AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 209–220

The minimal size of a graph with generalized connectivity $\kappa_3 = 2$ *

Shasha Li   Xueliang Li   Yongtang Shi

Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071
China
lss@cfc.nankai.edu.cn   lxl@nankai.edu.cn   shi@nankai.edu.cn

Abstract

Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. Chartrand et al. generalized the concept of connectivity as follows. The *k-connectivity*, denoted by $\kappa_k(G)$, of G is defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Thus $\kappa_2(G) = \kappa(G)$, where $\kappa(G)$ is the connectivity of G.

This paper mainly determines the minimal number of edges of a graph of order n with $\kappa_3 = 2$; that is, for a graph G of order n and size $e(G)$ with $\kappa_3(G) = 2$, it is proved that $e(G) \geq \lceil \frac{6}{5}n \rceil$, and the lower bound is sharp for all $n \geq 4$ apart from $n = 9, 10$, whereas for $n = 9, 10$ examples are given to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order n with generalized connectivity $\kappa_3 = 2$.

1 Introduction

We follow the terminology and notations of [1] and all graphs considered here are always finite and simple. As usual, we denote the numbers of vertices and edges in G by $n(G)$ and $e(G)$ (or simply n and e), and these two basic parameters are called the *order* and *size* of G, respectively. Let X be a set of vertices of G and $G[X]$ the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. A stable set in a graph is a set of vertices no two of which

* Supported by NSFC and the Fundamental Research Funds for the Central Universities.

are adjacent. A vertex with degree one in a tree is called a leaf. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney [4] provides an equivalent definition of the connectivity. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint uv-paths in G. Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

In [2], the authors generalized the concept of connectivity. Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \backslash S$). A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called an *internally disjoint set of trees connecting* S. The *k-connectivity*, denoted by $\kappa_k(G)$, of G is then defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-subsets S of $V(G)$. Obviously, $\kappa_2(G) = \kappa(G)$.

In [3], we focused on the investigation of $\kappa_3(G)$ and mainly studied the relationship between the 2-connectivity and the 3-connectivity of a graph. We gave sharp upper and lower bounds for $\kappa_3(G)$ for general graphs G, and showed that if G is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$. Moreover, we studied the algorithmic aspects for $\kappa_3(G)$ and gave an algorithm to determine $\kappa_3(G)$ for a general graph G.

In this paper, we determine the minimal number of edges of a graph with $\kappa_3 = 2$, i.e., for a graph G of order n and size $e(G)$ with $\kappa_3(G) = 2$, we obtain $e(G) \geq \lceil \frac{6}{5}n \rceil$, and the lower bound is sharp for all $n \geq 4$ but $n = 9, 10$, whereas for $n = 9, 10$ we give examples to show that $\lceil \frac{6}{5}n \rceil + 1$ is the best possible lower bound. This gives a clear picture on the minimal size of a graph of order n with generalized connectivity $\kappa_3 = 2$. Note that for a graph G of order n and size $e(G)$ with $\kappa(G) = 2$, we have $e(G) \geq n$, and a cycle of this order attains the lower bound.

2 Lower bound

Before proceeding, we recall a result in [3], which will be used frequently in the sequel.

Lemma 2.1. *If G is a connected graph with minimum degree δ, then $\kappa_3(G) \leq \delta$. In particular, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Now we give the lower bound.

Proposition 2.1. *Every graph G of order n with $\kappa_3(G) = 2$ has at least $\lceil \frac{6}{5}n \rceil$ edges.*

Proof. Since $\kappa_3(G) = 2$, by Lemma 2.1 we know that $\delta(G) \geq 2$ and no two vertices of degree 2 are adjacent. Denote by X the set of vertices of degree 2. We have that

X is a stable set. Put $Y = V(G) - X$ and obviously there are $2|X|$ edges joining X to Y. Assume that m' is the number of edges joining two vertices belonging to Y. It is clear that

$$e = 2|X| + m'. \tag{1}$$

Since every vertex of Y has degree at least 3 in G, it follows that $\sum_{v \in Y} d(v) = 2|X| + 2m' \geq 3|Y| = 3(n - |X|)$, so

$$5|X| + 2m' \geq 3n. \tag{2}$$

Combining (1) with (2), we have $\frac{5}{2}e = \frac{5}{2}(2|X| + m') = 5|X| + \frac{5}{2}m' \geq 5|X| + 2m' \geq 3n$, that is, $e \geq \frac{6}{5}n$. Since the number of edges is an integer, it follows that $e \geq \lceil \frac{6}{5}n \rceil$. The proof is complete. ∎

Remark 2.1: Furthermore, when n is a multiple of 5, in Proposition 2.1 equality holds if and only if $5|X| + \frac{5}{2}m' = 5|X| + 2m' = 3n$, that is, if and only if

(A) $m' = 0$, that is, Y is a stable set, and

(B) the maximum degree Δ is 3.

Moreover, in this case, inequality (2) becomes $5|X| = 3n$, that is, $|X| = \frac{3}{5}n$.

Remark 2.2: Obviously, for any graph G with $e(G) = \lceil \frac{6}{5}n(G) \rceil$, $\kappa_3(G) \leq 2$. The next two lemmas show that the number $e(G) = \lceil \frac{6}{5}n(G) \rceil$ cannot guarantee that $\kappa_3(G) = 2$.

Lemma 2.2. *For any connected graph G of order* 10 *and size* 12*, $\kappa_3(G) = 1$.*

Proof. Note that $e(G) = \lceil \frac{6}{5}n(G) \rceil$ and so $\kappa_3(G) \leq 2$. Assume, to the contrary, that there is a connected graph G of order 10 and size 12 with $\kappa_3(G) = 2$. Therefore by Remark 2.1, both X and Y are stable sets, $|X| = \frac{3}{5}n = 6$ and $|Y| = 4$, where X and Y are the sets of vertices of degrees 2 and 3, respectively. Let $X = \{x_1, \ldots, x_6\}$ and $Y = \{y_1, \ldots, y_4\}$.

Case 1: For every 2-subset $\{y_i, y_j\}$ of Y, there is a vertex in X that is adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 4$.

Note that every vertex in X has degree 2, and there are exactly six vertices in X and six 2-subsets of Y, namely

$$\{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_4\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_3, y_4\}.$$

Thus we may assume that G is isomorphic to the graph as shown in Figure 1. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_2 and x_4, contrary to our assumption.

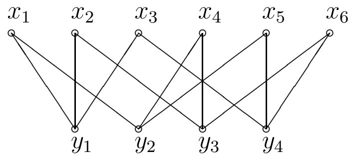

Figure 1: The graph for Case 1 of Lemma 2.2

Case 2: There exists a 2-subset of Y such that no vertex in X is adjacent to both of the vertices in that subset.

For this case, there must exist some 2-subset $\{y_i, y_j\}$ such that at least two vertices in X are adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 4$. Without loss of generality, we may assume that $\{y_i, y_j\} = \{y_1, y_2\}$. Since G is connected, we find that only two vertices in X are adjacent to both y_1 and y_2. Then we may assume that G is isomorphic to the graph as shown in Figure 2. Now consider the three vertices x_1, x_3 and x_5; we can obtain $\kappa_3(G) = 1$, contrary to our assumption.

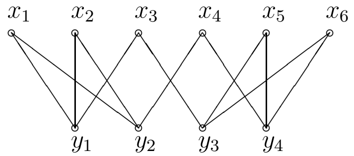

Figure 2: The graph for Case 2 of Lemma 2.2

The proof is complete. ∎

Remark 2.3: Note that there exists a graph G such that $n = 10$, $e(G) = 13$ and $\kappa_3(G) = 2$; see Figure 3.

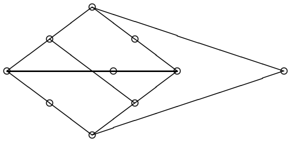

Figure 3: The graph G of order 10 and size 13 with $\kappa_3(G) = 2$.

Now we turn to the graphs of order 9 and size 11.

Lemma 2.3. *For any connected graph G of order 9 and size 11, $\kappa_3(G) = 1$.*

Proof. Assume, to the contrary, that there is a connected graph G of order $n = 9$ and size $e = 11$ with $\kappa_3(G) \geq 2$. By Lemma 2.1, we have the minimum degree

$\delta(G) \geq 2$. Denote by X the set of vertices of degree 2 in G. It follows that $2e = \Sigma_{v \in V(G)} d(v) \geq 2|X| + 3(n - |X|)$, so $|X| \geq 3n - 2e = 5$. On the other hand, by Lemma 2.1 again, we get that X is a stable set. Let m' be the number of edges joining two vertices belonging to Y, where $Y = V(G) - X$. It is clear that $e = 2|X| + m'$. So $|X| \leq \frac{e}{2} = 5.5$. Now we can conclude that $|X| = 5$, $|Y| = 4$, $m' = 1$ and every vertex in Y has degree exactly 3. Set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. Since $m' = 1$, without loss of generality, suppose that $y_1 y_2$ is the only edge in $G[Y]$.

Case 1: There is a vertex in X that is adjacent to both y_1 and y_2.

Note that G is a simple connected graph and every vertex in X has degree 2. It is not hard to see that G is isomorphic to the graph as shown in Figure 4. Then observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_2 and x_4, contrary to our assumption.

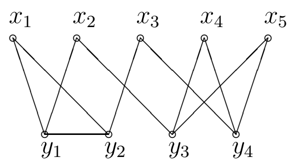

Figure 4: The graph for Case 1 of Lemma 2.3

Case 2: There is no vertex in X that is adjacent to both y_1 and y_2.

Subcase 2.1: For every 2-subset $\{y_i, y_j\}$ of Y other than $\{y_1, y_2\}$, there is a vertex in X that is adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 4$.

Note that there are exactly five vertices in X and five 2-subsets of Y other than $\{y_1, y_2\}$. Thus we may assume that G is isomorphic to the graph as shown in Figure 5. Consider the three vertices x_1, x_2 and x_5; we can get $\kappa_3(G) = 1$, contrary to our assumption.

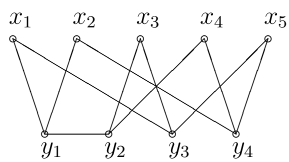

Figure 5: The graph for Subcase 2.1 of Lemma 2.3

Subcase 2.2: Apart from $\{y_1, y_2\}$, there exists another 2-subset such that no vertex in X is adjacent to both of the vertices in that subset.

In such a situation, there must exist some 2-subset $\{y_i, y_j\}$ such that at least two vertices in X are adjacent to both y_i and y_j, where $1 \leq i \neq j \leq 4$. If $\{y_i, y_j\} = \{y_3, y_4\}$, it is not hard to see that there must exist a vertex in X that is adjacent to both y_1 and y_2, contrary to the case. So without loss of generality, we may assume that $\{y_i, y_j\} = \{y_1, y_3\}$. Then we can get G isomorphic to the graph as shown in Figure 6. Observe that it is impossible to find two internally-disjoint trees connecting the vertices x_1, x_4 and x_5, contrary to our assumption.

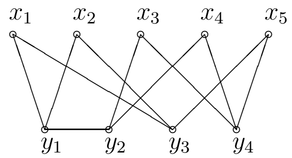

Figure 6: The graph for Subcase 2.2 of Lemma 2.3

The proof is complete. ∎

Remark 2.4: Notice that there exists a graph G such that $n = 9$, $e(G) = 12$ and $\kappa_3(G) = 2$; see Figure 7.

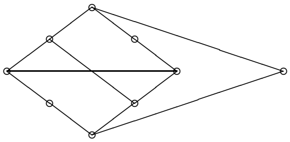

Figure 7: The graph G of order 9 and size 12 with $\kappa_3(G) = 2$.

In view of Lemmas 2.2 and 2.3 and Remarks 2.3 and 2.4, we can see that for $n = 9, 10$, the best possible lower bound is $\lceil \frac{6}{5}n \rceil + 1$. Naturally, for any positive integer n other than $n = 9, 10$, we want to know whether there is a graph of order n attaining the lower bound $\lceil \frac{6}{5}n \rceil$ in Proposition 2.1. For this purpose, we first construct a class of graphs.

Before giving the construction, we first give some notions. For any two integers a and $k \geq 1$, denote by $[a]_k$ an integer such that $1 \leq [a]_k \leq k$ and $a \equiv [a]_k \pmod{k}$. For a cycle $C = x_1x_2x_3 \ldots x_{k-1}x_kx_1$, we denote three special segments of C by $x_aCx_b = x_ax_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}x_b$, $\hat{x}_aCx_b = x_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}x_b$ and $\hat{x}_aC\hat{x}_b = x_{[a+1]_k}x_{[a+2]_k} \ldots x_{[b-1]_k}$, where $1 \leq a, b \leq k$. Denote by $|C|$ and $|P|$ the lengths of a cycle C and a path P, respectively.

Lemma 2.4. *For a positive integer $k \neq 2$, let $C = x_1y_1x_2y_2 \ldots x_{2k}y_{2k}x_1$ be a cycle of length $4k$. Add k new vertices $z_1, z_2, \ldots, z_k$ to C, and join z_i to x_i and x_{i+k}, for $1 \leq i \leq k$. The resulting graph is denoted by H. Then the 3-connectivity of H is 2, that is, $\kappa_3(H) = 2$.*

Proof. Since $\delta(H) = 2$, by Lemma 2.1 we can get $\kappa_3(H) \leq 2$. So the task is to show $\kappa_3(H) \geq 2$. By the definition of generalized connectivity, it suffices to prove that $\kappa(S) \geq 2$ for every 3-subset S of $V(H)$.

First, partition $V(H)$ into three types: $V_1 = \{x_1, x_2, \ldots, x_{2k}\}$, $V_2 = \{z_1, z_2, \ldots, z_k\}$ and $V_3 = \{y_1, y_2, \ldots, y_{2k}\}$. We proceed by considering all cases of S.

Case 1: $S = \{x_a, x_b, x_c\}$, where $1 \leq a < b < c \leq 2k$.

The three vertices divide the cycle C into three segments, at least one of which has length at most $|C|/3$. Without loss of generality, we may assume that $|x_a C x_b| \leq |C|/3$, that is, $|x_b C x_a| \geq 2|C|/3$. Let $b' = [b+k]_{2k}$. Note that $|x_b C x_{b'}| = |C|/2$, and so $x_{b'} \in V(\hat{x}_b C \hat{x}_a)$.

Subcase 1.1: $x_{b'} \in V(x_c C \hat{x}_a)$. In this case, $T_1 = x_a C x_b C x_c$ and $T_2 = x_c C x_{b'} C x_a \cup x_{b'} z_{[b]_k} x_b$ are two internally disjoint trees connecting S.

Subcase 1.2: $x_{b'} \in V(\hat{x}_b C \hat{x}_c)$. Let $a' = [a+k]_{2k}$. We can get $x_{a'} \in V(\hat{x}_b C \hat{x}_{b'})$, since $1 \leq |x_a C x_b| \leq |C|/3$, $|x_a C x_{a'}| = |C|/2$ and $|x_b C x_{b'}| = |C|/2$. Therefore $x_{a'} \in V(\hat{x}_b C \hat{x}_c)$, and then $T_1 = x_c C x_a C x_b$ and $T_2 = x_b C x_{a'} C x_c \cup x_{a'} z_{[a]_k} x_a$ are two internally disjoint trees connecting S.

Case 2: $S = \{z_a, z_b, z_c\}$, where $1 \leq a < b < c \leq k$.

Since $1 \leq a < b < c \leq k < a+k < b+k < c+k \leq 2k$, $x_a C x_b C x_c$ and $x_{a+k} C x_{b+k} C x_{c+k}$ are two disjoint segments of C. It is easy to find two internally disjoint trees connecting S: $T_1 = z_a x_a C x_b C x_c z_c \cup x_b z_b$ and $T_2 = z_a x_{a+k} C x_{b+k} C x_{c+k} z_c \cup x_{b+k} z_b$.

Case 3: $S = \{x_a, x_b, z_c\}$, where $1 \leq a < b \leq 2k$ and $1 \leq c \leq k$.

Observe that the two neighbors x_c and x_{c+k} of z_k divide the cycle into two segments $x_c C x_{c+k}$ and $x_{c+k} C x_c$.

Subcase 3.1: x_a and x_b lie in distinct segments. Without loss of generality, we may assume that $x_a \in V(x_c C x_{c+k})$ and $x_b \in V(x_{c+k} C x_c)$. Now $T_1 = x_a C x_{c+k} C x_b \cup x_{c+k} z_c$ and $T_2 = x_b C x_c C x_a \cup x_c z_c$ are two trees we want. Note that the subcase contains the situation that either x_c or x_{c+k} is exactly x_a or x_b.

Subcase 3.2: x_a and x_b lie in the same segment. Without loss of generality, suppose that $x_a, x_b \in V(\hat{x}_c C \hat{x}_{c+k})$. Let $b' = [b+k]_{2k}$. Since $|x_c C x_{c+k}| = |C|/2$, $|x_b C x_{b'}| = |C|/2$ and $x_b \in V(\hat{x}_c C \hat{x}_{c+k})$, we have $x_{b'} \in V(\hat{x}_{c+k} C \hat{x}_c)$ and $T_1 = x_a C x_b C x_{c+k} z_c$ and $T_2 = x_b z_{[b]_k} x_{b'} C x_c C x_a \cup x_c z_c$ are two internally disjoint trees connecting S.

Case 4: $S = \{x_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Since $1 \leq b < c \leq k < b+k < c+k \leq 2k$, the two neighbors x_b, x_{b+k} of z_b, together with two neighbors x_c, x_{c+k} of z_c, divide the cycle into four segments $x_b C x_c$, $x_c C x_{b+k}$, $x_{b+k} C x_{c+k}$ and $x_{c+k} C x_b$. Actually, it is easy to see that no matter which segment x_a lies in, the situations are equivalent. Therefore, without loss of generality, we may assume that $x_a \in V(x_b C x_c)$. We have $T_1 = x_a C x_c C x_{b+k} z_b \cup x_c z_c$ and $T_2 = z_c x_{c+k} C x_b C x_a \cup x_b z_b$ being two internally disjoint trees connecting S. Note that this case includes the situation that x_a is exactly x_b or x_c.

Next we consider the cases in which S contains the vertices in V_3.

Case 5: $S = \{y_a, y_b, y_c\}$, where $1 \le a < b < c \le 2k$.

Clearly, in this case, k is a positive integer at least 3. Among the three segments y_aCy_b, y_bCy_c and y_cCy_a of C, at least one of them has length not more than $|C|/3$. We may assume that $|y_aCy_b| \le |C|/3 = 4k/3$. Moreover, observe that x_{a+1} lies between y_a and y_b. We have $y_b \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$, since $|x_{a+1}Cy_b| < |y_aCy_b| \le 4k/3$ and $|x_{a+1}Cx_{[a+1+k]_{2k}}| = |C|/2 = 2k$.

Subcase 5.1: $y_c \in V(\hat{y}_bC\hat{x}_{[a+1+k]_{2k}})$. There is at least one vertex x_{b+1} between y_b and y_c. Since $x_{b+1} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$, it is clear that $x_{[b+1+k]_{2k}} \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{x}_{a+1})$, namely, $x_{[b+1+k]_{2k}} \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{y}_a)$. We can find two internally disjoint trees connecting S: $T_1 = y_ax_{a+1}Cy_b \cup y_cCx_{[a+1+k]_{2k}} \cup x_{a+1}z_{[a+1]_k}x_{[a+1+k]_{2k}}$ and $T_2 = y_bx_{b+1}Cy_c \cup x_{b+1}z_{[b+1]_k}x_{[b+1+k]_{2k}}Cy_a$.

Subcase 5.2: $y_c \in V(\hat{x}_{[a+1+k]_{2k}}C\hat{y}_a)$. There is at least one vertex x_a between y_c and y_a. Obviously, $x_{[a+k]_{2k}} \in V(\hat{x}_{a+1}C\hat{x}_{[a+1+k]_{2k}})$. Moreover, $x_aCy_b = |y_aCy_b| + 1 \le |C|/3 + 1 = 4k/3 + 1$ and $x_aCx_{[a+k]_{2k}} = |C|/2 = 2k$, where $k \ge 3$. So $y_b \in V(\hat{x}_aC\hat{x}_{[a+k]_{2k}})$. Now $T_1 = y_ax_{a+1}Cy_b \cup x_{a+1}z_{[a+1]_k}x_{[a+1+k]_{2k}}Cy_c$ and $T_2 = y_bCx_{[a+k]_{2k}}z_{[a]_k}x_a \cup y_cCx_ay_a$ are two internally disjoint trees connecting S.

Case 6: $S = \{y_a, y_b, x_c\}$, where $1 \le a < b \le 2k$ and $1 \le c \le 2k$.

Notice that y_a and y_b divide C into two segments y_aCy_b and y_bCy_a. Let $c' = [c+k]_{2k}$, and then two subcases arise.

Subcase 6.1: x_c and $x_{c'}$ lie in distinct segments. We may assume that $x_c \in V(y_aCy_b)$ and $x_{c'} \in V(y_bCy_a)$. Thus, $T_1 = y_aCx_cCy_b$ and $T_2 = y_bCx_{c'}Cy_a \cup x_cz_{[c]_k}x_{c'}$ are exactly two trees we want.

Subcase 6.2: x_c and $x_{c'}$ lie in the same segment. Without loss of generality, we may assume that $x_c, x_{c'} \in V(y_bCy_a)$ and they occur in cyclic order $y_a, y_b, x_c, x_{c'}$ on C. The segment y_aCy_b must contain a vertex x_{a+1} in V_1. Since $x_{a+1} \in V(\hat{x}_{c'}C\hat{x}_c)$, $x_{[a+1+k]_{2k}} \in V(\hat{x}_cC\hat{x}_{c'})$. So we can find two internally disjoint trees connecting S: $T_1 = y_ax_{a+1}Cy_b \cup x_{a+1}z_{[a+1]_k}x_{[a+1+k]_{2k}} \cup x_cCx_{[a+1+k]_{2k}}$ and $T_2 = y_bCx_cz_{[c]_k}x_{c'}Cy_a$.

Case 7: $S = \{y_a, y_b, z_c\}$, where $1 \le a < b \le 2k$ and $1 \le c \le k$.

If $k = 1$, then $C = x_1y_1x_2y_2x_1$ and $H = C \cup x_1z_1x_2$. So y_a, y_b and z_c are exactly y_1, y_2 and z_1, respectively. Now $T_1 = y_2x_1y_1 \cup x_1z_1$ and $T_2 = y_1x_2y_2 \cup x_2z_1$ are two internally disjoint trees connecting S.

Otherwise, $k \ge 3$, since $k \ne 2$. We know that y_a, y_b divide C into two segments y_aCy_b, y_bCy_a, and z_c has two neighbors x_c and x_{c+k}.

Subcase 7.1: x_c and x_{c+k} lie in distinct segments. Suppose that $x_c \in V(y_aCy_b)$ and $x_{c+k} \in V(y_bCy_a)$. Clearly $T_1 = y_aCx_cCy_b \cup x_cz_c$ and $T_2 = y_bCx_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

Subcase 7.2: x_c and x_{c+k} lie in the same segment. Without loss of generality, we may assume that $x_c, x_{c+k} \in V(y_bCy_a)$ and they occur in cyclic order y_a, y_b, x_c, x_{c+k} on C.

Subsubcase 7.2.1: Between y_a and y_b, there are at least two vertices in V_1. Clearly $x_{a+1} \neq x_b$, and $y_a, x_{a+1}, x_b, y_b, x_c, x_{[a+1+k]_{2k}}, x_{[b+k]_{2k}}$ and x_{c+k} are the cyclic order in which they occur on C. So we can find two internally disjoint trees connecting S:

$$\begin{aligned} T_1 &= y_a x_{a+1} z_{[a+1]_k} x_{[a+1+k]_{2k}} \cup y_b C x_c C x_{[a+1+k]_{2k}} \cup x_c z_c \quad \text{and} \\ T_2 &= y_b x_b z_{[b]_k} x_{[b+k]_{2k}} C x_{c+k} C y_a \cup x_{c+k} z_c. \end{aligned}$$

Subsubcase 7.2.2: Between y_a and y_b, there is only one vertex in V_1, i.e, $x_{a+1} = x_b$. Let $b' = [b+k]_{2k}$ and clearly $x_{b'} \in V(\hat{x}_c C \hat{x}_{c+k})$. Since $k \geq 3$, $V(\hat{x}_c C \hat{x}_{c+k})$ contains at least two vertices x_{c+1}, x_{c+k-1} in V_1. If $x_{c+1} \neq x_{b'}$, then $x_{[c+1+k]_{2k}} = x_{[c+k+1]_{2k}} \neq x_b \in V(\hat{x}_{c+k}) C \hat{y}_a$. So $T_1 = y_a x_b y_b \cup x_b z_{[b]_k} x_{b'} C x_{c+k} z_c$ and $T_2 = y_b C x_c y_c x_{c+1} z_{[c+1]_k} x_{[c+k+1]_{2k}} C y_a \cup x_c z_c$ are two internally disjoint trees connecting S. Otherwise, $x_{c+k-1} \neq x_{b'}$, i.e., $x_{[c-1]_{2k}} \neq x_b$. We have $x_{[c-1]_{2k}} \in V(\hat{y}_b C \hat{x}_c)$. So $T_1 = y_a x_b y_b \cup x_b z_{[b]_k} x_{b'} \cup z_c x_c C x_{b'}$ and $T_2 = y_b C x_{[c-1]_{2k}} z_{[c-1]_k} x_{c+k-1} y_{c+k-1} x_{c+k} C y_a \cup x_{c+k} z_c$ are two internally disjoint trees connecting S.

Case 8: $S = \{y_a, x_b, x_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq 2k$.

Let $b' = [b+k]_{2k}$ and $c' = [c+k]_{2k}$. If $b' = c$, i.e., $c = [b+k]_{2k}$, then without loss of generality, we may assume that $y_a \in V(x_b C x_c)$. We have $T_1 = y_a C x_c z_{[c]_k} x_b$ and $T_2 = x_c C x_b C y_a$ are two internally disjoint trees connecting S. Otherwise, $b' \neq c$. Without loss of generality, suppose $x_b, x_c, x_{b'}$ and $x_{c'}$ are the cyclic order in which they occur on C; then they divide C into four segments $x_b C x_c, x_c C x_{b'}, x_{b'} C x_{c'}$ and $x_{c'} C x_b$.

Subcase 8.1: $y_a \in V(x_b C x_c)$. We can find two internally disjoint trees connecting S: $T_1 = x_b C y_a \cup x_c C x_{b'} z_{[b]_k} x_b$ and $T_2 = y_a C x_c z_{[c]_k} x_{c'} C x_b$.

Subcase 8.2: $y_a \in V(x_c C x_{b'})$ or $y_a \in V(x_{c'} C x_b)$. It is easy to see that the two situations are actually equivalent. So we only consider the former. We can find two internally disjoint trees connecting S: $T_1 = x_b C x_c C y_a$ and $T_2 = y_a C x_{b'} C x_{c'} z_{[c]_k} x_c \cup x_{b'} z_{[b]_k} x_b$.

Subcase 8.3: $y_a \in V(x_{b'} C x_{c'})$. We can find two internally disjoint trees connecting S: $T_1 = x_b C x_c \cup x_b z_{[b]_k} x_{b'} C y_a$ and $T_2 = y_a C x_{c'} C x_b \cup x_{c'} z_{[c]_k} x_c$.

Case 9: $S = \{y_a, z_b, z_c\}$, where $1 \leq a \leq 2k$ and $1 \leq b < c \leq k$.

Observe that x_b, x_c, x_{b+k} and x_{c+k} divide the cycle into four segments $x_b C x_c$, $x_c C x_{b+k}$, $x_{b+k} C x_{c+k}$ and $x_{c+k} C x_b$. Actually, no matter which segment y_a lies in, the situations are equivalent. So without loss of generality, we may assume that $y_a \in V(x_b C x_c)$. Now $T_1 = y_a C x_c C x_{b+k} z_b \cup x_c z_c$ and $T_2 = z_c x_{c+k} C x_b C y_a \cup x_b z_b$ are two internally disjoint trees connecting S.

Case 10: $S = \{y_a, x_b, z_c\}$, where $1 \leq a \leq 2k$, $1 \leq b \leq 2k$ and $1 \leq c \leq k$.

Subcase 10.1: $b = c$ or $b = c + k$. Without loss of generality, we may assume that $b = c$ and $y_a \in V(x_{c+k}Cx_b)$. Therefore $T_1 = y_aCx_bz_c$ and $T_2 = x_bCx_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

Subcase 10.2: $b \neq c$ and $b \neq c + k$. Let $b' = [b + k]_{2k}$. We may assume that $x_b, x_c, x_{b'}$ and x_{c+k} are the cyclic order in which they occur on C. Moreover, they divide C into four segments x_bCx_c, $x_cCx_{b'}$, $x_{b'}Cx_{c+k}$ and $x_{c+k}Cx_b$.

If $y_a \in V(x_bCx_c)$, then $T_1 = y_aCx_cCx_{b'}z_{[b]_k}x_b \cup x_cz_c$ and $T_2 = z_cx_{c+k}Cx_bCy_a$ are two internally disjoint trees connecting S.

If $y_a \in V(x_cCx_{b'}Cx_{c+k})$, then $T_1 = x_bCx_cCy_a \cup x_cz_c$ and $T_2 = y_aCx_{c+k}Cx_b \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

If $y_a \in V(x_{c+k}Cx_b)$, then $T_1 = y_aCx_bCx_cz_c$ and $T_2 = x_bz_{[b]_k}x_{b'}Cx_{c+k}Cy_a \cup x_{c+k}z_c$ are two internally disjoint trees connecting S.

The proof is complete. ∎

Remark 2.5: Clearly, the order $n(H)$ of the graph H is $5k$ and the size $e(H)$ is $4k + 2k = 6k$. If $k = 2$, then H is a connected graph of order 10 and size 12. By Lemma 2.2, we can get $\kappa_3(H) = 1$. This is the reason we add the condition $k \neq 2$ to Lemma 2.4. Moreover, no graphs of order 10 can attain the lower bound.

Next we describe an operation on a vertex of degree 2 in a graph. For a vertex u of degree 2, to *smooth* u is to delete u and then add an edge between its neighbors. Obviously, performing such an operation results in the numbers of vertices and edges decreasing by one. Moreover, the degrees of the remaining vertices are unchanged.

Lemma 2.5. *Let G be a graph such that the set X of vertices of degree 2 is non-empty. Denote by G' the new graph obtained by smoothing a vertex in X; then we have $\kappa_3(G') \geq \kappa_3(G)$.*

Proof. Let u be a vertex in X and $\{w_1, w_2\}$ the neighbor set of u. Suppose that G' is obtained by smoothing u. Clearly, $V(G') = V(G) - u$. For any three vertices v_1, v_2 and v_3 of G', let $S = \{v_1, v_2, v_3\}$. Obviously, $S \subseteq V(G)$. Let T be a tree connecting S in G. Note that if v is a leaf of T, we can assume that $v \in S$. Otherwise, $T' = T - v$ is still a tree connecting S and it uses fewer vertices. Now if $u \in V(T)$, then we can see that $T' = T - u + w_1w_2$ is a tree connecting S in G'. If $u \notin V(T)$, the operation of smoothing u has nothing to do with T and so T is still a tree connecting S in G'. Therefore it is not hard to see that $\kappa_{G'}(S) \geq \kappa_G(S)$. From the definition of κ_3, the conclusion that $\kappa_3(G') \geq \kappa_3(G)$ follows. ∎

Remark 2.6: For a given G, if we successively do the operation of smoothing a vertex of degree 2 more than once, and the resulting graph is denoted by G', then we can also obtain $\kappa_3(G') \geq \kappa_3(G)$.

Now, we can get our main result.

Theorem 2.2. *If G is a graph of order n with $\kappa_3(G) = 2$, then $e(G) \geq \lceil \frac{6}{5}n \rceil$. Moreover, the lower bound is sharp for all $n \geq 4$ apart from $n = 9, 10$, whereas the best lower bound for $n = 9, 10$ is $\lceil \frac{6}{5}n \rceil + 1$.*

Proof. The lower bound $\lceil \frac{6}{5}n \rceil$ is clear from Proposition 2.1. The best lower bound $\lceil \frac{6}{5}n \rceil + 1$ for $n = 9, 10$ is given in Remarks 2.3 and 2.4. Note that all graphs considered here are always simple. Therefore any graph attaining the lower bound must have at least four vertices. Next, we will show that the lower bound $\lceil \frac{6}{5}n \rceil$ is best possible for all $n \geq 4$, $n \neq 9, 10$.

For $n = 8$, there is a graph G' of order n such that $\kappa_3(G') = 2$ as shown in Figure 8. Moreover, $e(G') = 10 = \lceil \frac{6}{5} \times 8 \rceil$, which means that G' attains the lower bound for $n = 8$.

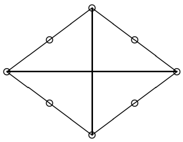

Figure 8: The graph G' attaining the lower bound for $n = 8$

Now, smooth a vertex of degree 2 in G'. Clearly, the resulting graph G'' is simple and $\delta(G'') = 2$. By Lemma 2.5, we can get $\kappa_3(G'') \geq (\kappa(G') = 2)$ and so clearly $\kappa_3(G'') = 2$. Moreover, $n = 8 - 1 = 7$ and $e = 10 - 1 = 9 = \lceil \frac{6}{5} \times 7 \rceil$. The graph G'' is what we want to find for $n = 7$. Similarly, the graph obtained from G'' by smoothing any one vertex of degree 2 attains the lower bound for $n = 6$.

Next, consider the graph H in Lemma 2.4. We know that $\kappa_3(H) = 2$, $n(H) = 5k$ and $e(H) = 6k = \frac{6}{5}n(H)$, for a positive integer $k \neq 2$. So H is exactly the graph of order $n = 5k$ which attains the lower bound.

For $k \geq 3$, let $k' = k - 1$ and then $n(H) = 5k' + 5$ and $e(H) = 6k' + 6$. Let X be the set of vertices of degree 2. Clearly $|X| = 3k' + 3 > 4$, where $k' \geq 2$. Now for the graph H, smooth successively any t vertices in X, for $1 \leq t \leq 4$. For any t, it is easy to check that no parallel edge can arise. Moreover, since $|X| > 4$, the minimum degree of the resulting graph H' is still 2. Combining Lemma 2.1 and Remark 2.6, we can get the 3-connectivity of the resulting graph H' is 2. Now let us consider the numbers of vertices and edges of H'.

When $t = 1$, $n(H') = 5k' + 4$ and $e(H') = 6k' + 5 = \lceil \frac{6}{5}(5k' + 4) \rceil$;

when $t = 2$, $n(H') = 5k' + 3$ and $e(H') = 6k' + 4 = \lceil \frac{6}{5}(5k' + 3) \rceil$;

when $t = 3$, $n(H') = 5k' + 2$ and $e(H') = 6k' + 3 = \lceil \frac{6}{5}(5k' + 2) \rceil$;

when $t = 4$, $n(H') = 5k' + 1$ and $e(H') = 6k' + 2 = \lceil \frac{6}{5}(5k' + 1) \rceil$.

Note that $k' \geq 2$. Therefore, for all $n \geq 4$ other than $n = 9, 10$, we can always find a graph of order n attaining the lower bound. ∎

Acknowledgement

The authors would like to thank the referees for comments and suggestions, which helped to improve the presentation of the paper.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] G. Chartrand, F. Okamoto and P. Zhang, Rainbow trees in graphs and generalized connectivity, *Networks* 55(4) (2010), 360–367.

[3] S. Li, X. Li and W. Zhou, Sharp bounds for the generalized connectivity $\kappa_3(G)$, *Discrete Math.* 310 (2010), 2147–2163.

[4] H. Whitney, Congruent graphs and the connectivity of graphs and the connectivity of graphs, *Amer. J. Math.* 54 (1932), 150–168.

(Received 20 Jan 2011; revised 26 June 2011)