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Properties of independent Roman domination in graphs*

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Abstract

A Roman dominating function on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number of G, $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. In this paper, we study independent Roman domination in graphs and obtain some properties, bounds and characterizations for the independent Roman domination number of a graph.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* $deg(x)$ of a vertex x denotes the number of neighbors of x in G. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. For a graph G and a subset of vertices S we denote by $G[S]$ the subgraph of G induced by S. A subset S of vertices is *independent* if $G[S]$ has no edge. A set $S \subseteq V(G)$ is an *independent dominating set* if S is independent and dominating set. The minimum cardinality of such a set is the *independent domination number* $i(G)$. For notation and graph theory terminology in general we follow [2].

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With K_n we denote the *complete graph* on n vertices and with C_n the *cycle* of length n. The *cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. An *r-partite graph* G is a graph whose vertex set $V(G)$ can be partitioned into r sets of pairwise nonadjacent vertices. For positive integers $p_1, p_2, \ldots, p_r$, the *complete r-partite graph* $K_{p_1,p_2,\ldots,p_r}$ is the r-partite graph with partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ such that $|V_i| = p_i$ for $1 \leq i \leq r$ and such that every two vertices belonging to different partite sets are adjacent to each other.

We recall that a leaf in a graph is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. Let $L(G)$ be the set of all leaves in a graph G.

For a graph G, let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \rightarrow \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a *Roman dominating function*, or just RDF, if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0; V_1; V_2)$ is called a γ_R-function if it is an RDF and $f(V(G)) = \gamma_R(G)$, [1, 5, 4].

Cockayne et al. in [1] introduced the concept of independent Roman domination in graphs. An RDF $f = (V_0; V_1; V_2)$ is called an *independent Roman dominating function* if the set $V_1 \cup V_2$ is an independent set. The *independent Roman domination number*, $i_R(G)$, is the minimum weight of an independent RDF on G. McRae [3] has shown that the decision problem corresponding to independent Roman dominating functions is NP-complete, even when restricted to bipartite graphs.

In this paper we study the independent Roman domination number in a graph G. We give some properties, bounds and characterizations for this new parameter. We make use of the following:

Proposition A [1]: *Let $f = (V_0; V_1; V_2)$ be any γ_R-function. Then*
(a) $G[V_1]$, the subgraph induced by V_1, has maximum degree 1;
(b) no edge of G joins V_1 and V_2.

We call an RDF $f = (V_0; V_1; V_2)$ in a graph G an i_R-function if it is an independent RDF and $f(V(G)) = i_R(G)$. Let $v \in S \subseteq V(G)$. A vertex u is called a private neighbor of v with respect to S if $u \in N[v] - N[S - v]$.

2 General results and bounds

It is obvious that any independent RDF on G is also an RDF. So for any graph G, $i_R(G) \geq \gamma_R(G)$. Our first aim is to give a characterization of all graphs with equal

Roman domination number and independent Roman domination number.

Lemma 1 *Let $f = (V_0; V_1; V_2)$ be an RDF for a graph G. If V_2 is independent, then there is an independent RDF g for G such that $w(g) \leq w(f)$.*

Proof. Let G be a graph and let $f = (V_0; V_1; V_2)$ be an RDF for G such that V_2 is independent. Let $f^1 = (V_0^1; V_1^1; V_2^1)$ where $V_0^1 = N(V_2)$, $V_1^1 = V_1$ and $V_2^1 = V_2$. If V_1 is independent then f^1 is an independent RDF for G. So suppose that V_1 is not independent. Let $x_1 \in V_1$ be a vertex such that $deg_{G[V_1]}(x_1) \geq 1$, and let $f^2 = (V_0^2; V_1^2; V_2^2)$, where $V_0^2 = V_0 \cup N_{G[V_1]}(x_1)$, $V_1^2 = V_1 \setminus N_{G[V_1]}[x_1]$ and $V_2^2 = V_2 \cup \{x_1\}$. It is obvious that V_2^2 is independent. If V_1^2 is not independent, we continue the above process. So there is an integer k such that $f^k = (V_0^k; V_1^k; V_2^k)$ and $V_1^k \cup V_2^k$ is independent. Since for $i = 1, 2, ..., k-1$, $w(f^{i+1}) \leq w(f^i)$, we deduce that f^k is an independent RDF for G with $w(f^k) \leq w(f)$. ■

Corollary 2 *For a graph G, $i_R(G) = \gamma_R(G)$ if and only if there is a γ_R-function $f = (V_0; V_1; V_2)$ such that V_2 is independent.*

Applying Corollary 2, the following is easily verified.

Proposition 3 *(1) $i_R(K_n) = \gamma_R(K_n)$.*
(2) $i_R(P_n) = i_R(C_n) = \gamma_R(P_n) = \gamma_R(C_n)$.
(3) $i_R(P_2 \Box P_n) = \gamma_R(P_2 \Box P_n)$.

Theorem 4 $i_R(K_{m,n}) = min\{m, n\} + 1$.

Proof. Let X and Y be the partite sets of $K_{m,n}$. First notice that if $x \in X$, then $(Y; X \setminus \{x\}; \{x\})$ is an independent RDF for G, and so $i_R(K_{m,n}) \leq min\{m, n\} + 1$. Now let $f = (V_0; V_1; V_2)$ be an independent RDF for $K_{m,n}$. It is obvious that either $V_1 \cup V_2 \subseteq X$ or $V_1 \cup V_2 \subseteq Y$. Further, $V_2 \neq \emptyset$. We deduce that the weight of f is at least $min\{m, n\} + 1$. Thus the result follows. ■

In the following we prove that for any graph G with maximum degree at most three, $i_R(G) = \gamma_R(G)$.

Theorem 5 *For any graph G with $\Delta(G) \leq 3$, $\gamma_R(G) = i_R(G)$.*

Proof. Let G be a graph with $\Delta(G) \leq 3$. Suppose to the contrary that $\gamma_R(G) < i_R(G)$. Let $f = (V_0; V_1; V_2)$ be a γ_R-function for G. By Corollary 2, V_2 is not independent. If $G[V_2]$ has a component containing a path P_3 on three vertices, then we label the central vertex of P_3 by 0, and its private neighbor with respect to V_2 by 1, and obtain an RDF for G with weight less than $\gamma_R(G)$. This contradiction leads that any component of $G[V_2]$ is a K_2. Let $G_1, G_2, ..., G_r$ be the components of $G[V_2]$ with $V(G_i) = \{v_i, u_i\}$ for $1 \leq i \leq r$. It follows that

$$g = (V_0 \cup \{u_1, ..., u_r\}; V_1 \cup (\cup_{i=1}^r N(u_i) \setminus \{v_i\}); V_2 \setminus \cup_{i=1}^r \{u_i\})$$

is an RDF for G with weight $\gamma_R(G)$. By Lemma 1, there is a $i_R(G)$-function g_1 such that $w(g_1) \leq w(g)$. We deduce that $\gamma_R(G) = i_R(G)$, a contradiction. ■

Corollary 6 *If $\gamma_R(G) < i_R(G)$, then $\Delta(G) \geq 4$.*

Next we give some upper bounds.

Theorem 7 *For any graph G of order n, $i_R(G) \leq n$. Further, the equality holds if and only if $G = mK_2 \cup \overline{K_l}$ for some integers m, l with $n = 2m + l$.*

Proof. Let G be a graph of order n. It is obvious that $(\emptyset; V(G); \emptyset)$ is an RDF for G. So Lemma 1 shows that $i_R(G) \leq n$.

For the next part, first it is obvious that if $G = mK_2 \cup \overline{K_l}$ for some integers m, l with $n = 2m + l$, then $i_R(G) = n$. Suppose that G is a graph of order n and $i_R(G) = n$. For $n = 1, 2$ the statement is obviously true. Thus assume that $n > 2$. We show that G has maximum degree at most one. Suppose to the contrary, that v is a vertex of degree at least 2. Let G_1 be the graph obtained from G by removing $G[N[v]]$. By the first part of the current theorem, $i_R(G_1) \leq |V(G_1)|$. Let $f = (V_0; V_1; V_2)$ be an i_R-function for G_1. Then $g = (V_0 \cup N(v); V_1; V_2 \cup \{v\})$ is an independent RDF for G with weight less than n, a contradiction. Thus G has maximum degree at most one, and so the result follows. ■

Theorem 8 *For any graph G with $\Delta(G) \geq 3$,*

$$i_R(G) \leq \gamma_R(G) + \frac{\gamma_R(G) - 2}{2}(\Delta(G) - 3).$$

Proof. Let G be a graph with $\Delta(G) \geq 3$. If $i_R(G) = \gamma_R(G)$, then the inequality follows since $\gamma_R(G) \geq 2$. Suppose that $i_R(G) > \gamma_R(G)$. Let $f = (V_0; V_1; V_2)$ be a γ_R-function for G. By Corollary 2, V_2 is not independent. Let $x_1y_1 \in E(G[V_2])$, and let Y_1 be the set of all private neighbors of y_1 with respect to V_2. Then $f_1 = (V_0^{(1)}; V_1^{(1)}; V_2^{(1)})$ is an RDF for G, where $V_0^{(1)} = (V_0 \setminus Y_1) \cup \{y_1\}$, $V_1^{(1)} = V_1 \cup Y_1$ and $V_2^{(1)} = V_2 \setminus \{y_1\}$. If $V_2^{(1)}$ is not independent, then we let $x_2y_2 \in E(G[V_2^{(1)}])$, and let Y_2 be the set of all private neighbors of y_2 with respect to $V_2^{(1)}$. Then

$$f_2 = (V_0^{(2)} = (V_0^{(1)} \setminus Y_2) \cup \{y_2\}; V_1^{(2)} = V_1^{(1)} \cup Y_2; V_2^{(2)} = V_2^{(1)} \setminus \{y_2\})$$

is an RDF for G. Continuing this process produces an RDF $f_m = (V_0^{(m)}; V_1^{(m)}; V_2^{(m)})$ for some integer m such that $V_2^{(m)}$ is independent. By Lemma 1, there is an i_R-function g such that $i_R(G) = w(g) \leq w(f_m)$. It follows that $i_R(G) \leq \gamma_R(G) + m(\Delta(G) - 3)$. But $m \leq |V_2| - 1 \leq \frac{\gamma_R(G)}{2} - 1 = \frac{\gamma_R(G)-2}{2}$. We deduce that $i_R(G) \leq \gamma_R(G) + \frac{\gamma_R(G)-2}{2}(\Delta(G) - 3)$. ■

We remark that for a graph G with $\Delta(G) = 3$, Theorem 8 implies that $i_R(G) = \gamma_R(G)$. This fact is also stated in Theorem 5. In the following we characterize all graphs G with $\Delta(G) > 3$ achieving equality in Theorem 8.

Theorem 9 *For a graph G with $\Delta(G) > 3$, $i_R(G) = \gamma_R(G) + \frac{\gamma_R(G)-2}{2}(\Delta(G) - 3)$ if and only if either $\gamma_R(G) = i_R(G) = 2$, or $\gamma_R(G) = 4$, $i_R(G) = 1 + \Delta(G)$.*

Proof. Let G be a graph with $\Delta(G) > 3$. First notice that if either $\gamma_R(G) = i_R(G) = 2$, or $\gamma_R(G) = 4$, $i_R(G) = 1+\Delta(G)$, then $i_R(G) = \gamma_R(G)+\frac{\gamma_R(G)-2}{2}(\Delta(G)-3)$. Suppose that $i_R(G) = \gamma_R(G) + \frac{\gamma_R(G)-2}{2}(\Delta(G) - 3)$. The result is obvious for $\gamma_R(G) = 2$. So suppose that $\gamma_R(G) \geq 3$. Since $\Delta(G) > 3$, $i_R(G) \neq \gamma_R(G)$, and so by Corollary 2, for any γ_R-function $f = (V_0; V_1; V_2)$ on G, V_2 is not independent. Let $f = (V_0; V_1; V_2)$ be a γ_R-function on G, and let A be the a maximum independent subset of V_2. We observe that $V_2 \setminus A \neq \emptyset$. We proceed with Fact 1.

Fact 1. $|N[V_2] \setminus N[A]| \leq (\Delta(G) - 1)|V_2 \setminus A|$.
To see this notice that $N[V_2] \setminus N[A] = N[V_2 \setminus A] \setminus N[A]$. Also $N[V_2] \setminus N[A] = \bigcup_{u\in V_2\setminus A} N[u] \setminus N[A]$. Now

$$\begin{aligned} |N[V_2] \setminus N[A]| &= |\bigcup_{u\in V_2\setminus A} N[u] \setminus N[A]| \\ &\leq \sum_{u\in V_2\setminus A} |N[u] - N[A]|. \end{aligned}$$

But for any $u \in V_2 \setminus A$ by maximality of A we find that u is adjacent to a vertex $v_1 \in A$. Thus $|N[u] \cap N[A]| \geq 2$. We deduce that $|N[u] \setminus N[A]| = |N[u]| - |N[A] \cap N[u]| \leq \Delta(G) - 1$. This completes a proof for Fact 1.

Let $g = (N(A); V(G) \setminus N[A]; A)$. Then g is an RDF for G. It follows that $w(g) = 2|A|+|V(G)\setminus N[A]|$. Since $V_1 \subseteq V(G)\setminus N[A]$, we observe that $V(G)\setminus N[A] = V_1 \cup (N[V_2] - N[A])$. Now

$$\begin{aligned} w(g) &= 2|A| + |V(G) \setminus N[A]| \\ &= 2|A| + |V_1 \cup (N[V_2] - N[A])| \\ &= 2|A| + |V_1| + |N[V_2] - N[A]|. \end{aligned}$$

Applying Fact 1, we obtain that

$$\begin{aligned} w(g) &\leq 2|A| + |V_1| + (\Delta(G) - 1)|V_2 \setminus A| \\ &= 2|A| + |V_1| + (\Delta(G) - 1)(|V_2| - |A|) \\ &= 2|A| - 2|V_2| + 2|V_2| + |V_1| + (\Delta(G) - 1)(|V_2| - |A|) \\ &= \gamma_R(G) + (|V_2| - |A|)(\Delta(G) - 3) \\ &= \gamma_R(G) + (\frac{\gamma_R(G) - |V_1|}{2} - |A|)(\Delta(G) - 3) \\ &= \gamma_R(G) + (\frac{\gamma_R(G) - |V_1| - 2|A|}{2})(\Delta(G) - 3). \end{aligned}$$

Since $\Delta(G) > 3$ we find that $\gamma_R(G) \geq 2$. It is a routine matter to see that $|V_1| + 2|A| \geq 2$. This leads that $w(g) \leq \gamma_R(G) + (\frac{\gamma_R(G)-2}{2})(\Delta(G) - 3)$. By Lemma 1, there is an i_R-function h for G such that

$$i_R(G) = w(h) \leq w(g) \leq \gamma_R(G) + (\frac{\gamma_R(G)-2}{2})(\Delta(G)-3) = i_R(G).$$

We conclude that

$$\gamma_R(G) + (\frac{\gamma_R(G)-2}{2})(\Delta(G)-3) = \gamma_R(G) + (\frac{\gamma_R(G)-|V_1|-2|A|}{2})(\Delta(G)-3) \qquad (1)$$

and

$$|N[V_2] \setminus N[A]| = (\Delta(G)-1)|V_2 \setminus A| \qquad (2)$$

The relation (1) implies that $|V_1| + 2|A| = 2$, and therefore $|V_1| = 0$ and $|A| = 1$. Then $G[V_2]$ is a complete graph. Let $G[V_2] = K_t$ where $t \geq 2$. Let $u \in V_2 \setminus A$. It follows that $|N[u] \setminus N[A]| \leq \Delta(G) + 1 - t$, and so

$$\begin{aligned} |\bigcup_{u \in V_2 \setminus A} N[u] \setminus N[A]| &\leq \sum_{u \in V_2 \setminus A} |N[u] \setminus N[A]| \\ &\leq (\Delta(G) + 1 - t)|V_2 \setminus A| \\ &\leq (\Delta(G) - 1)|V_2 \setminus A|. \end{aligned}$$

But relation (2) implies that $(\Delta(G) + 1 - t)|V_2 \setminus A| = (\Delta(G) - 1)|V_2 \setminus A|$, and therefore $t = 2$. Since $|V_1| = 0$, we deduce that $\gamma_R(G) = 4$ and so $i_R(G) = 1 + \Delta(G)$. ■

3 Relations with independent domination

It is obvious that for any graph G, $i(G) \leq i_R(G) \leq 2i(G)$. In this section we give a characterization of all graphs G which $i_R(G) = i(G) + k$ for $0 \leq k \leq i(G)$.

Theorem 10 *For a graph G of order n, $i_R(G) = i(G)$ if and only if $G = \overline{K_n}$*

Proof. It is obvious that $i_R(\overline{K_n}) = i(\overline{K_n})$. Let G be a graph of order n and $i_R(G) = i(G)$. Let $f = (V_0; V_1; V_2)$ be an i_R-function for G. It follows that

$$i(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = i_R(G).$$

We deduce that $V_2 = \emptyset$, and so $i(G) = n$. Thus, $G = \overline{K_n}$. ■

The next theorem gives a characterization for all graphs G with $i_R(G) = i(G)+1$.

Theorem 11 *For a graph G of order n, $i_R(G) = i(G) + 1$ if and only if G has a vertex of degree $n - i(G)$.*

Proof. Let G has a vertex v with $deg(v) = n - i(G)$. First let $G' = G[V(G) \setminus N[v]]$. By Theorem 7, $i_R(G') \le |V(G')|$. Let $(V_0'; V_1'; V_2')$ be an i_R-function for G'. It follows that $(N(v) \cup V_0{}'; V_1{}'; \{v\} \cup \{V_2\}')$ is an independent RDF G. We conclude that $i_R(G) \le i(G) + 1$.

For the converse suppose that $i_R(G) = i(G) + 1$. Let $g = (V_0; V_1; V_2)$ be an i_R-function. Since $V_1 \cup V_2$ is an independent dominating set, we deduce that $|V_2| \le 1$. If $V_2 = \emptyset$, then V_1 is not independent, which is a contradiction. So $|V_2| = 1$. Let $V_2 = \{u\}$. Then any vertex of V_0 is adjacent to u. So $deg(u) = |V_0| = n - |V_1| - |V_2| = n - i(G)$. ■

Now we are ready to give the main result of this section.

Theorem 12 *For a graph G of order n and an integer k with $2 \le k \le i(G)$, $i_R(G) = i(G) + k$ if and only if the following hold:*
(i) For any integer s with $1 \le s \le k-1$, there is no independent set U_t of cardinality t such that $1 \le t \le s$ and $|\bigcup_{v \in U_t} N[v]| = n - i(G) - s + 2t$,
(ii) There is an independent set W_l of cardinality l for some integer l with $1 \le l \le k$ such that $|\bigcup_{v \in W_l} N[v]| = n - i(G) - k + 2l$.

Proof. Let G be a graph of order n and $k \ge 2$ be an integer.
($\Rightarrow$) Let $i_R(G) = i(G) + k$. First we prove (i). Suppose to the contrary that there exist two integers s_0 and t_0 with $1 \le t_0 \le s_0 \le k - 1$ and G has an independent set U_{t_0} of cardinality t_0 such that $|\bigcup_{v \in U_t} N[v]| = n - i(G) - s + 2t$. Then $f = (N(U_{t_0}); V(G) \backslash N[U_{t_0}]; U_{t_0})$ is an RDF for G such that U_{t_0} is independent. By Lemma 1, there is a $i_R(G)$ function g such that $w(g) \le w(f) = i(G) + s_0 \le i(G) + k - 1$. This is a contradiction, since $i_R(G) = i(G) + k$. Thus (i) is true.
Now we prove (ii). From $2|V_2| + |V_1| = i(G) + k$ we have that $|V_1| + |V_2| \ge i(G)$. By Theorem 10, $|V_2| \ne 0$. Then $|V_2| = l$ $(1 \le l \le k)$, and so $|V_1| = i(G) + k - 2l$. Let $W_l = V_2$. There exists an independent set W_l of cardinality l such that

$$|\bigcup_{v \in W_l} N[v]| = n - |V_1| = n - (i(G) + k - 2l) = n - i(G) - k + 2l.$$

($\Longleftarrow$) Suppose to the contrary that $i_R(G) = i(G) + m$ where $m \le k - 1$. If $m = 1$ then Theorem 11 implies that G has a vertex of degree $n - i(G)$, and putting $s = t = 1$ in (i) produce a contradiction. So suppose that $m \ge 2$. By part (ii) of ($\Rightarrow$) G has a set W_l of cardinality l with $1 \le l \le k - 1$ such that

$$|\bigcup_{v \in W_l} N[v]| = n - i(G) - m + 2l.$$

By (i) for any integer s with $1 \le s \le k - 1$, G does not have a set U_t of cardinality t $(1 \le t \le s)$ such that $|\bigcup_{v \in U_t} N[v]| = n - i(G) - s + 2t$, a contradiction. On the other hand if $i_R(G) = i(G) + m$ where $m \ge k + 1$, then by part (i) of ($\Rightarrow$) for any integer s with $1 \le s \le m - 1$, G does not have a set U_t of cardinality t $(1 \le t \le s)$ such that

$$|\bigcup_{v \in U_t} N[v]| = n - i(G) - s + 2t.$$

This contradicts (ii). We deduce that $i_R(G) = i(G) + k$. ■

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References

[1] E. J. Cockayne, P. M. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, On Roman domination in graphs, *Discrete Math.* **278** (2004), 11–22.

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, NewYork, 1998.

[3] A. A. McRae, Private communication, March 2000.

[4] C. S. ReVelle and K. E. Rosing, Defendens imperium romanum: a classical problem in military strategy, *Amer Math. Monthly* **107** (2000), 585–594.

[5] I. Stewart, Defend the Roman Empire!, *Sci. Amer.* **281** (6) (1999), 136–139.

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