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On automorphisms of Cayley graphs and irregular groups

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Abstract

Let G be a group and $\mathrm{Cay}P(G) < \mathrm{Sym}(G)$ be the subgroup of all permutations that induce graph automorphisms on every Cayley graph of G. The group G is *graphically abelian* if the map $\nu : g \to g^{-1}$ belongs to $\mathrm{Cay}P(G)$; these groups have been classified. Also G is *irregular* if there exists $\sigma \in \mathrm{Cay}P(G)$ such that $\sigma \neq 1_G$, $\sigma(1) = 1$ and $\sigma \neq \nu$. We show G is irregular if and only if $G = \mathrm{Dic}(A, I)$; every non-abelian graphically abelian group is irregular; and if G is irregular but not graphically abelian, $\sigma \in \mathrm{Cay}P(G)$ and $\sigma(1) = 1$, then $\sigma \in \mathrm{Aut}(G)$. No irregular group has a GRR. If an irregular group G is not graphically abelian then there is exactly one irregular map σ and $\mathrm{Cay}P(G) \cong G \rtimes \langle\sigma\rangle$, or otherwise $\mathrm{Cay}P(G) \cong (G \rtimes \mathrm{Inn}(G)) \rtimes \langle\nu\rangle$.

1 Basic Definitions

A non-empty subset S of a group G is called a Cayley subset of G provided $1 \notin S$ and for all $s \in G$, if $s \in S$ then $s^{-1} \in S$. The Cayley graph, $\mathrm{Cay}(G, S)$, may be defined as follows. The vertices of $\mathrm{Cay}(G, S)$ are the elements of G. Vertices a and b are adjacent (denoted $a \sim b$), whenever $b = as$, for some $s \in S$. The neighborhood of a vertex $a \in G$ is the set $N(a) = \{as \mid s \in S\}$. A group Q is called a quaternion group if Q is isomorphic to a group with a presentation $\langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\rangle$. A non-abelian group G is generalized dicyclic provided that G has an abelian subgroup A of index 2, and A contains a distinguished involution I such that for any $y \in G \setminus A$, $y^2 = I$. It is a consequence that for all $a \in A$, $yay^{-1} = a^{-1}$. We then write $G = \mathrm{Dic}(A, I)$.

Throughout this paper we assume that G denotes a group, and S a Cayley subset of G. We let E denote an elementary 2-group, and Q denote the quaternion group. We make no restriction on the cardinality of G. If X is a set, $\mathrm{Sym}(X)$ will denote the set of all permutations of X.

Definition 1.1. If, for all Cayley subsets S of G, $\sigma \in \mathrm{Sym}(G)$ induces a graph automorphism $\sigma : \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S)$, then σ is a *Cayley permutation* of G. A Cayley permutation σ such that $\sigma \in \mathrm{Aut}(G)$ will be called a *Cayley automorphism.*

The Cayley permutations of G, under composition of maps, form a group denoted $\mathrm{Cay}P(G)$. Let $\mathrm{Cay}P_1(G) = \{\sigma \in \mathrm{Cay}P(G) \mid \sigma(1) = 1\}$. A goal of this paper is to determine the structure of $\mathrm{Cay}P(G)$. We will see below that this comes down to determining $\mathrm{Cay}P_1(G)$.

For any $g \in G$, we define $\lambda_g(x) = gx$, for all $x \in G$. Here λ_g may be called *left translation* by g. The map $\lambda : G \to \mathrm{Sym}(G)$ defined by $\lambda(g) = \lambda_g$ is the *left regular representation* of G. Let $L_G = \{\lambda_g \in \mathrm{Sym}(G) \mid g \in G\}$. If $a,\ b \in G$, then $a \sim b$ in $\mathrm{Cay}(G, S)$ if and only if $\lambda_g(a) \sim \lambda_g(b)$, for any $g \in G$. It follows that $\lambda_g \in \mathrm{Cay}P(G)$, for any $g \in G$, so that $L_G \leq \mathrm{Cay}P(G)$. In particular, for every $g \in G$, λ_g is a Cayley permutation.

For any $g \in G$, we define $\rho_g(x) = xg^{-1}$, for all $x \in G$. Here ρ_g may be called *right translation* by g^{-1}. The map $\rho : G \to \mathrm{Sym}(G)$ defined by $\rho(g) = \rho_g$ is the *right regular representation* of G. Let $R_G = \{\rho_g \mid g \in G\}$. Note that R_G need not be a subgroup of $\mathrm{Cay}P(G)$.

Proposition 1.2. *Let $\tau \in \mathrm{Cay}P(G)$. Then τ factors as $\tau = \lambda_g \circ \sigma$, where $\sigma : G \to G$ is an element of $\mathrm{Cay}P_1(G)$, and λ_g is left multiplication by g, for some $g \in G$.*

Proof. Suppose $\tau(1) = g$. Let $\sigma = \lambda_{g^{-1}} \circ \tau$. The reader may verify that $\sigma \in \mathrm{Cay}P_1(G)$. □

Definition 1.3. A group G is said to be *graphically abelian* if the inversion map $\nu : g \mapsto g^{-1}$ is a member of $\mathrm{Cay}P(G)$.

It is readily shown that $\rho_g = \nu\lambda_g\nu$, so that if $\nu \in \mathrm{Cay}P(G)$, that is, if G is a graphically abelian group, then $\rho_g \in \mathrm{Cay}P(G)$, and R_G is a subgroup of $\mathrm{Cay}P(G)$. Furthermore, in this case, L_G and R_G are conjugate subgroups of $\mathrm{Cay}P(G)$, since $R_G = \nu L_G \nu^{-1}$. In [5], Goldstone and Weld establish that every non-abelian graphically abelian group is of the form $E \times Q$, and moreover, that R_G is a subgroup of $\mathrm{Cay}P(G)$ if and only if G is graphically abelian. These considerations motivate the following definition.

Definition 1.4. A group G will be called an *irregular group* if there is a non-trivial Cayley permutation $\sigma : G \to G$ such that $\sigma \in \mathrm{Cay}P_1(G)$ and σ is not the inversion map.

Recall that a group is said to have a *graphical regular representation* or GRR, if, for some Cayley set S, we have $G = \mathrm{Aut}(\mathrm{Cay}(G, S))$. It is clear that any group which admits a GRR is not irregular, because if, for some Cayley set S, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = L(G)$, then G cannot have an irregular map.

The question of which finite groups admit a GRR has been much studied. Over a period of about ten years, mathematicians completed the classification of the finite groups which admit a GRR; see [1, 4, 8, 14, 16] and elsewhere. Babai [1], established the veracity of a conjecture by Watkins when he showed that every finite group of order $|G| > 4682$ which does not possess a GRR is generalized dicyclic. Babai conjectured that the same must be true for infinite groups.

In this paper we classify the groups for which $L_G < \mathrm{Cay}P(G)$ is a proper subgroup; we further show that an irregular map must be either a group automorphism or, in some cases, a group anti-automorphism. We show that every irregular group G is a generalized dicyclic group, and we determine $\mathrm{Cay}P(G)$.

Definition 1.5. Let G be a group and let $\sigma : G \to G$. We say that σ satisfies the *Cayley condition* if $\sigma(ab) = \sigma(a)b^{\pm 1}$, for all $a, b \in G$.

The following propositions and their corollaries are easily proved from the definitions. Details are left to the reader.

Proposition 1.6. *Let G be a group and $\sigma \in \mathrm{Cay}P(G)$. Then σ satisfies the Cayley condition.*

Proposition 1.7. *Let $\sigma \in \mathrm{Sym}(G)$ and assume σ satisfies the Cayley condition; then $\sigma \in \mathrm{Cay}P(G)$.*

Corollary 1.8. *Let $\sigma \in \mathrm{Sym}(G)$ and assume σ satisfies the Cayley condition, and that $\sigma(1) = 1$. Then for all $g \in G$, $\sigma(g) = g^{\pm 1}$ and $\sigma^2 = 1$. In particular, if $\sigma \in \mathrm{Cay}P_1(G)$, then these two statements are true.*

Corollary 1.9. *Let $\sigma : G \to G$ be a group automorphism and assume $\sigma(g) = g^{\pm 1}$ for all $g \in G$. Then $\sigma \in \mathrm{Cay}P_1(G)$*

Proposition 1.10. *Let $\sigma \in \mathrm{Cay}P_1(G)$. Let H be a subgroup of G. Then $\sigma(H) \subseteq H$. In particular, $\sigma |_H : H \to H$ is an element of $\mathrm{Cay}P_1(H)$.*

Definition 1.11. Let $g \in G$, and $\sigma \in \mathrm{Cay}P_1(G)$. We say g is *σ-stable* if $\sigma(g) = g$, and otherwise we say that g is *σ-unstable*, or, when the context is clear, simply that g is *stable* or *unstable*. Observe that g is σ-unstable implies that $g \neq g^{-1}$.

2 Local Consequences of Irregularity

Throughout this section G will denote an irregular group and $\sigma : G \to G$ an irregular map. For such a group there are a number of local consequences. We collect these results in this section.

Proposition 2.1. *An irregular group G must have both stable elements of order greater than two, and unstable elements of order greater than* 2.

Proof. The group G must contain unstable elements, otherwise $\sigma = 1_G$. If g is an unstable element, then $g \neq g^{-1}$, so $|g| \geq 2$. If all non-trivial stable elements g have order 2, then, if g is stable, $\sigma(g) = g = g^{-1}$. But then σ is the inversion map, contradiction! □

Proposition 2.2. *Let $g \in G$, and let $\sigma : G \to G$ be an irregular map. Then $\sigma(g^n) = \sigma(g)^n$, for all $n \in Z$. Hence, if g is stable, so are all the elements of the subgroup $\langle g \rangle$. In particular, note that g is stable if and only if g^{-1} is stable. Similarly, g is unstable if and only if g^{-1} is unstable.*

Proof. The proof is an easy induction argument. □

Proposition 2.3. *Let $\sigma : G \to G$ be an irregular map, x a stable element of G and y an unstable element of G. Then $yxy^{-1} = x^{-1}$. In particular, an irregular group must be non-abelian.*

Proof. Apply the Cayley condition to obtain $\sigma(y) = \sigma(xx^{-1}y) = \sigma(x)(x^{-1}y)^{\pm 1}$. Since x is stable and y is unstable this gives $y^{-1} = x(x^{-1}y)^{\pm 1}$. If $y^{-1} = x(x^{-1}y)$ we have $y^{-1} = y$, implying y is σ-stable, a contradiction. Therefore $y^{-1} = x(x^{-1}y)^{-1}$, which implies that $yxy^{-1} = x^{-1}$. Finally, note there must be some unstable element y and some stable element x for which $x^2 \neq 1$, and for which $yxy^{-1} = x^{-1}$. Then x and y do not commute, so G is non-abelian. □

Corollary 2.4. *Let G be an irregular group and let y be an unstable element of G, and x any stable element. Then x and y commute if and only if $x^2 = 1$. Moreover, if $z \in Z(G)$, then z is stable, and $z^2 = 1$.*

Proof. The result follows from the fact that $yxy^{-1} = x^{-1}$, so if x and y do commute, then $x = x^{-1}$. Next, let $z \in Z(G)$. If z is unstable, then since G is irregular, there exists some stable element x, with $x^2 \neq 1$, and $zxz^{-1} = x^{-1}$; hence $z \notin Z(G)$, a contradiction. Hence z must be stable. Now, there must be some unstable element $y \in G$, and $zy = yz$; hence by what was proved above, $z^2 = 1$. □

Corollary 2.5. *Let G be an irregular group and let $H < G$, and suppose every element of H is stable. Then H is abelian.*

Proof. Let G be an irregular group. Then there exists some unstable element $y \neq y^{-1} \in G$. If $h \in H$, by assumption h is stable, and $yhy^{-1} = h^{-1}$. Thus the conjugation map κ_y defined by $\kappa_y(h) = yhy^{-1}$ maps H to itself. But κ_y is the inversion map, and hence H must be abelian. □

Proposition 2.6. *Let G be an irregular group. The product of two unstable elements y_1 and y_2 must be stable. Every unstable element has order 4.*

Proof. Assume the hypotheses. Let $x \in G$ be stable and $x \neq x^{-1}$. If y_1y_2 were unstable, then conjugation by y_1y_2 would invert x. But $(y_1y_2)x(y_1y_2)^{-1} = y_1(y_2xy_2^{-1})y_1^{-1} = y_1x^{-1}y_1^{-1} = x$. Thus y_1y_2 must be stable. Let y be any unstable element. Then $y^2 = yy$ is stable. It follows from Proposition 2.2 that $\sigma(y^2) = (\sigma(y))^2$ implies $y^2 = y^{-2}$; hence $y^4 = 1$. But $y^2 \neq 1$, hence $|y| = 4$. □

Proposition 2.7. *Let G be an irregular group. Let x and z denote stable elements of G of order greater than 2, and suppose the product $y = xz$ is unstable. Then $\langle x, z\rangle \cong Q$, and $\langle x, y\rangle \cong Q$.*

Proof. By Proposition 2.6, the unstable element $y = xz$ has order 4, and since z is stable, by Proposition 2.3, we have $yzy^{-1} = z^{-1}$, while on the other hand $yzy^{-1} = (xz)z(xz)^{-1} = xzx^{-1}$. It follows that $xz = z^{-1}x$. Now $y^2 = (xz)(z^{-1}x) = x^2$. Since $| y |= 4$, this in turn implies that $|x| = 4$.

Putting $Q = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\rangle$, we see that y and x satisfy the relations for generators of Q, and may verify that the map $h : Q \to \langle y, x\rangle$, defined by $h(a) = y$ and $h(b) = x$, is an isomorphism. Since $\langle x, z\rangle = \langle x, y\rangle$, the proof is complete. □

3 The Core of an Irregular Map

Definition 3.1. Let $\sigma : G \to G$ be an irregular map. Let $\mathrm{Core}(\sigma) < G$ be the subgroup of G which is generated by the unstable elements of G.

The subgroup $\mathrm{Core}(\sigma)$ may be equal to G itself, or it may be a proper subgroup of G. The distinction between these two possibilities will be critical in classifying irregular maps.

Definition 3.2. An irregular map $\sigma : G \to G$ is said to be a *Type I irregular map* or simply *Type I map* if $\mathrm{Core}(\sigma) = G$, and a *Type II map* otherwise.

Let $g \in G$, and let $\sigma : G \to G$ be a Type II map. Observe that if g is σ-unstable then $g \in \mathrm{Core}(\sigma)$, and that $g \notin \mathrm{Core}(\sigma)$ implies that g is stable. This simple observation will be used repeatedly.

Proposition 3.3. *Let $\sigma : G \to G$ be a Type II map. Suppose $g \in G \setminus \mathrm{Core}(\sigma)$, and let y be an unstable element of G. Then $\langle g, y\rangle \cong Q$, and hence it follows that*
(i) $gyg^{-1} = y^{-1}$; (ii) $ygy^{-1} = g^{-1}$; (iii) $g^2 = y^2$; and (iv) $|g| = 4$.

Proof. Put $H = \langle g, y\rangle$. It suffices to show that $H \cong Q$. If $g^{-1}y \in \mathrm{Core}(\sigma)$ then since also $y \in \mathrm{Core}(\sigma)$, we would have $g = y(g^{-1}y)^{-1} \in \mathrm{Core}(\sigma)$, a contradiction. We conclude that $g^{-1}y \notin \mathrm{Core}(\sigma)$, and hence both g and $g^{-1}y$ are stable elements. Their product, y, is unstable; hence by Proposition 2.7, $\langle g, g^{-1}y\rangle = \langle g, y\rangle = H$, so that $H \cong Q$. The four results now follow from the definition of Q. □

Proposition 3.4. *Let $\sigma : G \to G$ be a Type II map. Then for any elements $g_1,\ g_2 \in G$, we have $g_1, g_2 \notin \mathrm{Core}(\sigma)$ implies that $g_1g_2 \in \mathrm{Core}(\sigma)$.*

Proof. By Proposition 2.1, G contains an unstable element y. By Proposition 1.10, y^{-1} is also unstable. Since $g_1, g_2 \notin \mathrm{Core}(\sigma)$, both are stable. Proposition 3.3 implies $(g_1g_2)y(g_1g_2)^{-1} = g_1y^{-1}g_1^{-1} = y$. Now if $g_1g_2 \notin \mathrm{Core}(\sigma)$, then by Proposition 3.3, $(g_1g_2)y(g_1g_2)^{-1} = y^{-1}$. This implies that $y = y^{-1}$, but y unstable implies $y \neq y^{-1}$, a contradiction. We conclude that $g_1g_2 \in \mathrm{Core}(\sigma)$. □

Corollary 3.5. *Let $\sigma : G \to G$ be a Type II irregular map. Then $[G : \mathrm{Core}(\sigma)] = 2$, and consequently $\mathrm{Core}(\sigma) \lhd G$.*

Proof. Left to the reader. □

4 The Stable Core of an Irregular Map

Definition 4.1. Let $\sigma : G \to G$ be an irregular map. Let $\mathrm{StCore}(\sigma) = \{g \in \mathrm{Core}(\sigma) \mid \sigma(g) = g\}$. We call this set the *stable core of the irregular map* σ.

The reader will easily verify the following proposition.

Proposition 4.2. *Let* $\sigma : G \to G$ *be an irregular map. Then* $1 \in \mathrm{StCore}(\sigma)$ *and* $x \in \mathrm{StCore}(\sigma)$ *implies* $x^{-1} \in \mathrm{StCore}(\sigma)$.

In fact, $\mathrm{StCore}(\sigma)$ must be a subgroup of G. We will show this first for a Type I map, via a somewhat extended argument by contradiction, and then establish the result for a Type II map in Section 5.

Proposition 4.3. *Let* $\sigma : G \to G$ *be a Type I map. Then* $Z(G) = \{g \in G \mid g^2 = 1\}$ *and* $Z(G) \subseteq \mathrm{StCore}(\sigma)$.

Proof. The second assertion restates Corollary 2.4, which shows that $Z(G) \subseteq \{g \in G \mid g^2 = 1\}$. Conversely, let $g \in G$ with $g^2 = 1$. Then by Proposition 2.3, for any unstable $y \in G$, we have $ygy^{-1} = g^{-1} = g$, and hence $g \in C_G(y)$. Thus g commutes with all of the generators of $\mathrm{Core}(\sigma)$, and since σ is Type I, $\mathrm{Core}(\sigma) = G$. This establishes that $g \in Z(G)$. ☐

Proposition 4.4. *Let* G *be a group and suppose* G *admits a Type I map* σ. *Let* x *be stable element of* G *such that* $x \neq x^{-1}$. *Then* G*, acting by conjugation, acts transitively on the Cayley set* $S_x = \{x, x^{-1}\}$. *The stabilizer of* x *under this action is* $C_G(x)$. *Furthermore,* $[G : C_G(x)] = 2$. *In particular, this implies that* $C_G(x) \lhd G$. *Here* $C_G(x)$ *consists entirely of stable elements, and* $C_G(x)$ *is abelian.*

Proof. Let y be an unstable element of G. By Proposition 2.3, $yS_xy^{-1} = S_x$, that is, y acts transitively on S_x. Since G is generated by the unstable elements, it follows that G acts transitively on S_x. The stabilizer of x is the set $\{g \in G \mid gxg^{-1} = x\} = C_G(x)$. It follows from the orbit stabilizer theorem that $[G : C_G(x)] = |S_x| = 2$. Let $g \in C_G(x)$ and assume g is unstable. Then $gxg^{-1} = x^{-1}$ by Proposition 2.3, a contradiction. That $C_G(x)$ is abelian follows from Corollary 2.5. ☐

Proposition 4.5. *Let* $\sigma \in \mathrm{Cay}P_1(G)$ *be a Type I map. Assume that* $\mathrm{StCore}(\sigma) \not\leq G$. *Let* $x \in G$ *be stable,* $x \neq x^{-1}$*, and let* $y \in G$ *be unstable. Then* xy *and* xy^{-1} *are stable elements of* G*;* $H = \langle x, y\rangle \cong Q$*; and* $x^2 = y^2$.

Proof. Assume the hypotheses. Proposition 4.4 implies that if $x \neq x^{-1}$ is a stable element of G, then $C_G(x) \subseteq \mathrm{StCore}(\sigma)$. Since $\mathrm{StCore}(\sigma)$ is not a subgroup of G it follows that $C_G(x) \subsetneq \mathrm{StCore}(\sigma)$. Therefore there is a stable element x_0 of G with $x_0 \notin C_G(x)$; hence $x_0 \notin Z(G)$. By Corollary 2.4, $x_0^2 \neq 1$, which implies that $x_0 \neq x_0^{-1}$. By Proposition 4.4, x acts by conjugation on the Cayley set $S_{x_0} = \{x_0, x_0^{-1}\}$. It must follow that $x^{-1}x_0^{-1}x = x_0$. Now, if $x^{-1}y$ were unstable, conjugation would invert x_0. Computation shows that $(x^{-1}y)x_0(x^{-1}y)^{-1} = x_0$. But $x_0 \neq x_0^{-1}$, and hence $x^{-1}y$ must be stable. Since x and $x^{-1}y$ are stable elements with unstable product $x(x^{-1}y) = y$, it follows from Proposition 2.7 that $\langle x, x^{-1}y\rangle \cong Q$. But $\langle x, x^{-1}y\rangle = \langle x, y\rangle = H$, so $H \cong Q$. Moreover, it then follows that $x^2 = y^2$. Since $H \cong Q$, H has a unique involution, which will be denoted by I. Then $I \in Z(H)$, and $x^{-1}y = (Ix)y = (xy)^{-1}$. Therefore $(xy)^{-1}$ is stable, and xy must also be stable. □

Proposition 4.6. *Let $\sigma \in \mathrm{Cay}P_1(G)$ be a Type I map, and assume that* $\mathrm{StCore}(\sigma) \nleq G$. *Then G contains an involution, denoted I, with the property that for any $g \in G$ with $g \neq g^{-1}$, $g^2 = I$. Here I is unique with respect to this property. Furthermore, for any $g \in G$ such that $g \neq g^{-1}$, $|g| = 4$.*

Proof. By Proposition 2.1, G contains a stable element $g_0 \neq g_0^{-1}$, and an unstable element y_0. Let $I = (y_0)^2$. By Proposition 4.5, if g is stable with $g \neq g^{-1}$, then $g^2 = y_0^2 = I$. In particular $g_0^2 = I$. Now, if y is unstable, then again by Proposition 4.5, $y^2 = g_0^2 = I$. Therefore, for any $g \in G$ with $g \neq g^{-1}$, $g^2 = I$. It follows that if $g \neq g^{-1}$, then $|g| = 4$. □

We now have enough information to determine the structure of G. We will see that for a Type I map σ, the assumption that $\mathrm{StCore}(\sigma)$ is not a subgroup of G will imply that $G \cong E \times Q$ and that $\mathrm{Core}(\sigma) \neq G$; hence σ is a Type II map, a contradiction.

Definition 4.7. A group G is a group of *restricted type* if for all $g \in G$, $|g|$ divides 4; G contains an involution which we denote by I, such that for all $g \in G$, if $|g| = 4$, then $g^2 = I$; and every involution is central.

Proposition 4.8. *Let $\sigma : G \to G$ be a Type I map. Assume that* $\mathrm{StCore}(\sigma)$ *is not a subgroup of G. Then G is a group of restricted type.*

Proof. The first two conditions follow from Proposition 4.6, while the third follows from Proposition 4.3. □

Proposition 4.9. *Let G be a group of restricted type. Then every subgroup of G is a normal subgroup.*

Proof. Let H be a subgroup of a group G of restricted type. If every non-trival $h \in H$ is an involution, then, by Definition 4.7, $H < Z(G)$, and hence $H \lhd G$. Suppose there exists $h \in H$ with $h^2 \neq 1$. Then from Definition 4.7 it follows that $|h| = 4$, that $h^2 = I$, and consequently $I \in H$. Consider the subgroup $\langle I\rangle$ of G. Since $I \in Z(G)$, $\langle I\rangle < Z(G)$, which implies that $\langle I\rangle \lhd G$. Now let $\pi : G \to G/\langle I\rangle$ be the canonical map. For any $g \in G$, denote $\pi(g)$ by $\overline{g}$. Let $g \in G$; then either $g^2 = 1 \Rightarrow \overline{g}^2 = 1$ or $g^4 = 1 \Rightarrow g^2 = I \Rightarrow \overline{g}^2 = 1$. It follows that $G/\langle I\rangle$ is an elementary 2-group, whence $G/\langle I\rangle$ is abelian, and therefore every subgroup of $G/\langle I\rangle$ is normal. From this we conclude that $\pi(H) = H/\langle I\rangle \lhd G/\langle I\rangle$. It follows that $H \lhd G$. □

Proposition 4.10. *Let G be a non-abelian group of restricted type. Let H be any subgroup of G with $H \cong Q$. Then G is the internal direct product of subgroups E and H where E is an elementary (abelian) 2-group.*

Proof. Assume the hypotheses. By Proposition 4.9, every subgroup of G is a normal subgroup. Since G is non-abelian, by the theorem of Baer (see [2]), this implies that $G \cong A \times B \times Q$, where A is an elementary 2-group and B is an abelian torsion group all of whose elements have odd order. But a group of restricted type contains no non-trivial element of odd order, so $G \cong A \times Q$. We may interpret this result as representing G as an internal direct product of normal subgroups E and H, with $H \cong Q$. □

Lemma 4.11. *Let $\sigma : G \to G$ be an irregular map. Let $E < G$ such that if $c \in E$ then $c^2 = 1$. Then each right coset of E in G consists entirely of stable elements, or entirely of unstable elements.*

Proof. Let $g \in G$ be stable and consider an element $cg \in Eg$. By Proposition 4.3, $cg = gc$. The Cayley condition says $\sigma(gc) = \sigma(g)c^{\pm 1} = gc = cg$. So all elements of Eg are stable. Now suppose g is unstable and that some $cg \in Eg$ is stable. Since $cg \in Eg$, the right cosets $E(cg)$ and Eg are equal. But all elements in the coset $E(cg)$ are then stable, a contradiction! □

Proposition 4.12. *Let $\sigma : G \to G$ be a Type I map. Then* $\mathrm{StCore}(\sigma)$ *is a subgroup of G.*

Proof. By contradiction. Assume that $\mathrm{StCore}(\sigma) \not< G$. By Proposition 2.1, G contains an unstable element y and a stable element $x \neq x^{-1}$. Letting $H = \langle x, y \rangle$, by Proposition 4.5, $H \cong Q$. By Proposition 4.8, G is a group of restricted type. By Proposition 2.3, G is not abelian. Apply Proposition 4.10 to conclude that $G = E \times H$, where $E < G$ is an elementary 2-group.

Consider the element xy of G. By Proposition 4.5, xy is stable. The elements x and y of H generate 2 of the 3 distinct subgroups of order 4 of $H \cong Q$. The element xy generates the remaining subgroup of order 4.

Both x and xy are stable, while y is unstable. By Proposition 2.2, x^{-1} and $(xy)^{-1}$ are stable, while y^{-1} is unstable. The identity, 1, is stable, and I is stable as well. Apply Lemma 4.11. There are eight right cosets of E in G. The unstable elements are members of $Ey \cup Ey^{-1}$. We have $Ey \cup Ey^{-1} \subseteq E\langle y \rangle$. However, $E\langle y \rangle = E \cup EI \cup Ey \cup Ey^{-1}$, so that $[G : E\langle y \rangle] = 2$. Therefore $E\langle y \rangle$ is a proper subgroup of G. Since the unstable elements of G all lie in $E\langle y \rangle$, it follows that $\mathrm{Core}(\sigma) \subseteq E\langle y \rangle \subsetneq G$. This is a contradiction since, for any Type I map, we have $G = \mathrm{Core}(\sigma)$. The contradiction establishes that $\mathrm{StCore}(\sigma) < G$. □

Theorem 4.13. *Let G be a group with Type I map $\sigma : G \to G$. Then $[G : \mathrm{StCore}(\sigma)] = 2$; hence $\mathrm{StCore}(\sigma) \lhd G$, and $\mathrm{StCore}(\sigma)$ is abelian.*

Proof. By Proposition 2.1, G contains a stable element $x \neq x^{-1}$. By Proposition 4.4, $C_G(x)$ consists of stable elements so $C_G(x) < \mathrm{StCore}(\sigma) \subsetneq G$. The latter inclusion is proper because, again by Proposition 2.1, G does contain an unstable element. By Proposition 4.4, $[G : C_G(x)] = 2$, so that $C_G(x)$ is a maximal normal subgroup of G. It follows that $C_G(x) = \mathrm{StCore}(\sigma)$. We therefore conclude that $[G : \mathrm{StCore}(\sigma)] = 2$, and it follows that $\mathrm{StCore}(\sigma) \lhd G$. Since $H = \mathrm{StCore}(\sigma) < G$ is contains only stable elements, it is abelian, by Corollary 2.5. □

5 Type II maps

Proposition 5.1. *Let $\sigma : G \to G$ be a Type II map. Then σ maps* $\mathrm{Core}(\sigma)$ *onto itself, and $\sigma\,|_{\mathrm{Core}(\sigma)} \in \mathrm{Cay}P_1(\mathrm{Core}(\sigma))$. There are two possibilities: either* $\mathrm{StCore}(\sigma)$ *contains an element x with $x^{-1} \neq x$, and $\sigma\,|_{\mathrm{Core}(\sigma)}$ is a Type I map on* $\mathrm{Core}(\sigma)$*; or for all $x \in \mathrm{StCore}(\sigma)$, $x^2 = 1$. In the latter case $\sigma\,|_{\mathrm{Core}(\sigma)}$ is the inversion map.*

Proof. Details are left to the reader. □

Proposition 5.2. *Let $\sigma : G \to G$ be a Type II map. Then* $\mathrm{StCore}(\sigma) < \mathrm{Core}(\sigma)$*.*

Proof. Let $\sigma : G \to G$ be a Type II map. By Proposition 5.1, there are only two possibilities. If $\sigma\,|_{\mathrm{Core}(\sigma)}$ is a Type I map, then by Theorem 4.12, $\mathrm{StCore}(\sigma) < \mathrm{Core}(\sigma)$. Since $\mathrm{StCore}(\sigma) = StCore(\sigma\,|_{\mathrm{Core}(\sigma)})$, it follows that $\mathrm{StCore}(\sigma)$ is a subgroup of $\mathrm{Core}(\sigma)$. If, for all $x \in \mathrm{StCore}(\sigma)$, $x^2 = 1$, then it follows that $\mathrm{StCore}(\sigma)$ is, trivially, inverse closed. Now let x_1, $x_2 \in \mathrm{StCore}(\sigma)$. Since $\mathrm{StCore}(\sigma) \subseteq \mathrm{Core}(\sigma)$, and since $\mathrm{Core}(\sigma) < G$, it follows that $x_1x_2 \in \mathrm{Core}(\sigma)$. If $x_1x_2 \notin \mathrm{StCore}(\sigma)$, it follows that x_1x_2 is unstable. Let $H = \langle x_1, x_2\rangle$. By Proposition 2.6, $H \cong Q$. Since x_1 and x_2 form a pair of generators of $H \cong Q$, it follows that $|x_1| = |x_2| = 4$. This contradicts the fact that $x_1^2 = x_2^2 = 1$. The contradiction establishes the fact that x_1x_2 is stable, and thus $\mathrm{StCore}(\sigma)$ is closed under multiplication and is a subgroup of $\mathrm{Core}(\sigma)$. □

Let $\sigma : G \to G$ be an irregular map. We have shown, whether or not σ is of Type I or II, that $\mathrm{StCore}(\sigma)$ is a subgroup of $\mathrm{Core}(\sigma)$. From Corollary 2.5 and Proposition 2.6 we immediately conclude:

Corollary 5.3. *Let $\sigma : G \to G$ be an irregular map. Then* $\mathrm{StCore}(\sigma)$ *is abelian. If y_1 and y_2 are unstable elements of G then $y_1y_2 \in \mathrm{StCore}(\sigma)$.*

Proposition 5.4. *Let $\sigma : G \to G$ be an irregular map. Then* $[\mathrm{Core}(\sigma) : \mathrm{StCore}(\sigma)] = 2$*, implying in particular that* $\mathrm{StCore}(\sigma) \lhd \mathrm{Core}(\sigma)$*.*

Proof. If σ is a Type I map then the result is given in Theorem 4.13, so assume σ is a Type II map. As in Proposition 5.1, there are two possibilities. In the first case, $\sigma\,|_{\mathrm{Core}(\sigma)}$ is a Type I map of $\mathrm{Core}(\sigma)$, and the present proposition again follows from Theorem 4.13. In the second, for all $x \in \mathrm{StCore}(\sigma)$, we have $x^2 = 1$. Consider any two right cosets $\mathrm{StCore}(\sigma)y_1$ and $\mathrm{StCore}(\sigma)y_2$, not equal to $\mathrm{StCore}(\sigma)$ itself. Then y_1 and y_2 must be unstable elements of G. By Proposition 2.2, y_2 is unstable implies that y_2^{-1} is unstable. By Proposition 2.6, $y_1y_2^{-1} \in \mathrm{StCore}(\sigma)$. This implies $\mathrm{StCore}(\sigma)y_1\mathrm{StCore}(\sigma)y_2$, and in turn that $[\mathrm{Core}(\sigma) : \mathrm{StCore}(\sigma)] \leq 2$. But by definition of $\mathrm{StCore}(\sigma)$, G must contain an unstable element $y \in \mathrm{Core}(\sigma)\setminus\mathrm{StCore}(\sigma)$, so $\mathrm{Core}(\sigma) \neq \mathrm{StCore}(\sigma)$, and hence $[\mathrm{Core}(\sigma) : \mathrm{StCore}(\sigma)] = 2$. □

Corollary 5.5. *Let $\sigma : G \to G$ be a Type II map. Then* $[G : \mathrm{StCore}(\sigma)] = 4$.

Proof. This is immediate from Corollary 3.5 and Proposition 5.4. ☐

6 Type I Maps are Group Automorphisms

Proposition 6.1. *Let G be a non-abelian group, and let σ be a non trivial automorphism of G such that $\sigma(g) = g^{\pm 1}$ for all $g \in G$. Then σ is a Type I map.*

Proof. By Corollary 1.9, $\sigma \in \mathrm{Cay}P_1(G)$. Observe that σ is not the inversion map, because G is non-abelian. Suppose that σ is a Type II map. Since $\sigma \in \mathrm{Aut}(G)$, the set $H = \{g \in G \mid \sigma(g) = g\} < G$. Since σ is Type II, G contains an element $g \notin \mathrm{Core}(\sigma)$, and $g \notin \mathrm{Core}(\sigma)$ implies g is stable. By Proposition 2.1, G contains an unstable element y. Since $\mathrm{Core}(\sigma) < G$, $y \in \mathrm{Core}(\sigma)$, $g \notin \mathrm{Core}(\sigma)$ implies $gy \notin \mathrm{Core}(\sigma)$, hence gy is stable, so $gy \in H$. Since g is stable, $g \in H \quad \Rightarrow g^{-1} \in H \quad \Rightarrow g^{-1}(gy) = y \in H$, so that y is stable, contradiction. We conclude that σ must be of Type I. ☐

Proposition 6.2. *Let G be a group and $\sigma : G \to G$ a Type I map. Let y be any unstable element of G, and let $A = \mathrm{StCore}(\sigma)$. Then $G = \mathrm{Dic}(A, y^2)$. Further, the subgroup A is not an elementary 2-group.*

Proof. Assume the hypotheses. Then by Proposition 4.12 and Theorem 4.13, $\mathrm{StCore}(\sigma)$ is an abelian subgroup of index 2 in G. Since y is unstable, $y \in G \setminus A$. By Proposition 2.6, y has order 4, so y^2 is an involution of $\mathrm{StCore}(\sigma)$. Let $I = y^2$. Then any other element of $G \setminus A$ has the form ay. The reader can verify that $(ay)^2 = y^2 = I$. Finally, by Proposition 2.1, G has a stable element of order larger than 2, and since for a Type I map, $G = \mathrm{Core}(\sigma)$; hence every stable element of G is a member of $\mathrm{StCore}(\sigma)$, and it follows that $\mathrm{StCore}(\sigma)$ is not an elementary 2-group. ☐

In the remainder of the section we show that every Type I map is a group automorphism. Below, we summarize without proof some well known facts about generalized dicyclic groups.

Proposition 6.3. *Let $G = \mathrm{Dic}(A, I)$, and let $y \notin A$ be the remaining generator of G. Then for any $g \in G \setminus A$, $g^2 = y^2 = I$; $|g| = 4$; $g^{-1} = Ig$; for all $a \in A$, $ga = a^{-1}g$; $g_1, g_2 \notin A$ implies $g_1 g_2 \in A$; and for any $z \in G$, if $z^2 = 1$ then $z \in Z(G)$.*

Proposition 6.4. *Let $G = \mathrm{Dic}(A, I)$ and let $x \in A$. Define a map $\alpha : G \to G$ as follows: if $g \in A$, then let $\alpha(g) = g$, otherwise let $\alpha(g) = xg$. Then α is an automorphism of G.*

Proof. Assume the hypotheses. It is clear that α is a bijection. To show that $\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2)$ there are four cases to check, corresponding to whether $g_i \in A$ for $i = 1, 2$. We leave the details to the reader. ☐

Corollary 6.5. *Let $G = \mathrm{Dic}(A, I)$. Let g_1, g_2 be any two elements of G that are not in A. Then there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha\mid_A = 1_A$ and $\alpha(g_1) = g_2$. There exists an automorphism $\beta : G \to G$ with $\beta(g) = g$ if $g \in A$ and $\beta(g) = g^{-1}$ if $g \notin A$.*

Proof. Take $x = g_2 g_1^{-1}$ in Proposition 6.4. Then the so-constructed automorphism α of G satisfies $\alpha\mid_A = 1_A$ and $\alpha(g_1) = x g_1 = g_2 g_1^{-1} g_1 = g_2$. For the second part, apply Proposition 6.4 with $x = I$. □

Theorem 6.6. *Let $\sigma : G \to G$ be a Type I map. Then σ is a group automorphism. Conversely, every dicyclic group $G = \mathrm{Dic}(A, I)$ admits one irregular group automorphism $\sigma : G \to G$, provided that A is not an elementary 2-group.*

Proof. By Proposition 6.2, we know that every irregular group G with Type I map is generalized dicyclic with $A = \mathrm{StCore}(\sigma)$. By Corollary 6.5, we know there is an automorphism $\alpha : G \to G$ that fixes all elements of A and inverts the others. But σ satisfies this description, so we conclude σ is a group automorphism.

For the converse, let $G = \mathrm{Dic}(A, I)$, and let σ be the automorphism of G defined in Corollary 6.5. Then for any $g \in G$, $\sigma(g) = g^{\pm 1}$, so by Corollary 1.9, $\sigma \in \mathrm{Cay}P_1(G)$. Since σ fixes elements of A and inverts elements in $G \setminus A$, σ is neither the inversion map nor the identity map, hence σ is irregular. □

Corollary 6.7. *Let $G = \mathrm{Dic}(A, I)$, and let $Aut(G, A)$ denote the subgroup of automorphisms of G that fix A point wise. Then $Aut(G, A) \cong A$.*

Proof. For any $a \in A$ define a map by $\tau_a : g \mapsto ag$, for $g \in G \setminus A$, and by $\tau_a : g \mapsto g$ for $g \in A$. By Corollary 6.5, $\tau_a \in \mathrm{Aut}(G, A)$. Consider the map $\tau : A \to \mathrm{Aut}(G, A)$ defined by $\tau(a) = \tau_a$. It may be verified that τ is in fact an isomorphism of A with $\mathrm{Aut}(G, A)$. □

7 Groups Admitting Type II Irregular Maps

Proposition 7.1. *Let $\sigma : G \to G$ be a Type II map, and y a chosen unstable element of G. Then for any $g \notin \mathrm{StCore}(\sigma)$, $g^2 = y^2$.*

Proof. If $g \in \mathrm{Core}(\sigma)$, then since $g \notin \mathrm{StCore}(\sigma)$, g must be unstable. Now, by Proposition 5.4, $\mathrm{StCore}(\sigma)$ has two cosets in $\mathrm{Core}(\sigma)$. It follows that $g = yx$, for some stable $x \in \mathrm{StCore}(\sigma)$. Consider g^2. Using Proposition 2.3, it is easy to see that $g^2 = (yx)^2 = y^2$: This proves the proposition if $g \in \mathrm{Core}(\sigma)$. Now suppose $g \notin \mathrm{Core}(\sigma)$. Then immediately from Proposition 3.3, $g^2 = y^2$. □

Proposition 7.2. *Let $\sigma \in \mathrm{Cay}P_1(G)$ be a Type II map, and y a chosen unstable element of G. Set $I = y^2$. Then $I \in \mathrm{StCore}(\sigma)$, $|I| = 2$ and $I \in Z(G)$.*

Proof. Left to the reader. □

Proposition 7.3. *Let σ be a Type II map. Then for any $g \notin \mathrm{StCore}(\sigma)$, $gxg^{-1} = x^{-1}$ for all $x \in \mathrm{StCore}(\sigma)$.*

Proof. Let y be a chosen unstable element and set $I = y^2$. Assume that $g \in \mathrm{Core}(\sigma)$; then since $g \notin \mathrm{StCore}(\sigma)$, the result follows from Proposition 2.3. On the other hand, if $g \notin \mathrm{Core}(\sigma)$, and $x \in \mathrm{StCore}(\sigma)$, we may conclude that $gx \notin \mathrm{StCore}(\sigma)$, since otherwise we would be forced to have $g \in \mathrm{StCore}(\sigma)$, a contradiction. Consider the product $(gx)^2 = (gxg^{-1})g^2x = (gxg^{-1})Ix$; by Proposition 7.1 and since I is central, we may write $(gx)^2 = ((gxg^{-1})xI$. But $gx \notin \mathrm{StCore}(\sigma)$, so applying Propositions 7.1 and 7.2, we have $(gx)^2 = I$, and hence $I = (gxg^{-1})xI$. This in turn implies that $(gxg^{-1}) = x^{-1}$. □

Corollary 7.4. *Let σ be a Type II map. Then $\mathrm{StCore}(\sigma) \lhd G$.*

Proof. Let $g \in G$. If $g \in \mathrm{StCore}(\sigma)$ then of course $g \in N_G(\mathrm{StCore}(\sigma))$. If, on the other hand $g \notin \mathrm{StCore}(\sigma)$, the map $x \to gxg^{-1}$ is simply the inversion map of $\mathrm{StCore}(\sigma)$ to itself. Thus $g \in N_G(\mathrm{StCore}(\sigma))$ in this case as well. □

Proposition 7.5. *Let σ be a Type II map. Then for all $g \in G$, $g^2 \in \mathrm{StCore}(\sigma)$.*

Proof. Let y be some unstable element of G. If $g \notin \mathrm{StCore}(\sigma)$ then by Proposition 7.1, $g^2 = y^2$, and $y^2 \in \mathrm{StCore}(\sigma)$ by Proposition 2.6. On the other hand, if $g \in \mathrm{StCore}(\sigma)$ the result is immediate since $\mathrm{StCore}(\sigma)$ is a subgroup. □

Corollary 7.6. *Let σ be a Type II map, then $G/\mathrm{StCore}(\sigma) \cong Z_2^2$.*

Proof. By Corollary 5.5 we know$G/\mathrm{StCore}(\sigma)$ has order 4. By Proposition 7.5 we know $G/\mathrm{StCore}(\sigma)$ is an elementary two group. □

Proposition 7.7. *Let $\sigma : G \to G$ be a Type II map, then $\mathrm{StCore}(\sigma)$ is an elementary 2-group.*

Proof. Since by Corollary 7.4, $\mathrm{StCore}(\sigma) \lhd G$, for all $h \in G$, the map $\kappa_h : g \to hgh^{-1}$ is an element of $\mathrm{Aut}(\mathrm{StCore}(\sigma))$. Since $\mathrm{StCore}(\sigma)$ is abelian, if $g \in \mathrm{StCore}(\sigma)$, the automorphism κ_g is the identity map. Letting $\alpha : G \to \mathrm{Aut}(\mathrm{StCore}(\sigma))$ be the map $\alpha(g) = \kappa_g$, then $\mathrm{StCore}(\sigma) < Ker(\alpha)$. Let $\pi : G \to G/\mathrm{StCore}(\sigma)$ be the canonical epimorphism. Since $Ker(\pi) = \mathrm{StCore}(\sigma) < Ker(\alpha)$ we conclude there is a homomorphism $\beta : G/\mathrm{StCore}(\sigma) \to \mathrm{Aut}(\mathrm{StCore}(\sigma))$ such that the diagram in Figure 1 commutes.

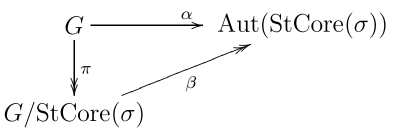

Figure 1: The diagram commutes.

Since $G/\mathrm{StCore}(\sigma) \cong Z_2^2$, we may choose non-trivial elements a, b and $c \in G$ such that $\pi(a)$, $\pi(b)$, $\pi(c)$ are three distinct non-trivial elements of $G/\mathrm{StCore}(\sigma)$, and $\pi(a)\pi(b) = \pi(c)$. Since β is a homomorphism, and the diagram commutes it follows that $\kappa_{ab} = \alpha(ab) = \alpha(c) = \kappa_c$. Now according to Proposition 7.3, if $g \notin \mathrm{StCore}(\sigma)$ then κ_g is the inversion map ν. Hence the equation $\kappa_{ab} = \kappa_c$ implies that $\nu\nu = \nu$, from which it follows that $\nu = 1 \in \mathrm{Aut}(\mathrm{StCore}(\sigma))$. Thus for all $x \in \mathrm{StCore}(\sigma)$, $\nu(x) = x^{-1} = x$, which implies that $\mathrm{StCore}(\sigma)$ is an elementary 2-group. □

Proposition 7.8. *Let $\sigma : G \to G$ be a Type II map; then* $\mathrm{StCore}(\sigma) = \{g \in G \mid g^2 = 1_G\}$. *Furthermore,* $\mathrm{StCore}(\sigma) = Z(G)$.

Proof. By Proposition 7.7, $\mathrm{StCore}(\sigma) \subseteq \{g \in G \mid g^2 = 1_G\}$. Conversely, suppose $g^2 = 1$. Although g is stable, we must also show that $g \in \mathrm{Core}(\sigma)$. Suppose that $g \notin \mathrm{Core}(\sigma)$; then by Proposition 3.3, $|g| = 4$; hence $g^2 \neq 1$, a contradiction. We conclude that $g \in \mathrm{Core}(\sigma)$, and hence $g \in \mathrm{StCore}(\sigma)$. This shows that $\mathrm{StCore}(\sigma) = \{g \in G \mid g^2 = 1_G\}$.

Now $\mathrm{StCore}(\sigma)$ is itself abelian, so to verify that $\mathrm{StCore}(\sigma) < Z(G)$, it suffices to show that $x \in \mathrm{StCore}(\sigma)$ commutes with all $g \notin \mathrm{StCore}(\sigma)$. Let $g \in G \setminus \mathrm{StCore}(\sigma)$. Then by Proposition 7.3, $gxg^{-1} = x^{-1}$, but $x^{-1} = x$, hence $gx = xg$. Hence $\mathrm{StCore}(\sigma) < Z(G)$.

Conversely, let $x \in Z(G)$. By Corollary 2.4 $x^2 = 1$ and x is stable. By Proposition 3.3, all elements of G not in $\mathrm{Core}(\sigma)$ have order 4. It follows that $x \in \mathrm{Core}(\sigma)$. Since x is stable, it follows that $x \in \mathrm{StCore}(\sigma)$. We conclude that $Z(G) < \mathrm{StCore}(\sigma)$. This completes the proof that $\mathrm{StCore}(\sigma) = Z(G)$. □

Proposition 7.9. *Let $\sigma : G \to G$ be a Type II map; then G is a non-abelian group of restricted type. It follows that G contains subgroups E and H, where E is an elementary two group and $H \cong Q$, such that $G = E \times H$.*

Proof. Let $g \in G$. If $g \in \mathrm{Core}(\sigma)$ and $g \notin \mathrm{StCore}(\sigma)$ then $|g| = 4$, by Proposition 2.6. If $g \notin \mathrm{Core}(\sigma)$, then by Proposition 3.3, $|g| = 4$. If $g \in \mathrm{StCore}(\sigma)$ then $|g| = 2$, by Proposition 7.7. We conclude the order of every element of G is a divisor of 4. By Proposition 2.1, there is some unstable $y \in G$. Let $g \in G$ be an element of order 4. Since $\mathrm{StCore}(\sigma)$ is an elementary 2-group, by Proposition 7.7 it follows that $g \notin \mathrm{StCore}(\sigma)$, and hence by Proposition 7.1, $g^2 = y^2$. By Proposition 7.2, $y^2 = I$ is an involution. Finally, if $g \in G$ and $g^2 = 1$, then by Proposition 7.8, $g \in Z(G)$. This completes the demonstration that G is of restricted type. Furthermore, G is nonabelian by Proposition 2.3. The remaining assertions now follow from Proposition 4.10. □

Corollary 7.10. *Every irregular group is a generalized dicyclic group. Moreover if G admits a Type II map then G is a non-abelian graphically abelian group.*

Proof. If G admits a Type I map, we are done by Proposition 6.2. Suppose G admits a Type II map. Then the previous proposition asserts that $G \cong E \times Q$, where E is an elementary 2-group. Such a group is a generalized dicyclic group. By a theorem of [5], a non-abelian group G is graphically abelian if and only if $G \cong E \times Q$. □

8 The Structure of $\mathrm{Cay}P(G)$

Throughout this section E denotes an elementary 2-group, $\mathrm{Inn}(G)$ is the group of inner automorphisms of G, κ_h is the conjugation map u $g \mapsto hgh^{-1}$, and ν is the inversion map $g \mapsto g^{-1}$. $\mathrm{CAut}(G)$ is the group $\mathrm{Cay}P_1(G) \cap \mathrm{Aut}(G)$.

Proposition 8.1. *Let $\sigma \in \mathrm{Cay}P_1(G)$. Then for all $g \in G$, we have $\sigma\lambda_g\sigma^{-1} \in L_G$ if and only if $\sigma \in \mathrm{Aut}(G)$, and hence $L_G \lhd L_G\mathrm{CAut}(G)$. Moreover, $\sigma \in \mathrm{CAut}(G)$ implies $\sigma\lambda_g\sigma^{-1} = \lambda_{\sigma(g)}$.*

Proof. Assume $\sigma\lambda_g\sigma^{-1} = \lambda_{g'} \in L_G$. Then $\sigma\lambda_g(1_G) = \lambda_{g'}\sigma(1_G) = g'$; hence $\sigma(g) = g'$, and $\sigma\lambda_g = \lambda_{\sigma(g)}\sigma$. Now for any $x \in G$, it follows that on the one hand $(\lambda_{\sigma(g)}\sigma)(x) = \lambda_{\sigma(g)}(\sigma(x)) = \sigma(g)\sigma(x)$, while on the other hand $(\sigma\lambda_g)(x) = \sigma(\lambda_g(x)) = \sigma(gx)$, so that σ must be an automorphism of G. The proof of the converse is left to the reader. ☐

Proposition 8.2. *If G is irregular, $\mathrm{Cay}P_1(G) \ntriangleleft G$.*

Proof. Suppose for every $\sigma \in \mathrm{Cay}P_1(G)$ and for every $\lambda \in L_G$, we have $\lambda\sigma\lambda^{-1} = \sigma' \in \mathrm{Cay}P_1(G)$. Then $\lambda_g\sigma\lambda_{g^{-1}}(1_G) = \sigma'(1_G) = 1_G$. However $\lambda_g\sigma\lambda_{g^{-1}}(1_G) = \lambda_g(\sigma(g^{-1})) = g\sigma(g^{-1}) = g(\sigma(g))^{-1}$, since by Corollary 2.2 σ carries inverses to inverses. Thus $g\sigma(g)^{-1} = 1_G \Rightarrow \sigma(g)^{-1} = g^{-1} \Rightarrow \sigma(g) = g$. But g is arbitrary, so σ is the identity map. But G is irregular, a contradiction. ☐

Proposition 8.3. *If G admits a Type II map, then $\mathrm{Inn}(G) < \mathrm{Cay}P_1(G)$; $|\mathrm{Inn}(G)| = 4$; and $\mathrm{Cay}P_1(G) \cong Z_2^3$.*

Proof. By Theorem 7.9, $G = E \times H$, where $H \cong Q$. Let $\sigma \in \mathrm{Cay}P_1(G)$. Then $\sigma\mid_E$ is the identity map of E and $\sigma\mid_H \in \mathrm{Cay}P_1(H)$. It may be verified that the restriction map $\sigma \to \sigma\mid_H$ induces an isomorphism $\mathrm{Cay}P_1(G) \cong \mathrm{Cay}P_1(H)$. Since $H \cong Q$, it suffices to demonstrate that $\mathrm{Cay}P_1(Q) \cong Z_2^3$. Let $K < \mathrm{Sym}(G)$ be defined by $K = \{\sigma \in \mathrm{Sym}(G) \mid \sigma(g) = g^{\pm 1}\}$. By Corollary 1.8, every element of $\mathrm{Cay}P_1(G)$ has this property so $\mathrm{Cay}P_1(Q) < K$. Putting $Q = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}\rangle$, we observe that if $\sigma \in K$ then σ acts on each of the three sets $\{a^{\pm 1}\}$, $\{b^{\pm 1}\}$, $\{(ab)^{\pm 1}\}$. It follows that $\mid K \mid = 8$. Since for each $\sigma \in K$, $\sigma^2 = 1$, it follows that $K \cong Z_2^3$.

Let $\sigma \in \mathrm{Inn}(Q)$. It may be readily verified that $\sigma(g) = g^{\pm 1}$ for all $g \in G$; hence by Corollary 1.9, $\sigma \in \mathrm{Cay}P_1(Q)$ and evidently $\sigma \in K$. Thus $\mathrm{Cay}P_1(Q)$ has at least four elements. Since Q is graphically abelian, by [5] the inversion map $\nu \in \mathrm{Cay}P_1(Q)$. Now $\nu \notin \mathrm{Aut}(Q)$; it is in fact an anti-automorphism. Thus $\mathrm{Cay}P_1(Q)$ contains at least five elements of K, so $\mathrm{Cay}P_1(Q) = K$. ☐

Corollary 8.4. *Let $G = E \times Q$. Let $\sigma \in \mathrm{Cay}P_1(G)$. If $\sigma \in \mathrm{Inn}(G)$ and $\sigma \neq 1$, then σ is a Type I map. If $\sigma \notin \mathrm{Inn}(G)$, then σ is an anti-automorphism of G, and if $\sigma \neq \nu$, σ is a Type II map of G. In particular, CAut$(E \times Q) = \mathrm{Inn}(E \times Q)$.*

Proof. The first assertion follows from Proposition 8.3, in view of Proposition 6.1. To prove the second assertion, let $\sigma \in \mathrm{Cay}P_1(G)$ and assume $\sigma \notin \mathrm{Inn}(G)$. Since $[\mathrm{Cay}P_1(G) : Inn(G)] = 2$, it follows that $\sigma\nu \in \mathrm{Inn}(G)$. Since $\sigma\nu$ is an automorphism of G and ν is an anti-automorphism of G, σ is an anti-automorphism of G. Since G is non-abelian, $\sigma \notin \mathrm{Aut}(G)$; hence by Theorem 6.6, σ cannot be a Type I map, and so if $\sigma \neq \nu$, then σ must be a Type II map. ☐

Corollary 8.5. *If $G \cong E \times Q$, then $L_G\mathrm{CAut}(G) \lhd \mathrm{Cay}P(G)$.*

Proof. By Proposition 1.2, $\mathrm{Cay}P(G) = L_G\mathrm{Cay}P_1(G)$. By Corollary 8.4, G admits Type II maps, and $\mathrm{Inn}(G) = \mathrm{CAut}(G)$. By Proposition 8.3, $[\mathrm{Cay}P_1(G) : \mathrm{Inn}(G)] = 2$, and hence every Cayley permutation is one of exactly two types, $\tau = \lambda_g\kappa_h$ or $\tau = \lambda_g\kappa_h\nu$, for some $g, h \in G$. It follows that $[\mathrm{Cay}P(G) : L_GCAut(G)] = 2$, and hence that $L_GCAut(G) \lhd \mathrm{Cay}P(G)$. ☐

Corollary 8.6. *If σ is a Type II map, then $\sigma L_G\sigma = R_G$.*

Proof. Since σ is a Type II map, Proposition 8.3 implies that $\sigma = \kappa\nu$, for some Type I map κ. Proposition 8.3 implies that $\nu\kappa = \kappa\nu$. With the help of Proposition 8.1, the reader may verify that for all $g \in G$, $(\kappa\nu)\lambda_g(\kappa\nu)^{-1} = \nu(\kappa\lambda_g\kappa^{-1})\nu = \nu\lambda_{\kappa(g)}\nu = \rho_{\kappa(g)}$. ☐

It now follows that $\mathrm{Cay}P(G)$ is an internal direct product of its subgroups:

Corollary 8.7. *Let $G \cong E\times Q$. Then $\mathrm{Cay}P(G) = L_GCAut(G) \rtimes \langle\nu\rangle \cong (G \rtimes Z_2^2) \rtimes Z_2$.*

Proof. Apply Proposition 8.1 and Corollaries 8.5 and 8.6. ☐

Lemma 8.8. *Let G be a group and assume σ_1, σ_2 are distinct Type I maps on G. Let $g \in G$ be such that $g^2 \neq 1$. Then g cannot be both σ_1-stable and σ_2-stable.*

Proof. Suppose g is an element of order greater than 2 and that g is both σ_1-stable and σ_2-stable. Then $g \in \mathrm{StCore}(\sigma_1)$. Consider $C_G(g)$. By Corollary 5.3, $\mathrm{StCore}(\sigma_1)$ is abelian, and since $g \in \mathrm{StCore}(\sigma_1)$, $\mathrm{StCore}(\sigma_1) < C_G(g)$.

On the other hand, if $y \in G$ is not an element of $\mathrm{StCore}(\sigma_1)$, then y is σ_1-unstable, so $ygy^{-1} = g^{-1}$, by Proposition 2.3. Since $g \neq g^{-1}$, $y \notin C_G(g)$, so $C_G(g) \neq G$. Since $[G : \mathrm{StCore}(\sigma_1)] = 2$, we conclude that $\mathrm{StCore}(\sigma_1) = C_G(g)$. Symmetrically we may conclude that $\mathrm{StCore}(\sigma_2) = C_G(g)$. It follows that $\mathrm{StCore}(\sigma_1) = \mathrm{StCore}(\sigma_2)$. Therefore an element g^* is σ_1-stable if and only if g^* is σ_2-stable. It follows that $\sigma_1 = \sigma_2$, a contradiction. ☐

Proposition 8.9. *Suppose σ_1 and σ_2 are distinct Type I maps of G. Then G is a non-abelian graphically abelian group, that is $G \cong E \times Q$. In consequence G admits Type II maps.*

Proof. Assume the hypotheses. Let $g \in G$ with $g \neq g^{-1}$, and assume $|g| \neq 4$. It follows from Proposition 2.6 that g may not be unstable; hence g is both σ_1-stable and σ_2-stable. This contradicts Lemma 8.8. We conclude that if g is not an involution then $|g| = 4$. Then for any $g \in G$, $|g|$ divides 4.

Now if $g \in G$ is an involution, then by Proposition 4.3, $g \in Z(G)$. We have now shown that G satisfies two of the defining conditions for a group of restricted type, Definition 4.7. It remains to be seen that G contains a distinguished involution I such that for all elements $g \in G$ with $|g| = 4$, $g^2 = I$.

Since σ_1 is a Type I map, $G = \mathrm{Dic}(A, I)$, where $A = \mathrm{StCore}(\sigma_1)$, and I is a distinguished involution of A; similarly, since σ_2 is also a Type I map, $G = \mathrm{Dic}(B, J)$, where $B = \mathrm{StCore}(\sigma_2)$, and J is a distinguished involution of B. Now $G \setminus B$ contains an element z such that $G = \langle B, z\rangle$ and $z^2 = J$. And A and B are proper subgroups of G. Since the union of two proper subgroups may not equal G, there exists $g_0 \in G$ with $g_0 \notin A \cup B$. Therefore g_0 is both σ_1-unstable and σ_2-unstable; hence $g_0^2 = I$ and $g_0^2 = J$, and thus $I = J$. Now let g be any element of order 4. Then by Lemma 8.8, g must be *either* σ_1-unstable *or* σ_2-unstable. In the first case, by Proposition 6.2, $g^2 = I$, and in the second $g^2 = J = I$. This establishes that G is of restricted type. By Proposition 4.10, $G \cong E \times Q$. By Corollary 8.4, G admits Type II maps. ☐

Corollary 8.10. *If $G = \mathrm{Dic}(A, I)$, and G is not graphically abelian, then the structure of G as a generalized dicyclic group is unique; moreover $\mathrm{Cay}P_1(G) \cong Z_2$.*

Proof. Let σ be a Type I map. Since G is not graphically abelian, G does not admit any Type II maps. By Proposition 8.9, σ is unique. By Corollary 1.8, $\sigma^2 = 1$. ☐

Corollary 8.11. *Let G be an irregular group and suppose G does not admit Type II maps. Then $\mathrm{Cay}P(G) \cong L_G \rtimes \mathrm{Cay}P_1(G) \cong G \rtimes Z_2$.*

Proof. By Proposition 1.2, $\mathrm{Cay}P(G) = L_G \mathrm{Cay}P_1(G)$. Clearly $L_G \cap \mathrm{Cay}P_1(G) = \langle 1\rangle$. By Proposition 6.6, every Type I map is a group automorphism. Proposition 8.1 implies $L_G \lhd L_G \mathrm{Cay}P_1(G)$. By Proposition 8.10, $\mathrm{Cay}P_1(G) \cong Z_2$, and clearly $L_G \cong G$. ☐

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