

Feedback numbers of Kautz undirected graphs*

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Abstract

The feedback number $f(d, n)$ of the Kautz undirected graph $UK(d, n)$ is the minimum number of vertices whose removal results in an acyclic graph. This paper shows $\lceil (d^{n+1} - d^{n-1} - \frac{1}{2}d(d+1) + 1)/(2d-1) \rceil \leq f(d, n) \leq d^n - (\lfloor \frac{d^2}{4} \rfloor + 1)d^{n-2}$, which implies that $f(2, n) = 2^{n-1}$, as obtained by Kráľovič and Ružička [*Information Processing Letters* 86 (4) (2003), 191–196].

1 Introduction

Let $G = (V, E)$ be a simple graph, i.e., loopless and without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. It is well-known that the cycle rank of a graph G is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if G has v vertices, ε edges, and ω connected components, then the minimum number of edges whose deletion from G leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G) = \varepsilon - v + \omega$ (see, for example Xu [21]). A corresponding problem is the removal of vertices. A subset $F \subset V(G)$ is called a *feedback vertex set* if the subgraph $G - F$ is acyclic, that is, if $G - F$ is a forest. The minimum cardinality of all feedback vertex sets is called the *feedback number* (or *decycling number* proposed first by Beineke and Vandell [5]) of G . A feedback vertex set of this cardinality is called a *minimum feedback vertex set*.

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Determining the feedback number of a graph G is equivalent to finding the greatest order of an induced forest of G , proposed first by Erdős, Saks and Sós [8], since the sum of the two numbers equals the order of G . Thus, it has attracted the interest of many researchers, both for general graphs and for specific graphs. A review of recent results and open problems on the decycling number is provided by Bau and Beineke [4].

Apart from its graph-theoretical interest, the minimum feedback vertex set problem has important application to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [16], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters' placement problem (see [7, 9]).

In fact, the problem of finding a feedback number is NP-hard for general graphs [11]. The best known approximation algorithm for this problem has approximation ratio 2 [1]. There are also polynomial time algorithms for a number of topologies, such as reducible graphs [18], cocomparability graphs [11], convex bipartite graphs [11], cyclically reducible graphs [19], and interval graphs [14]. Recently, the lower and the upper bounds of the feedback numbers have been established and improved on some graphs, such as hypercubic graphs, meshes, toroids, butterflies, cube-connected cycles, hypercubes, directed split-stars, de Bruijn digraphs, generalized de Bruijn digraphs and de Bruijn undirected graphs (see [1, 2, 3, 7, 9, 10, 14, 15, 17, 18, 19, 22, 23, 24]).

In this paper, we consider an important class of interconnection networks, namely, Kautz networks first proposed by Kautz in 1969 [12]. Structurally, Kautz networks are very similar to de Bruijn networks, and thus have as many desirable properties as de Bruijn networks (see, for example, Section 3.3 in [20]). Moreover, Kautz networks are an improvement over de Bruijn networks, and have also been thought of as good candidates for the next generation of parallel system architectures, after the hypercube networks [6].

For two given integers $d \geq 2$ and $n \geq 1$, the Kautz digraph $K(d, n)$ is defined as follows. The vertex set of $K(d, n)$ is

$$V(d, n) = \{x_1x_2 \dots x_n \mid x_i \in \{0, 1, 2, \dots, d\}, x_i \neq x_{i+1}, i = 1, 2, \dots, n-1\}$$

and the edge set $E(d, n)$ consists of all edges from one vertex $x_1x_2 \dots x_n$ to d other vertices $x_2x_3 \dots x_n\alpha$, where $\alpha \in \{0, 1, \dots, d\}$ and $\alpha \neq x_n$. A pair of edges in $K(d, n)$ is said to be symmetric if they have the same end-vertices but different orientations.

It is clear that $K(d, n)$ is d -regular, $|V(d, n)| = d^n + d^{n-1}$ and $|E(d, n)| = d^{n+1} + d^n$. Moreover, $K(d, n)$ has $\frac{1}{2}d(d+1)$ pairs of symmetric edges.

The Kautz undirected graph, denoted by $UK(d, n)$, is an undirected graph obtained from $K(d, n)$ by deleting the orientation of all edges and omitting multiple edges.

It is clear that $UK(d, n)$ has $d^{n+1} + d^n - \frac{1}{2}d(d+1)$ edges, the maximum degree is $2d$ for $n \geq 3$ and the minimum degree is $2d - 1$ for $n \geq 2$.

Use $f(d, n)$ to denote the feedback number of $UK(d, n)$. Kráľovič and Ružička [13] determined that $f(2, n) = 2^{n-1}$. In this paper, by refining their technique, we obtain

the following result. For any $d \geq 2$ and $n \geq 3$, the following holds:

$$\left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d-1} \right\rceil \leq f(d, n) \leq d^n - \left(\left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}.$$

This result implies that $f(2, n) = 2^{n-1}$, due to Královič and Ružička [13].

The main result of the theorem is the upper bound of $f(d, n)$. To obtain this upper bound, we first construct a cycle-free set in Section 2; its complementary set is a feedback vertex set. Thus the upper bound can be established in Section 3.

2 Constructing a Cycle-free Set

Throughout this paper, we follow Xu [21] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a graph and $S \subset V(G)$. The symbol $N_G(S)$ denotes the set of neighbors of S , namely, $N_G(S) = \{x \in V(G - S) : xy \in E(G), y \in S\}$. The subgraph induced by S is denoted by $G[S]$. The set S is independent if no two of vertices in S are adjacent in G , and is cycle-free if $G[S]$ has no cycles.

Throughout this paper, we assume that $G = \text{UK}(d, n)$ with $n \geq 3$. For $d \geq 2$, let $I_d = \{0, 1, 2, \dots, d\}$, and J_d be the set of integers in the interval $[\lfloor \frac{d}{2} \rfloor + 1, d]$. For $x = x_1 x_2 \dots x_n \in V(G)$, let

$$\begin{aligned} N_G^{(L)}(x) &= \{\alpha x_1 x_2 \dots x_{n-1} \mid \alpha \in I_d, \alpha \neq x_1\}, \\ N_G^{(R)}(x) &= \{x_2 x_3 \dots x_n \beta \mid \beta \in I_d, \beta \neq x_n\}. \end{aligned}$$

Then $N_G(x) = N_G^{(L)}(x) \cup N_G^{(R)}(x)$. Define $\lfloor \frac{d}{2} \rfloor + 1$ subsets of $V(G)$ as follows.

$$\begin{aligned} F_0 &= \{x_1 x_2 x_3 \dots x_n \in V(G) \mid x_1 = 0\}, \\ F_1 &= \{x_1 x_2 x_3 \dots x_n \in V(G) \mid x_1 = 1, x_2 \in \{0\} \cup J_d\}, \\ F_2 &= \{x_1 x_2 x_3 \dots x_n \in V(G) \mid x_1 = 2, x_2 \in J_d\}, \\ &\vdots \\ F_{\lfloor \frac{d}{2} \rfloor} &= \{x_1 x_2 x_3 \dots x_n \in V(G) \mid x_1 = \lfloor \frac{d}{2} \rfloor, x_2 \in J_d\}, \\ F &= F_0 \cup F_1 \cup F_2 \cup F_3 \cup \dots \cup F_{\lfloor \frac{d}{2} \rfloor}. \end{aligned} \tag{2.1}$$

Lemma 2.1. $|F| = d^{n-1} + (\lfloor \frac{d^2}{4} \rfloor + 1) d^{n-2}$.

Proof. By the definition of F in (2.1), it is easy to verify that $F_i \cap F_j = \emptyset$ for $0 \leq i \neq j \leq \lfloor \frac{d}{2} \rfloor$, and

$$\begin{aligned} |F_0| &= d^{n-1}, \quad |F_1| = (d - \lfloor \frac{d}{2} \rfloor + 1) d^{n-2} = (\lceil \frac{d}{2} \rceil + 1) d^{n-2}, \\ |F_i| &= (d - \lfloor \frac{d}{2} \rfloor) d^{n-2} = \lceil \frac{d}{2} \rceil d^{n-2} \text{ for each } i = 2, 3, \dots, \lfloor \frac{d}{2} \rfloor. \end{aligned}$$

It follows that

$$\begin{aligned} |F| &= |F_0| + |F_1| + |F_2| + \dots + |F_{\lfloor \frac{d}{2} \rfloor}| \\ &= d^{n-1} + (\lceil \frac{d}{2} \rceil + 1) d^{n-2} + (\lfloor \frac{d}{2} \rfloor - 1) \lceil \frac{d}{2} \rceil d^{n-2} \\ &= d^{n-1} + \lfloor \frac{d}{2} \rfloor \lceil \frac{d}{2} \rceil d^{n-2} + d^{n-2} \\ &= d^{n-1} + (\lfloor \frac{d^2}{4} \rfloor + 1) d^{n-2}, \end{aligned}$$

as required. \square

Lemma 2.2. F_i is an independent set of $\text{UK}(d, n)$ for each $i = 0, 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$.

Proof. Vertices in F_i are all starting with the same entry; it is clear that no edge connects any two of them in $\text{UK}(d, n)$. Thus F_i is an independent set for each $i = 0, 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$. \square

Lemma 2.3. $F_0 \cup F_1$ is a cycle-free set of $\text{UK}(d, n)$.

Proof. Let $G = \text{UK}(d, n)$. For the sake of contradiction, we assume that there exists a cycle in $G[F_0 \cup F_1]$. Then the length of C is even since both F_0 and F_1 are independent sets by Lemma 2.2. Let $C = (v_1, v_2, \dots, v_{2m+1})$, where $v_{2m+1} = v_1$. Without loss of generality, we may suppose $v_1 \in F_0$. Then $v_{2i-1} \in F_0$ and $v_{2i} \in F_1$ for each $i = 1, 2, \dots, m+1$.

If C is a directed cycle in the digraph $K(d, n)$, then $v_1 = 0101\dots 01$ when n is even and $v_1 = 0101\dots 010$ when n is odd. Thus C is a digraph cycle of length 2, a contradiction.

If C is not a directed cycle, then there must be a vertex u with in-degree 2. Suppose $x, y \in N_G^{(L)}(u)$ and x, y are vertices in C . Without loss of generality, we may suppose $u \in F_0$ and $u = 0u_2\dots u_n$. Then $N_G^{(L)}(u) = \{\alpha 0u_2\dots u_{n-1} \mid \alpha \in I_d, \alpha \neq 0\}$. But x, y have the same first entry 1, which means $x = y$, a contradiction.

The lemma follows. \square

Lemma 2.4. F is a cycle-free set of $\text{UK}(d, n)$.

Proof. Let $G = \text{UK}(d, n)$, and for each $i = 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$, let

$$U_i = F_0 \cup F_1 \cup F_2 \cup \dots \cup F_i \text{ and } G_i = G[U_i].$$

To prove that F is a cycle-free set of $\text{UK}(d, n)$, we only need to prove that $G_{\lfloor \frac{d}{2} \rfloor}$ is acyclic. The proof proceeds by induction on $i \geq 1$.

By Lemma 2.3, G_1 is acyclic. Assume the induction hypothesis for $i-1$ with $i \geq 2$. Let v be any vertex in F_i . We only need to prove that only one of the neighbors of v is in G_{i-1} since F_i is an independent set by Lemma 2.2.

Let $v = i\alpha x_3 x_4 \dots x_n$, where $\alpha \in J_d$. Then

$$\begin{aligned} N_G^{(L)}(v) &= \{\gamma i\alpha x_3 x_4 \dots x_{n-1} \mid \gamma \in I_d, \gamma \neq i\}, \\ N_G^{(R)}(v) &= \{\alpha x_3 x_4 \dots x_n \beta \mid \beta \in I_d, \beta \neq x_n\}. \end{aligned}$$

Since $\alpha > i$, $N_G^{(R)}(v) \cap U_{i-1} = \emptyset$. Moreover, $N_G^{(L)}(v) \cap U_{i-1} \neq \emptyset$ implies $\gamma = 0$ in $N_G^{(L)}(v)$, i.e., $N_G^{(L)}(v) \cap U_{i-1} = \{0i\alpha x_3 x_4 \dots x_{n-1}\}$, a single vertex in F_0 . This is due to, in F_k ($k \geq 1$), any vertex $u = k\alpha x_3 x_4 \dots x_n$ with $\alpha \in J_d$ but clearly satisfying $i \notin J_d$. Thus $u = 0i\alpha x_3 x_4 \dots x_{n-1}$ is a unique neighbor of v in G_{i-1} .

So we have completed the proof of the lemma. \square

3 Establishing Bounds

In this section, we prove the main result of our paper. We first restate a lower bound on the size of the feedback vertex set in any graph, due to Beineke and Vandell [5].

Lemma 3.1. *For feedback vertex set F in a graph $G = (V, E)$ with maximum degree Δ , the following holds:*

$$|F| \geq \left\lceil \frac{|E| - |V| + 1}{\Delta - 1} \right\rceil.$$

Theorem 3.1. *For any $d \geq 2$ and $n \geq 3$, the following holds:*

$$\left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d-1} \right\rceil \leq f(d, n) \leq d^n - \left(\left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}.$$

Proof. Noting that $\text{UK}(d, n)$ has $d^n + d^{n-1}$ vertices and $d^{n+1} + d^n - \frac{1}{2}d(d+1)$ edges, and the maximum degree is $2d$ for $n \geq 3$, by Lemma 3.1, we immediately obtain the lower bound:

$$f(d, n) \geq \left\lceil \frac{d^{n+1} - d^{n-1} - \frac{d(d+1)}{2} + 1}{2d-1} \right\rceil. \quad (3.2)$$

We now establish the upper bound

$$f(d, n) \leq d^n - \left(\left\lfloor \frac{d^2}{4} \right\rfloor + 1 \right) d^{n-2}. \quad (3.3)$$

Let $\overline{F} = V(\text{UK}(d, n)) \setminus F$, where F is defined in (2.1). Then \overline{F} is a feedback vertex-set of $\text{UK}(d, n)$ since F is a cycle-free set of $\text{UK}(d, n)$ by Lemma 2.4. It follows from Lemma 2.1 that

$$\begin{aligned} f(d, n) &\leq |\overline{F}| = |V(d, n)| - |F| \\ &= (d^n + d^{n-1}) - (d^{n-1} + (\left\lfloor \frac{d^2}{4} \right\rfloor + 1) d^{n-2}) \\ &= d^n - (\left\lfloor \frac{d^2}{4} \right\rfloor + 1) d^{n-2}. \end{aligned}$$

as required. □

Corollary 3.1. (Královič and Ružička [13]) *If $n \geq 3$, then $f(2, n) = 2^{n-1}$.*

Proof. On the one hand, by (3.3) we have

$$f(2, n) \leq 2^n - 2 \cdot 2^{n-2} = 2^{n-1}.$$

On the other hand, by (3.2) we have

$$f(2, n) \geq \left\lceil \frac{2^{n+1} - 2^{n-1} - 2}{3} \right\rceil = \left\lfloor 2^{n-1} - \frac{2}{3} \right\rfloor = 2^{n-1}.$$

Thus, $f(2, n) = 2^{n-1}$. □

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