod 4$, or by having two sets of size k_i , $i = 2$ and/or $3 \pmod 4$.

Example 1 Consider $C = \{0, 1, 3, 10, 12\} \pmod{13}$. This has differences $(a_i - b_i) \pmod{13}$ where $a_i \neq b_i$, $a_i, b_i \in C$. Since both $(a_i - b_i) \pmod{13}$ and $(b_i - a_i) \pmod{13}$ both occur, and since 13 is odd, the number of even (and odd) elements in Λ will be the same, 10. So C is a $(13; 5)$ PC (proto-crystal) set.

a_1/b_1	0	1	3	10	12
0	*	1	3	10	12
1	12	*	2	9	11
3	10	11	*	7	9
10	3	4	6	*	2
12	1	2	4	11	*

So the totality of differences (multiset) is

$$\Lambda = [1, 1, 2, 2, 2, 3, 3, 4, 4, 6, 7, 9, 9, 10, 10, 11, 11, 11, 12, 12].$$

Definition 2 Two $2-(n; k_1, k_2; \mu)$ subsets C_1 and C_2 , of a set V of size n , which have sizes k_1 and k_2 , respectively, will be said to be *crystal sets* when Λ , the totality (multiset) of all the differences from both of the subsets, has each element occurring zero or an even number of times.

By counting the differences we see that $\mu = k_1(k_1 - 1) + k_2(k_2 - 1)$, so we usually write $2-(n; k_1, k_2)$ crystal sets.

Example 2 Consider $C_1 = \{0, 1, 3, 10, 12\}$ and $C_2 = \{1, 3, 10, 12\} \pmod{13}$. These have differences $(a_i - b_i) \pmod{13}$, as follows, where $a_i \neq b_i$, $a_i, b_i \in C_j$,

a_1/b_1	0	1	3	10	12	a_2/b_2	1	3	10	12
0	*	1	3	10	12	1	*	2	9	11
1	12	*	2	9	11	3	11	*	7	9
3	10	11	*	7	9	10	4	6	*	2
10	3	4	6	*	2	12	2	4	11	*
12	1	2	4	11	*					

so the totality of differences (multiset) is $\Lambda =$

$$\{1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 6, 6, 7, 7, 9, 9, 9, 9, 10, 10, 11, 11, 11, 11, 11, 11, 12, 12\}.$$

Here Λ has each difference an even number of times so we have $2\text{-}(13; 5, 4)$ crystal sets. We note $\mu = 32$.

Example 3 It is possible to have a protocrystal set that is by itself a crystal set. For example, consider $\{0, 1, 2, 4\} \pmod{7}$.

2.2 Weighing matrices

A weighing matrix $W = W(n, k)$ is an $n \times n$ square matrix with entries $0, \pm 1$, having k non-zero entries per row and column and inner product of distinct rows equal to zero. Therefore W satisfies $WW^T = kI_n$. The number k is called the weight of W . Weighing matrices were first studied because of a statistical application in weighing experiments. Later a conjecture of Seberry Wallis, that if $n \equiv 0 \pmod{4}$, weighing matrices $W(n, k)$ exist for all $k = 0, \dots, n$ [10], sparked further work. Further conjectures concerning weighing matrices have been studied extensively; see [7] and references therein. A well-known necessary condition for the existence of $W(2n, k)$ matrices states that if there exists a $W(2n, k)$ matrix with n odd, then $k < 2n$ and k is the sum of two squares. The two circulant construction for weighing matrices is described in the theorem below, taken from [3], and is of special interest because of its applications in signal processing.

Theorem 1 *If there exist two circulant matrices A_1, A_2 of order n , with $0, \pm 1$ elements, satisfying $A_1A_1^t + A_2A_2^t = kI_n$, where k is an integer, then there exists a $W(2n, k)$, given as*

$$W(2n, k) = \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_1^t \end{pmatrix} \text{ or } W(2n, k) = \begin{pmatrix} A_1 & A_2R \\ -A_2R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

In this paper we study *crystal sets* which give positions of the zeros for $W(2n, 2n - a)$ constructed from two circulant matrices of order n , that is, the $2\text{-}(n; k_1, k_2)$ crystal sets. If n is odd the weight $k = k_1 + k_2$ is equal to $2n - a = x^2 + y^2$, with x, y integers.

2.3 Sequences with Zero Periodic Autocorrelation Function

Given the sequence $A = \{a_1, a_2, \dots, a_n\}$, of length n , the *non-periodic autocorrelation function* $\text{NPAF}_A(s)$ is defined as

$$\text{NPAF}_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1, \quad (1)$$

Given A as above, of length n , the *periodic autocorrelation function* $\text{PAF}_A(s)$ is defined, reducing $i + s$ modulo n , as

$$\text{PAF}_A(s) = \sum_{i=1}^n a_i a_{i+s}, \quad s = 0, 1, \dots, n-1. \quad (2)$$

Two sequences, $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$, both of length n , which will be useful in this paper have

$$\text{NPAF}_A(s) + \text{NPAF}_B(s) = 0, \quad s = 1, 2, \dots, n, \quad (3)$$

or

$$\text{PAF}_A(s) + \text{PAF}_B(s) = 0, \quad s = 1, 2, \dots, n, \quad (4)$$

and are said to have *zero non-periodic auto-correlation function* or *zero periodic auto-correlation function* respectively.

2.4 Trivial and Foundational Crystal Sets

We use the following notation:

$|N| = n$ is odd;

N is the set $\{0, 1, \dots, n-1\}$;

\emptyset denotes the empty set;

C is a protocystal set which is a crystal set, $|C| = k$;

PC is a protocystal set;

C^C is the complement of a crystal set in N , equal to all the elements of N which are not in C , $|C^C| = n - k$.

Theorem 2 *Two sets which are identical (or one a shift of the other, that is, its elements are formed from the first set by adding a constant modulo the size of the set) can be used as crystal sets.*

Alternatively, if PC is any protocystal set, then $\{PC, PC\}$ that is, all 2 -($n; k, k$) exist provided $2n - 2k$ is the sum of two squares.

Lemma 2 *Both $\{0\}$ and $\{\emptyset\}$ are possible 2 -($n; 1, 0$) crystal sets, for n odd and $2n - 1$ the sum of two squares.*

Lemma 3 *The following are always possible parameters for two crystal sets, where n is odd:*

- (i) \emptyset, C , are $2-(n; 0, k)$ for $2n - k$ the sum of two squares;
- (ii) $N \setminus \{0\}, C$ are $2-(n; n - 1, k)$ for $(n - k - 1)$ the sum of two squares;
- (iii) $\{0\}, C$ are $2-(n; 1, k)$ for $2n - 1 - k$ the sum of two squares;
- (iv) N, C are $2-(n; n, k)$ for $n - k$ the sum of two squares.

Remark 1 Let n be odd and C_1 and C_2 be two protocystal sets of sizes k_1 and k_2 respectively. We recall from the properties of weighing matrices that C_1 and C_2 can only be $2-(n, k_1, k_2)$ crystal sets if $2n - k_1 - k_2$, the number of non-zero elements, is the sum of two squares.

However it is possible that if C_1 and C_2 are not $2-(n, k_1, k_2)$, that is, $2n - k_1 - k_2$ is not the sum of two squares, but

- (i) C_1 and C_2^C could be $2-(n, k_1, n - k_2)$ or
 - (ii) C_2 and C_1^C could be $2-(n, k_2, n - k_1)$ or
 - (iii) C_1^C and C_1^C could $2-(n, n - k_1, n - k_2)$
- if (i) $n - k_1 + k_2$, or (ii) $n + k_1 - k_2$, or (iii) $(k_1 + k_2)$, respectively, are the sum of two squares.

Example 4 We note that for $n = 11$, there are no $2-(11, 2, 6)$ crystal sets because $2n - k_1 - k_2 = 14$ is not the sum of two squares. In fact:

- neither $k_1 = 2$ nor $k_2 = 6$ is the sum of two squares;
- $2n - k_1 - k_2 = 14, n - k_1 + k_2 = 15$: neither is the sum of two squares;
- $n + k_1 - k_2 = 7, k_1 + k_2 = 8$ and 8 is two squares;
- $k_1 = 2 \neq k_2 = 6$. This tells us that we only need to search for sets of size $11 - 2$ and $11 - 6$, as in (iii).

Lemma 4 If $A = \{a_1, a_2, \dots, a_{k_1}\}$ and $B = \{b_1, b_2, \dots, b_{k_2}\} \pmod{n}$, with n odd, are two crystal sets $(n; k_1, k_2; \mu)$, then A^C and B^C are two crystal sets.

Proof. Let $2L$ be the set of all differences from the set $N = \{0, 1, 2, \dots, n - 1\}$ and Λ_1 be the set of differences from A and B . Then for n odd, $2L$ contains $1, 2, 3, \dots, n - 1, 2n$ times, n odd. Hence A^C and B^C will contain each difference an even number of times. □

Remark 2 In Lemma 4, if A has k_1 elements and B has k_2 elements, then A^C has $n - k_1$ elements and B^C has $n - k_1$ elements. In order to minimize any searches for crystal sets, we can consider the pair A, B or A^C, B^C , whichever has $\min(k_2, n - k_1)$.

Example 5 For $n = 11$, let A, B have $(k_1, k_2) = 6, 8$. Hence $(n - k_1, n - k_2) = (11 - 6, 11 - 8) = (5, 3)$ for A^C and B^C . So we could choose to search for the crystal sets with sizes 3 and 5, knowing that if they do not exist and Λ does not have each element an even number of times, then there are no crystal sets with sizes 6 and 8. If each element does occur an even number of times then A^C and B^C will be crystal sets.

Lemma 5 *A $2\text{-}\{n; k_1, k_2; \Lambda\}$ crystal set corresponds to even $\text{PAF}_A(j) + \text{PAF}_B(j)$ for all $j \in \Lambda$.*

Proof. Suppose the crystal set needed to give the zero positions in the first row of the circulant matrices $A = \text{circ}\{a_1, \dots, a_n\}$ and $B = \text{circ}\{b_1, \dots, b_n\}$, where the non-zero positions are marked $*$, meaning ± 1 . Write $C = [AB]$. Then if $i \in \Lambda$, it must occur $2\lambda_i$ times. That means that in the inner product of row 1 of C with row i of C , a zero element occurs in the same $2\lambda_i$ columns of C .

Rearranging the columns of C to obtain C^* , we see that row 1 and row i may be written as

$$\begin{array}{cccc} \underbrace{00 \dots 00}_{k_1+k_2} & \underbrace{* * \dots * *}_{2n-k_1-k_2} & & \\ \underbrace{0 \dots 0}_{2\lambda_i} & \underbrace{* \dots *}_{(k_1+k_2-2\lambda_i)} & \underbrace{0 \dots 0}_{(k_1+k_2-2\lambda_i)} & \underbrace{* \dots *}_{(2n-2k_1-2k_2+2\lambda_i)} \end{array}$$

So the inner product of row 1 of C^* and row i of C^* has an even number of non-zero terms. Rearranging the columns back to C gives $\text{PAF}_A(j) + \text{PAF}_B(j)$ is even, for all $i \in \Lambda$. □

Corollary 2 *Let $n = q^2 + q + 1$, q a prime power. Then there exists a $2\text{-}\{n; q^2, 1; \Lambda\}$ crystal set where Λ is the elements $1, 2, \dots, q^2, q$, each $q(q - 1)$ times.*

Proof. For the first row of A , put the zeros in the positions given by the complement of the elements in the $(q^2 + q + 1, q + 1, 1)$ difference set (from the projective plane) in the $(q^2 + q + 1, q^2, q(q - 1))$ difference set. For the first row of B , make the first element 0.

Now $n = q^2 + q + 1$, $k_1 = q^2$, $k_2 = 1$ and $2\lambda_i = q(q - 1)$ for all i . So

$$\begin{aligned} \text{PAF}(i) &= 2q^2 + 2q + 2 - 2q^2 - 2 + 2\lambda_i \\ &= 2q + q^2 - q \\ &= q(q + 1), \end{aligned}$$

which is always even. □

Theorem 3 *Suppose there exists a cyclic difference set with parameters (v, k, λ) , λ even, v odd. Then there exists a $2\text{-}\{v; k, 1; \Lambda\}$ crystal set where Λ is the elements $1, 2, \dots, v - 1$, each $\lambda = \frac{k(k-1)}{v-1}$ times.*

Proof. Same as above, noting $\text{PAF}(i) = 2v - 2k - 2 + \frac{k(k-1)}{v-1}$ is even.

Remark 3 There are combinations, for example, a difference set repeated and $2\text{-}\{v; k_1, k_2; 2\lambda\}$ sds which give similar results.

Example 6 There is a $(7, 4, 2)$ difference set $\{0, 1, 2, 4\}$, which can be used to give the crystal set $\{0, 1, 2, 4\} \oplus \{0\}$ and the first rows:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & - & 1 & - & - \end{array}$$

which can be used in the two circulant construction.

Example 7 There is a $(13, 9, 6)$ difference set $\{0, 1, 2, 4, 5, 6, 7, 8, 10\}$ which can be used to give the first two rows in the two circulant construction:

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 \end{array}$$

3 Crystalization

In [4], Kotsireas and Koukouvinos have found a number of new $W(2n, 2n - 5)$ weighing matrices constructed from two circulants, for n odd. They used A and B to denote the first rows of the two circulants used to construct a $W(2n, 2n - 5)$ as per Theorem 1. In [4] it was observed, experimentally, that if one fixes the locations of four of the five zeros for the sequences to make a $W(2n, 2n - 5)$ as shown in (5),

$$\begin{array}{l} A = 0 \ 0 \ a_3 \ \dots \ a_n \\ B = 0 \ 0 \ b_3 \ \dots \ b_n \end{array} \quad (5)$$

then the fifth zero, for the pattern $(3, 2)$ or $(2, 3)$, can only appear in precisely the position $\frac{n+3}{2}$ in one or the other of A and B . Thus we have more generally:

Theorem 4 *The crystalization pattern $(t, 1)$ in length n , n odd, is only possible if there is a single crystal set (n, t, Λ) .*

Proof. The second sequence, B , with a single zero, has the corresponding protocrytal set $\{0\}$, which has each difference occurring an even number of times, that is, zero times. Hence if there is a single crystal set (n, t, Λ) , we have two sets which have Λ as the totality of differences and they are all even. \square

Lemma 6 *The two sets $C_1 = \{0, j, n - j\}$ and $C_2 = \{j, n - j\}$ are 2- $(n; 3, 2)$ crystal sets, n odd.*

Proof. Here C_1 has differences $\pm j, \pm(n - j), \pm(n - 2j)$, and C_2 has differences $\pm(n - 2j)$ and modulo n , n odd. Thus each difference occurs an even number of times and so we have two crystal sets or 2- $(n; 3, 2)$ crystal sets. \square

Remark 4 These can be used to give the potential zeros of the first rows of two $0, \pm 1$ circulant matrices giving a $W(2n, 2n - 5)$ for $2n - 5$ the sum of two squares, n odd.

Remark 5 Kotsiras and Koukouvinos [4] have had great success when searching for cases where there is a total of $2n - j$ zeros, $j \equiv 1 \pmod{4}$.

3.1 Crystalization Pattern (k, ℓ) or $2-(n; k, \ell)$ Crystal Sets

In future, if the number of zeros in the sequences (first rows) A and B are both equal to ℓ , we will say the sequences have pattern (ℓ, ℓ) ; if the number of zeros in A and B is k and ℓ , with $k > \ell$, respectively, then we will say the sequences have pattern (k, ℓ) [2]. This is the same as saying the structural pattern (k, ℓ) means

$$\text{there are } k \text{ zeros in } [a_1, \dots, a_n] \text{ and } \ell \text{ zeros in } [b_1, \dots, b_n].$$

This pattern of the zeros has been called the $(n; k, \ell)$ *crystalization* of the zeros. The positions of the non-zero elements in any sequence has been called *the support*.

We will generalize the notion of *crystalization* as outlined in [4] by using *crystal sets*.

Theorem 5 *Suppose C_1 and C_2 are $2-(n; k_1, k_2)$ crystal sets. Then C_1 and C_2 can be used to place the zeros for the (k_1, k_2) structural pattern for the construction of two circulant matrices which may give a weighing matrix.*

Proof. Form the totality, Λ , of the differences from the elements of the crystal sets. Suppose difference i occurs λ_i times in Λ . If the elements of the crystal sets are the zero elements of the first rows of two circulant $0, \pm 1$ matrices of order n (even or odd), then the inner product of row 1 and row i will have $(2n - 2(k_1 + k_2) + \lambda_i)$, an even number, of non-zero entries.

This is the same as saying $\text{PAF}(A, i) + \text{PAF}(B, i)$ is even for all $i = 1, \dots, \frac{n-1}{2}$. It must be even so the number of non-zero entries is able to be even, and so, the number of $+1$ s and -1 s can cancel to give inner product of rows k and $k + i - 1$ to be zero. \square

Corollary 3 *The two first rows of two circulant weighing matrices must have their zeros in the positions of crystal sets.*

3.2 Crystals Sets from Difference Sets and SDS

From Seberry Wallis [9] we see that $2-(n; k_1, k_2; \lambda)$ sds are similar to $2-(n; k_1, k_2)$ crystal sets, except that each non-zero difference in Λ must occur the same number of times, λ , and occurs for both even and odd entries.

Theorem 6 *Suppose there exist $2-(n; k_1, k_2; \lambda)$ sds (for reference see [8, 9]) with λ even; then they form $2-(n; k_1, k_2; \lambda)$ crystal sets, and the complementary $2-(n; n - k_1, n - k_2; 2n - 2k_1 - 2k_2 + \lambda)$ sds or $2-(n; n - k_1, n - k_2)$ crystal sets.*

Similarly a *difference set* (n, k, λ) is a single set with each non-zero difference in Λ occurring the same number of times, λ .

Theorem 7 *Every (n, k, λ) difference set with λ even is a single (n, k) PC.*

Thus we can sometimes combine difference sets to give crystal sets:

Theorem 8 *A (n, k_1, λ_1) difference set together with a (n, k_2, λ_2) difference set, where $\lambda_1 + \lambda_2$ is even, gives $2-(n; k_1, k_2)$ crystal sets.*

Definition 3 We call the sets (n, n) PC, $(n, 0)$ PC, $(n, 1)$ PC and $(n, n-1)$ PC, which always exist, the *trivial cases*. For convenience we will write them as (n, ψ) PS. We note that in all trivial cases all entries of Λ occur an even number of times.

Theorem 9 *Suppose there exists a (n, k, λ) difference set, n odd. Then its complementary $(n, n-k, n-2k+\lambda)$ difference set also exists. If λ is odd (respectively even), then $n-2k+\lambda$ will be even (or odd respectively). We suppose the (n, k, λ) difference set, n odd, has λ even (if not we will use the complementary set). Then the (n, k, λ) (λ even) difference set and any (n, ψ) PC trivial set give $2-(n; k, \psi)$ crystal sets.*

Theorem 10 *Suppose $n = q^2 + q + 1$, q a prime power. Then there exist $2-(q^2 + q + 1; q^2, 1)$ crystal sets. If $n = q^2 + q + 1$ is a prime there exists a $W(2(q^2 + q + 1), q^2 + 1)$.*

Proof. For these values of q there is a projective plane of order q which gives a $(q^2 + q + 1, q^2, q(q - 1))$ difference set. The Legendre construction [11, p9] shows there is a $\{0, \pm 1\}$ circulant matrix of order n , n an odd prime, with $n - 1$ non-zero elements for each row and column and inner products of rows -1 . The incidence matrix of the projective plane has inner product of all its rows 1. Thus these two circulant matrices give the first rows for our 2-circulant matrices to construct the $2-CW(2(q^2 + q + 1), q^2 + 1)$. □

Example 8 $\{1, 2, 4\}$ is a $2-(7, 2, 1)$ difference set. So we use this to make the complementary $2-(7, 4, 2)$ difference set and with the trivial set with 1 element we find the 2-complementary sequences:

$$0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ : \ 0 \ 1 \ 1 \ - \ 1 \ - \ -$$

This example was shown us by I. Kotsireas.

3.3 Crystalization Pattern $(3, 2)$ or $2-(n; 3, 2)$ Crystal Sets, n odd

Remark 6 From Lemma 6 we see that these crystal sets exist for all odd size sets. This greatly reduces the search space in looking for $(0, \pm 1)$ with zero periodic autocorrelation function as we have cut the search space from 3^{2n-2} to 2^{2n-5} .

3.4 The partition $(4, 1)$

Remark 7 Kotsireas and Koukouvinos [4] mentioned the possibility of the pattern $(4, 1)$ and Kotsireas provided the only known example. These results inspired us to consider the more general question of when the partition $(4, 1)$ could exist.

We note that the general pattern for four zeros in one set is

$$0, \underbrace{*, \dots, *}_j, 0, \underbrace{*, \dots, *}_k, 0, \underbrace{*, \dots, *}_\ell, 0, \underbrace{*, \dots, *}_m$$

where

$$n = j + k + \ell + m + 4. \tag{6}$$

□

This general arrangement means we can write the zeros as occurring at positions $x_1 = 0, x_2 = j + 1, x_3 = j + k + 2, x_4 = j + k + \ell + 3$. We assume j, k, ℓ, m are all nonnegative and each is less than or equal to $n - 4$. So the differences we obtain are

$(x_i - x_j)$	0	$j + 1$	$j + k + 2$	$j + k + \ell + 3$
0	*	$j + 1$	$j + k + 2$	$j + k + \ell + 3$
$j + 1$	$-j - 1$	*	$k + 1$	$k + \ell + 2$
$j + k + 2$	$-j - k - 2$	$-k - 1$	*	$\ell + 1$
$j + k + \ell + 3$	$-j - k - \ell - 3$	$-k - \ell - 2$	$-\ell - 1$	*

and each must occur an even number of times, that is, 0 or 2 or 4 ...

Remark 8 We observe that if $j = k = \ell = m$ then Equation (6) becomes $4j = n - 4$, which is not possible for j, k, ℓ, m all non-negative integers when n is odd.

Remark 9 We note none of j, k, ℓ, m can be $-1 \pmod n$, as each of them is non-negative and $\leq n - 4$. For any of them to be non-zero it would have to be the equivalent of $(n - 1) \pmod n$. This is not possible. This is exclusion by the pigeonhole principle.

Lemma 7 *Suppose n is odd. Then if the element given by $(x_i - x_j) \pmod n$ is even, the element given by $(x_j - x_i) \pmod n$ will be odd (and vice versa). Hence $j + 1 \not\equiv (-j - 1) \pmod n$.*

Lemma 8 *Suppose n is odd. Then it is only possible for two of j, k, ℓ and m to be equal if $7|n$.*

Proof. Without any loss of generality we will write $j = k = a - 1$ and $j + k + \ell + 3 = b$ (that is, $b = n - m - 1$). Then the differences from $x_1 = 0, x_2 = a, x_3 = 2a$ and $x_b = b$ are given in the following table:

$(x_i - x_j) \pmod n$	0	a	$2a$	b
0	*	a	$2a$	b
a	$-a$	*	a	$b - a$
$2a$	$-2a$	$-a$	*	$b - 2a$
b	$-b$	$a - b$	$2a - b$	*

Those that have not already paired are:

$$2a, -2a, b, -b, b - a, a - b, b - 2a, 2a - b.$$

We note that $2a \neq b$ since that causes the zero difference to occur. This also occurs if $2a$ is set equal to $2a - b$.

We try setting $2a$ equal to each of the other differences in turn. We have, from Lemma 7, that $2a \neq -2a$. Now $2a \neq b$ as this would leave the differences a and $-a$ to be paired, which is not possible by Lemma 7.

Setting $2a = b - a$ gives the differences $\{2a, 3a, 2a, a, -2a, -3a, -2a, -a\}$ or just $\{3a, a, -3a, -a\}$ to be paired, which implies $2|n$. Setting $2a = a - b$ gives the same result.

Setting $2a = b - 2a$ gives the differences $\{2a, 4a, 3a, 2a, -2a, -4a, -3a, -a\}$ or just $\{3a, 4a, -4a, -3a\}$ to be paired, which implies $2|n$ or $7|n$. Setting $2a = -b$ gives the same result.

Thus, since n is odd, we have the result. □

Theorem 11 *The general pattern (4, 1) described above can only exist for n , odd, if n is divisible by 7. This means we can only have $\{0, 1, 2, 4\}$ modulo 7 or $\{0, \alpha, 2\alpha, 4\alpha\}$ modulo 7α .*

Proof. The “1” in the partition is obtained by having zeros on the main diagonal of B . From the above array there are a total of 12 differences which arise from the first set. We consider their equality with the first, $j + 1$, one by one.

Case 1 By Lemma 7, $j + 1 \not\equiv (-j - 1) \pmod{n}$.

Case 2 Suppose $j + 1 = j + k + 2$. Then $k = -1 \equiv n - 1 \pmod{n}$. This is excluded by the previous remark.

Case 3 Suppose $j + 1 = j + k + \ell + 3$. This is equivalent to saying $k + \ell + 2 = 0$. This is also excluded, since all are non-negative.

Case 4 Suppose $j + 1 = k + 1$. This is covered by Lemma 8.

Case 5 Suppose $j + 1 = \ell + 1$. This is covered by Lemma 8.

Case 6 Suppose $j + 1 = -k - 1$. This means $j + k \equiv -2$ and so is excluded by the pigeonhole principle and that all are non-negative.

Case 7 Suppose $j + 1 = -j - k - \ell - 3$. Then $j = m$; this is covered by Lemma 8.

Case 8 Suppose $j + 1 = -k - \ell - 2$, that is, $j + k + \ell + 3 = n$. Here, using Equation (6), we have $n = m + 1$, which is not possible as $m = -1$ is excluded by the pigeonhole principle.

Case 9 Suppose $j + 1 = -\ell - 1$. Then $j + \ell \equiv -2 \pmod{n}$ and so is excluded by the pigeonhole principle.

Case 10 Suppose $j + 1 = -j - k - 2$. Then $2j + k + 3 \equiv 0$. This is not possible as j and k are non-negative.

Case 11 Suppose $j + 1 = k + \ell + 2$. Using Equation (6) this means $j + j + 3 + m \equiv n \pmod{n}$.

To simplify the visualization of this case we will rewrite the above array using $k + \ell = j - 1$ and then use symbols to identify obviously even numbers of entries. Thus we have

$$\begin{array}{c|cccc}
 (x_i - x_j) \pmod{n} & 0 & j + 1 & j + k + 2 & j + k + \ell + 3 \\
 \hline
 0 & * & j + 1 & j + k + 2 & 2j + 2 \\
 j + 1 & -j - 1 & * & k + 1 & j + 1 \\
 j + k + 2 & -j - k - 2 & -k - 1 & * & \ell + 1 \\
 j + k + \ell + 3 & -2j - 2 & -j - 1 & -\ell - 1 & *
 \end{array}$$

Thus we have the following; as yet unpaired differences is Λ :

$$\Lambda_1 = \{k+1, \ell+1, 2j+2, j+k+2, -k-1, -\ell-1, -2j-2, -j-k-2\}.$$

The only possibilities for $k+1$ are $2j+2, j+k+2, -\ell-1, -2j-2$ or $-j-k-2$. So we can have the following cases:

Case 11.1 Suppose $k+1 = j+k+2$; then $j = -1$ which is not possible by the pigeonhole principle.

Case 11.2 Suppose $k+1 = -\ell-1$. Then $k+\ell+2 = 0$. This is not possible.

Case 11.3 Suppose $k+1 = -2j-2$. Then $2j+k+3 = 0$. So $k = m$. This is covered by Lemma 8. This is not possible.

Case 11.4 Suppose $k+1 = -j-k-2$. However this is not possible as all the remaining differences cannot be paired.

Case 11.5 Suppose $k+1 = 2j+2$. Then we form Λ :

$$\Lambda_1 = \{k+1, \ell+1, k+1, 3j+3, -2j-2, -\ell-1, -2j-2, -3j-3\}.$$

This means the unpaired elements are

$$\{\ell+1, 3j+3, -\ell-1, -3j-3\}.$$

This means that in order to pair them, $\ell+1 = 3j+3$ or $\ell+1 = -3j-3$. Now $\ell+1 = -3j-3$ gives $3j+\ell+4 = 0$, which is not possible.

The remaining pair is $\ell+1 = 3j+3$ or $3j = \ell-2$ or $3j \leq n-6$, which is possible. Working backwards we have $\Lambda =$

$$\{j+1, j+1, -j-1, -j-1, 2j+2, 3j+3, 2j+2, 3j+3, -2j-2, -3j-3, -2j-2, -3j-3\}.$$

Hence the only surviving case is that of n divisible by 7. We replace $3j+3$ by $-4j-4$ to clarify the following. This means we can only have $\{0, 1, 2, 4\}$ modulo 7 or $\{0, \alpha, 2\alpha, 4\alpha\}$ modulo 7α . \square

We have shown that the general pattern (4.1), that is, $2 - (2n; 4, 1)$ crystal sets can only exist when $7|n$. This leads us to speculate that that the patterns $(q^2, 1)$, that is, $2 - (2n; q^2, 1)$ crystal sets, will only exist when $q^2 + q + 1$ is a prime and $(q^2 + q + 1)|n$.

4 Search space size reduction in the search for Proto-crystal sets.

4.1 Significance of these results

In a naive search for crystal sets we would first decide to search for all protocystal set sizes for k_1, k_2 from zero to n .

Next we would see that there is no need to look for $k_2 = 0$ unless k_1 is a square.

Next we note from Remark 1 that we can reduce our search by only considering this remark and its consequences; the overall search is limited to $(\frac{n-1}{2})^2$ cases to establish existences (of course there will be far more considering inequivalence).

Now we see that $k_1 = k_2$ is a special case. We also see that $k_1 = 1, k_2 = 0$ is a special case; Lemma 6 tells us these always exist.

The search is now further reduced by applying Corollary 1.

Crystal Sets under $n = 9$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$								
	k_1	k_2	$2n$ $-k_1$ $-k_2$	n $-k_1$ $+k_2$	k_1 $+k_2$	k_1 $= k_2$	a^2 $+b^2$	Reference
1	1	5		13			$2^2 + 3^2$	Remark 1
2	2	2		9		Y	$3^2 + 0^2$	Theorem 2
3	2	3	13				$2^2 + 3^2$	Remark 1
4	2	6	10				$1^2 + 3^2$	Remark 1
5	3	3	9			Y	$3^2 + 0^2$	Theorem 2
6	3	6	9				$3^2 + 0^2$	Remark 1
7	3	7	8				$2^2 + 2^2$	Remark 1
8	4	4	10			Y	$3^2 + 1^2$	Theorem 2
9	4	5	9				$3^2 + 0^2$	Remark 1
10	4	8	13				$3^2 + 2^2$	Remark 1
11	4	9	5				$1^2 + 2^2$	Remark 1

Table 1 $n = 9$: Values for which k_1 and k_2 can give crystal sets

Crystal Sets under $n = 11$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$								
	k_1	k_2	$2n$ $-k_1$ $-k_2$	n $-k_1$ $+k_2$	k_1 $+k_2$	k_1 $+k_2$	a^2 $+b^2$	Reference
1	1	5	16				$4^2 + 0^2$	Remark 1
2	2	2	18			Y	$3^2 + 3^2$	Theorem 2
3	2	3	17				$4^2 + 1^2$	Remark 1
4	2	6			8		$2^2 + 2^2$	Remark 1
5	2	7	13				$2^2 + 3^2$	Remark 1
6	3	3	16			Y	$4^2 + 0^2$	Theorem 2
7	3	6	13				$2^2 + 3^2$	Remark 1
8	3	7			10		$1^2 + 3^2$	Remark 1
9	4	4			8	Y	$2^2 + 2^2$	Theorem 2
10	4	5	13				$2^2 + 3^2$	Remark 1
11	4	8	10				$3^2 + 1^2$	Remark 1
12	4	9	9				$3^2 + 0^2$	Remark 1
13	5	5			10	Y	$1^2 + 3^2$	Theorem 2
14	5	8	9				$3^2 + 0^2$	Remark 1
15	5	9	8				$2^2 + 2^2$	Remark 1

Table 2 $n = 11$: Values for which k_1 and k_2 can give crystal sets

We assume the set is of size n and we search for sets of size $k < n$.

5 Algorithm

```

... pseudocode
MAIN(v)
1 ← input
2 if  $v \geq 2$ 
3 then GENERATE SUBSETS UNDER  $V(v)$ 

```

```

BOOL DETERMINE( $Total[1000]$ ,  $NUM$ )
1  $a[100]$ ,  $b[100] \leftarrow 0$ 
2  $i, j, k, cnt, No \leftarrow 0$ 
3 for  $i \leftarrow 0$  to  $NUM$ 
4 do for  $J \leftarrow 0$  to  $cnt$ 
5 do if  $a[j] == Total[i]$ 
6 then break
7 if  $j == cnt$ 
8 then  $a[cnt + 1] \leftarrow Total[i]$ 
9  $b[cnt - 1] ++$ 
10 else  $b[j] ++$ 
11 for  $k \leftarrow 0$  to  $cnt$ 
12 do if  $b[k] \bmod 2 == 1$ 
13 then break
14 else  $No ++$ 
15 if  $No == cnt$ 
16 then return true
17 else return false

```

```

SORTING( $TotalSet[1000]$ ,  $num$ )
1  $i, j, k, x \leftarrow 0$ 
2  $k \leftarrow num/2$ 
3 while  $k \geq 1$ 
4 do for  $i \leftarrow k$  to  $num$ 
5 do  $x \leftarrow TotalSet[i]$ 
6  $j \leftarrow i - k$ 
7 while  $j \geq 0$  and  $x \leq TotalSet[j]$ 
8 do  $TotalSet[j + k] \leftarrow TotalSet[j]$ 
9  $j \leftarrow j - k$ 
10  $TotalSet[j + k] \leftarrow x$ ;
11  $k \leftarrow k/2$ 

```

```

CRYSTALLIZATION( $n$ )
1  $i, j, q, p, t \leftarrow 0$ 
2  $M \leftarrow pow(2, n - 1) + 1$ 
3 malloc  $SubSet[i]$ 
4 malloc  $Length[i]$ 
5 ----- Generate subsets under v
6  $a, b \leftarrow 0$ 
7 position  $\leftarrow 0$ 
8 set[100]  $\leftarrow 0$ 
9 set[position]  $\leftarrow 0$ 
10 for  $i \leftarrow 0$  to  $2^{n-1}$ 
11 do if set[0] == 0
12 then  $SubSet[a][b] \leftarrow set[0]$ 
13  $b \leftarrow b + 1$ 
14 else break
15 for  $i \leftarrow 1$  to position
16 do  $SubSet[a][b] \leftarrow set[i]$ 
17  $b \leftarrow b + 1$ 
18  $Length[a][0] = b$ 
19 if set[position] <  $n - 1$ 
20 then set[position + 1]  $\leftarrow set[position] + 1$ 
21 position  $\leftarrow position + 1$ 
22 if position  $\neq 0$ 
23 then position  $\leftarrow position - 1$ 
24 set[position]  $\leftarrow set[position] + 1$ 
25 else break
26 ----- Calculate the differences
27 for  $p \leftarrow 0$  to  $M - 1$ 
28 do if  $Length[p][0] \leq (n - 1)/2$ 
29 then if  $(Length[p][0] * (Length[p][0] - 1)) \bmod 4 == 0$ 
30 then for  $q \leftarrow 0$  to  $M - 1$ 
31 do  $Totality[1000] \leftarrow 0$ 
32  $Num \leftarrow 0$ 
33 if  $(Length[q][0] * (Length[q][0] - 1)) \bmod 4 == 0$ 
34 then  $Totality[Num] \leftarrow (SubSet[p][i] - SubSet[p][j]) \bmod n$ 
35
36 SORTING( $Totality$ ,  $Num$ )
37 if Determine( $Totality$ ,  $Num$ )
38 then Print Crystal Sets
39
40 for  $p \leftarrow 0$  to  $M - 1$ 
41 do if  $Length[p][0] \leq (n - 1)/2$ 
42 then if  $(Length[p][0] * (Length[p][0] - 1)) \bmod 4 \neq 0$ 
43 then for  $q \leftarrow 0$  to  $M - 1$ 
44 do  $Totality[1000] \leftarrow 0$ 
45  $Num \leftarrow 0$ 
46 if  $(Length[q][0] * (Length[q][0] - 1)) \bmod 4 \neq 0$ 
47 then  $Totality[Num] \leftarrow (SubSet[p][i] - SubSet[p][j]) \bmod n$ 
48 SORTING( $Totality$ ,  $Num$ )
49 if Determine( $Totality$ ,  $Num$ )
50 then Print Crystal Sets
51 for  $i \leftarrow 0$  to  $M$ 
52 do free( $SubSet[i]$ )

```

6 Further Research

Prove the conjecture:

Conjecture 1 *The patterns $(q^2, 1)$, that is, $2 - (2n; q^2, 1)$ crystal sets will only exist when $q^2 + q + 1$ is a prime and $(q^2 + q + 1) | n$.*

Find further ways to cut down the search space. Find more infinite families of crystal sets.

Appendices

A More permissible values of n , k_1 and k_2

A.1 $n = 13$

Crystal Sets under $n = 13$, Universal Set = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}								
	k_1	k_2	$\frac{2n}{-k_1 - k_2}$	$\frac{n}{-k_1 + k_2}$	$\frac{k_1}{+k_2}$	$\frac{k_1}{=k_2}$	$\frac{a^2}{+b^2}$	Reference
1	1	9	16				$4^2 + 0^2$	Remark 1
2	2	2		13		Y	$2^2 + 3^2$	Theorem 2
3	2	3			5		$1^2 + 2^2$	Remark 1
4	2	6	18				$3^2 + 3^2$	Remark 1
5	2	7			9		$3^2 + 3^2$	Remark 1
6	3	3	20			Y	$4^2 + 2^2$	Theorem 2
7	3	6	17				$4^2 + 1^2$	Remark 1
8	3	7	16				$4^2 + 0^2$	Remark 1
9	3	10	13				$2^2 + 3^2$	Remark 1
10	4	4	18			Y	$3^2 + 3^2$	Theorem 2
11	4	5	17				$4^2 + 1^2$	Remark 1
12	4	8		17			$4^2 + 1^2$	Remark 1
13	4	9	13				$2^2 + 3^2$	Remark 1
14	4	12	10				$1^2 + 3^2$	Remark 1
15	4	13	9				$3^2 + 0^2$	Remark 1
16	5	5	16			Y	$4^2 + 0^2$	Theorem 2
17	5	8	13				$2^2 + 3^2$	Remark 1
18	5	9		17			$4^2 + 1^2$	Remark 1
19	6	6		13		Y	$2^2 + 3^2$	Theorem 2
20	6	7	13				$2^2 + 3^2$	Remark 1
21	6	10	10				$1^2 + 3^2$	Remark 1
22	6	11	9				$3^2 + 0^2$	Remark 1

A.2 $n = 15$

Crystal Sets under $n = 15$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$								
	k_1	k_2	$2n$ $- k_1$ $- k_2$	$n -$ k_1 $+ k_2$	k_1 $+ k_2$	k_1 $= k_2$	a^2 $+ b^2$	Reference
1	1	5	24	19				Remark 1
2	1	8			9		$3^2 + 0^2$	Remark 1
3	1	9	20				$4^2 + 2^2$	Remark 1
4	2	2	26			Y	$5^2 + 1^2$	Theorem 2
5	2	3	25				$5^2 + 0^2$	Remark 1
6	2	6			8		$2^2 + 2^2$	Remark 1
7	2	7		20			$4^2 + 2^2$	Remark 1
8	2	10	18				$3^2 + 3^2$	Remark 1
9	2	11	17				$4^2 + 1^2$	Remark 1
10	3	3	24	15		Y		Theorem 2
11	3	6		18			$3^2 + 3^2$	Remark 1
12	3	7	20				$4^2 + 2^2$	Remark 1
13	3	10	17				$4^2 + 1^2$	Remark 1
14	3	11	16				$4^2 + 0^2$	Remark 1
15	4	4			8	Y	$2^2 + 2^2$	Theorem 2
16	4	5	21	16			$4^2 + 0^2$	Remark 1
17	4	8	18				$3^2 + 3^2$	Remark 1
18	4	9	17				$4^2 + 1^2$	Remark 1
19	4	12			16		$4^2 + 0^2$	Remark 1
20	4	13	13				$2^2 + 3^2$	Remark 1
21	5	5	20			Y	$4^2 + 2^2$	Theorem 2
22	5	8	17				$4^2 + 1^2$	Remark 1
23	5	9	16				$4^2 + 0^2$	Remark 1
24	5	12	13				$2^2 + 3^2$	Remark 1
25	5	13			18		$3^2 + 3^2$	Remark 1
26	6	6	18			Y	$3^2 + 3^2$	Theorem 2
27	6	7	17				$4^2 + 1^2$	Remark 1
28	6	10			16		$4^2 + 0^2$	Remark 1
29	6	11	13				$2^2 + 3^2$	Remark 1
30	6	14	10				$1^2 + 3^2$	Remark 1
31	6	15	9				$0^2 + 3^2$	Remark 1
32	7	7	16			Y	$4^2 + 0^2$	Theorem 2
33	7	10	13				$2^2 + 3^2$	Remark 1
34	7	11			18		$3^2 + 3^2$	Remark 1
35	7	14	9				$0^2 + 3^2$	Remark 1
36	7	15	8				$2^2 + 2^2$	Remark 1

A.3 $n = 17$

Crystal Sets under $n = 17$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$								
	k_1	k_2	$2n - k_1$ $-k_2$	$n - k_1$ $+k_2$	k_1 $+ k_2$	k_1 $= k_2$	a^2 $+b^2$	<i>Reference</i>
1	1	9		25			$5^2 + 0^2$	Remark 1
2	2	2		17		Y	$4^2 + 1^2$	Theorem 2
3	2	3	29				$5^2 + 2^2$	Remark 1
4	2	6	26				$5^2 + 1^2$	Remark 1
5	2	7	25				$5^2 + 0^2$	Remark 1
6	2	10		25			$5^2 + 0^2$	Remark 1
7	2	11			13		$3^2 + 2^2$	Remark 1
8	3	3		17		Y	$4^2 + 1^2$	Theorem 2
9	3	6	25				$5^2 + 0^2$	Remark 1
10	3	7			10		$3^2 + 1^2$	Remark 1
11	3	10			13		$3^2 + 2^2$	Remark 1
12	3	11	20				$4^2 + 2^2$	Remark 1
13	4	4	26			Y	$5^2 + 1^2$	Theorem 2
14	4	5	25				$5^2 + 0^2$	Remark 1
15	4	8		13			$3^2 + 2^2$	Remark 1
16	4	9		13			$3^2 + 2^2$	Remark 1
17	4	12	18				$3^2 + 3^2$	Remark 1
18	4	13	17				$4^2 + 1^2$	Remark 1
19	5	5		17		Y	$4^2 + 1^2$	Theorem 2
20	5	8			13		$2^2 + 3^2$	Remark 1
21	5	9	20				$4^2 + 2^2$	Remark 1
22	5	12	17				$4^2 + 1^2$	Remark 1
23	5	13	16				$4^2 + 0^2$	Remark 1
24	6	6		17		Y	$4^2 + 1^2$	Theorem 2
25	6	7		18			$3^2 + 3^2$	Remark 1
26	6	10	18				$3^2 + 3^2$	Remark 1
27	6	11	17				$4^2 + 1^2$	Remark 1
28	6	14			20		$4^2 + 2^2$	Remark 1
29	6	15	13				$4^2 + 3^2$	Remark 1
30	7	7	20			Y	$4^2 + 2^2$	Theorem 2
31	7	10	17				$4^2 + 1^2$	Remark 1
32	7	11	16				$4^2 + 0^2$	Remark 1
33	7	14	13				$2^2 + 3^2$	Remark 1
34	7	15		25			$5^2 + 0^2$	Remark 1
35	8	8	18			Y	$3^2 + 3^2$	Theorem 2
36	8	9	17				$4^2 + 1^2$	Remark 1
37	8	12			20		$4^2 + 2^2$	Remark 1
38	8	13	13				$3^2 + 2^2$	Remark 1
39	8	16	10				$3^2 + 0^2$	Remark 1
40	8	17	9				$3^2 + 0^2$	Remark 1

B Examples of Permissible n, k_1 and k_2

B.1 $n = 9$

Crystal Sets under $n = 9$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$			
	k_1	k_2	<i>Sample</i>
1	1	5	$\{0,1\};\{0,1,5\}$
2	2	2	$\{0,1\};\{0,8\}$
3	2	3	$\{0,1\};\{0,1,5\}$
4	2	6	$\{0,1\};\{0,1,2,4,5,7\}$
5	3	3	$\{0,1,2\};\{0,7,8\}$
6	3	6	$\{0,1,2\};\{0,1,2,3,5,6\}$
7	3	7	$\{0,1,3\};\{0,1,3,4,5,6,8\}$
8	4	4	$\{0,1,2,3\};\{0,1,5,7\}$
9	4	5	$\{0,1,2,3\};\{0,1,2,3,6\}$
10	4	8	$\{0,1,3,6\};\{0,1,2,3,4,5,6,8\}$
11	4	9	$\{0,1,3,6\};\{0,1,2,3,4,5,6,8\}$

B.2 $n = 11$

Crystal Sets under $n = 11$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$			
	k_1	k_2	<i>Sample</i>
1	1	5	$\{0\};\{0,1,2,4,7\}$
2	2	2	$\{0,1\};\{0,10\}$
3	2	3	$\{0,1\};\{0,1,6\}$
4	2	6	$\{0,1\};\{0,1,2,4,5,8\}$
5	2	7	$\{0,1\};\{0,1,2,3,4,6,7\}$
6	3	3	$\{0,1,3\};\{0,1,9\}$
7	3	6	$\{0,1,2\};\{0,1,3,6,7,9\}$
8	3	7	$\{0,1,2\};\{0,1,2,3,5,7,10\}$
9	4	4	$\{0,1,2,4\};\{0,2,9,10\}$
10	4	5	$\{0,1,2,3\};\{0,1,2,3,7\}$
11	4	8	$\{0,1,2,3\};\{0,1,2,3,5,6,7,9\}$
12	4	9	$\{0,1,3,5\};\{0,1,2,3,5,7,8,9,10\}$
13	5	5	$\{0,1,2,3,4\};\{0,1,5,7,8\}$
14	5	8	$\{0,1,2,3,5\};\{0,1,3,4,6,7,8,10\}$
15	5	9	$\{0,1,2,3,5\};\{0,1,2,3,4,5,6,8,9\}$

B.3 $n = 13$

Crystal Sets under $n = 13$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$			
	k_1	k_2	Sample
1	1	9	$\{0\}; \{0, 1, 2, 3, 4, 5, 7, 9, 10\}$
2	2	2	$\{0, 1\}; \{0, 12\}$
3	2	3	$\{0, 1\}; \{0, 6, 7\}$
4	2	6	$\{0, 1\}; \{0, 1, 2, 3, 5, 7\}$
5	2	7	$\{0, 1\}; \{0, 1, 2, 4, 5, 7, 11\}$
6	3	3	$\{0, 1, 2\}; \{0, 1, 12\}$
7	3	6	$\{0, 1, 2\}; \{0, 1, 2, 3, 4, 8\}$
8	3	7	$\{0, 1, 2\}; \{0, 1, 2, 3, 5, 6, 10\}$
9	3	10	$\{0, 1, 4\}; \{0, 1, 2, 3, 4, 6, 7, 8, 9, 11\}$
10	4	4	$\{0, 1, 2, 3\}; \{0, 1, 4, 9\}$
11	4	5	$\{0, 1, 2, 3\}; \{0, 1, 2, 3, 8\}$
12	4	8	$\{0, 1, 2, 3\}; \{0, 1, 2, 3, 4, 7, 8, 10\}$
13	4	9	$\{0, 1, 2, 3\}; \{0, 1, 2, 3, 4, 6, 7, 8, 12\}$
14	4	12	$\{0, 1, 3, 9\}; \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
15	4	13	$\{0, 1, 4, 6\}; \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
16	5	5	$\{0, 1, 2, 3, 4\}; \{0, 1, 2, 3, 12\}$
17	5	8	$\{0, 1, 2, 3, 4\}; \{0, 1, 2, 3, 5, 6, 8, 11\}$
18	5	9	$\{0, 1, 2, 3, 4\}; \{0, 1, 2, 3, 5, 6, 7, 9, 10\}$
19	6	6	$\{0, 1, 2, 3, 4, 5\}; \{0, 1, 2, 3, 4, 12\}$
20	6	7	$\{0, 1, 2, 3, 4, 5\}; \{0, 1, 2, 3, 4, 5, 9\}$
21	6	10	$\{0, 1, 2, 3, 4, 5\}; \{0, 1, 2, 3, 4, 5, 7, 8, 9, 11\}$
22	6	11	$\{0, 1, 2, 3, 6, 8\}; \{0, 1, 3, 4, 6, 7, 8, 9, 10, 11, 12\}$

B.4 $n = 15$

Crystal Sets under $n = 15$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$			
	k_1	k_2	Sample
1	1	5	$\{0\};\{0,1,5,6,10\}$
2	1	8	$\{0\};\{0,1,2,4,6,7,10,14\}$
3	1	9	$\{0\};\{0,1,2,3,4,5,6,8,12\}$
4	2	2	$\{0,1\};\{0,14\}$
5	2	3	$\{0,1\};\{0,1,8\}$
6	2	6	$\{0,1\};\{0,1,2,3,8,12\}$
7	2	7	$\{0,1\};\{0,1,2,5,6,9,11\}$
8	2	10	$\{0,1\};\{0,1,2,3,4,5,8,9,10,12\}$
9	2	11	$\{0,1\};\{0,1,2,3,4,6,8,9,11,12,13\}$
10	3	3	$\{0,1,2\};\{0,1,14\}$
11	3	6	$\{0,1,2\};\{0,1,2,4,6,9\}$
12	3	7	$\{0,1,2\};\{0,1,2,5,8,10,13\}$
13	3	10	$\{0,1,2\};\{0,1,2,3,4,5,6,8,9,10\}$
14	3	11	$\{0,1,2\};\{0,1,2,3,5,6,7,8,9,12,13\}$
15	4	4	$\{0,1,2,3\};\{0,1,2,14\}$
16	4	5	$\{0,1,2,3\};\{0,1,7,13,14\}$
17	4	8	$\{0,1,2,3\};\{0,1,2,3,4,6,7,12\}$
18	4	9	$\{0,7,8,12\};\{0,1,2,3,4,5,8,10,12\}$
19	4	12	$\{0,7,8,12\};\{0,1,2,3,4,5,7,8,9,10,11,13\}$
20	4	13	$\{0,7,9,10\};\{0,1,2,3,5,6,7,9,10,11,12,13, 14\}$
21	5	5	$\{0,1,2,3,4\};\{0,1,2,3,14\}$
22	5	8	$\{0,1,2,3,4\};\{0,1,2,3,4,6,9,11\}$
23	5	9	$\{0,1,2,3,4\};\{0,1,2,4,5,7,8,10,12\}$
24	5	12	$\{0,3,6,9,11\};\{0,1,2,3,4,5,8,9,10,11,12,13, 14\}$
25	5	13	$\{0,3,6,9,11\};\{0,1,2,3,4,5,6,7,8,9,10,11, 14\}$
26	6	6	$\{0,1,2,3,4,5\};\{0,1,2,3,4,14\}$
27	6	7	$\{0,1,2,3,4,5\};\{0,1,2,3,4,5,10\}$
28	6	10	$\{0,1,2,3,4,6\};\{0,1,2,3,4,5,6,8,10,13\}$
29	6	11	$\{0,7,8,12,13,14\};\{0,1,2,3,5,6,8,11,12,13, 14\}$
30	6	14	$\{0,3,6,9,10,12\};\{0,1,2,3,4,5,6,7,8,9,10, 11,12,13\}$
31	6	15	$\{0,1,2,3,7,9\};\{0,1,2,3,4,5,6,7,8,9,10,11, 12,13,14\}$
32	7	7	$\{0,1,2,3,4,5,6\};\{0,1,2,3,4,5,14\}$
33	7	10	$\{0,1,2,3,4,5,6\};\{0,1,2,3,5,6,7,9,11,14\}$
34	7	11	$\{0,1,2,3,4,5,6\};\{0,1,2,3,4,5,7,8,9,12,13\}$
35	7	14	$\{0,1,2,4,5,8,10\};\{0,1,2,3,4,5,6,7,8,9,10, 11,12,13\}$
36	7	15	$\{0,1,2,4,5,8,10\};\{0,1,2,3,4,5,6,7,8,9,10, 11,12,13,14\}$

B.5 $n = 17$

Crystal Sets under $n = 17$, Universal Set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$			
	k_1	k_2	Sample
1	1	9	$\{0\};\{0,1,2,3,4,5,6,10,12\}$
2	2	2	$\{0,1\};\{0,16\}$
3	2	3	$\{0,1\};\{0,1,9\}$
4	2	6	$\{0,1\};\{0,1,2,3,6,13\}$
5	2	7	$\{0,1\};\{0,1,2,4,5,8,14\}$
6	2	10	$\{0,1\};\{0,1,2,3,4,6,8,10,11,14\}$
7	2	11	$\{0,1\};\{0,1,2,3,4,5,6,7,9,12,14\}$
8	3	3	$\{0,1,2\};\{0,1,16\}$
9	3	6	$\{0,1,2\};\{0,1,3,7,10,11\}$
10	3	7	$\{0,1,2\};\{0,2,3,5,7,12,13\}$
11	3	10	$\{0,1,2\};\{0,2,3,5,9,10,13,14,15,16\}$
12	3	11	$\{0,1,2\};\{0,1,2,3,4,5,6,7,9,10,13\}$
13	4	4	$\{0,1,2,3\};\{0,1,2,16\}$
14	4	5	$\{0,1,2,3\};\{0,1,2,3,10\}$
15	4	8	$\{0,1,2,3\};\{0,1,2,3,5,6,7,11\}$
16	4	9	$\{0,1,2,3\};\{0,1,2,3,5,6,8,10,16\}$
17	4	12	$\{0,1,2,3\};\{0,1,2,3,4,5,6,7,9,10,12,13\}$
18	4	13	$\{0,1,2,3\};\{0,1,2,3,4,6,7,8,10,11,12,13,15\}$
19	5	5	$\{0,1,2,3,4\};\{0,1,2,3,16\}$
20	5	8	$\{0,1,2,3,4\};\{0,1,2,3,4,5,7,8,14\}$
21	5	9	$\{0,1,2,3,4\};\{0,1,2,3,5,6,10,14\}$
22	5	12	$\{0,1,2,3,4\};\{0,1,2,3,4,5,6,7,9,10,11,12\}$
23	5	13	$\{0,1,2,3,5\};\{0,1,2,3,4,6,7,8,10,11,12,14,15\}$
24	6	6	$\{0,1,2,3,4,5\};\{0,1,2,3,4,16\}$
25	6	7	$\{0,1,2,3,4,5\};\{0,1,2,3,4,5,11\}$
26	6	10	$\{0,1,2,3,4,5\};\{0,1,2,3,4,6,11,12,13,14\}$
27	6	11	$\{0,1,2,3,4,5\};\{0,1,2,3,4,6,8,12,13,15,16\}$
28	6	14	$\{0,1,2,3,4,8\};\{0,1,2,3,4,5,6,7,8,9,10,11,13,14\}$
29	6	15	$\{0,1,2,3,5,8\};\{0,1,2,3,4,5,6,7,8,9,10,12,13,14,15\}$
30	7	7	$\{0,1,2,3,4,5,6\};\{0,1,2,3,4,5,16\}$
31	7	10	$\{0,1,2,3,4,5,6\};\{0,1,2,3,4,7,9,10,11,14\}$
32	7	11	$\{0,1,2,3,4,5,6\};\{0,1,2,3,4,8,9,10,11,14,16\}$
33	7	14	$\{0,1,2,3,4,5,7\};\{0,1,2,3,4,5,6,7,9,10,11,12,14,15\}$
34	7	15	$\{0,1,2,3,5,7,15\};\{0,1,2,3,4,5,6,7,8,10,11,12,13,14,15\}$
35	8	8	$\{0,1,2,3,4,5,6,7\};\{0,1,2,3,4,5,6,16\}$
36	8	9	$\{0,1,2,3,4,5,6,7\};\{0,1,2,3,4,5,6,7,12\}$
37	8	12	$\{0,2,4,6,8,9,10,11\};\{0,1,2,3,4,5,6,7,10,11,12,14\}$
38	8	13	$\{0,1,2,3,4,5,6,8\};\{0,1,2,3,4,5,6,7,8,9,11,12,15\}$
39	8	16	$\{0,1,2,3,4,8,9,12\};\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,16\}$
40	8	17	$\{0,1,2,3,4,8,9,12\};\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$

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