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2-domination in bipartite graphs with odd independence number

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Abstract

For a positive integer k, a set of vertices S in a graph G is said to be a k-dominating set if each vertex x in $V(G) - S$ has at least k neighbors in S. The cardinality of a smallest k-dominating set of G is called the k-domination number of G and is denoted by $\gamma_k(G)$. The independence number of a graph G is denoted by $\alpha(G)$. In [*Australas. J. Combin.* 40 (2008), 265–268], Fujisawa, Hansberg, Kubo, Saito, Sugita and Volkmann proved that a connected bipartite graph G satisfies $\gamma_2(G) \leq \left\lfloor \frac{3\alpha(G)}{2} \right\rfloor$. They also characterized the bipartite graphs G with $\gamma_2(G) = \frac{3\alpha(G)}{2}$ and therefore $\alpha(G)$ even. In this note, we give a characterization of the bipartite graphs G with $\alpha(G)$ odd satisfying $\gamma_2(G) = \frac{3\alpha(G)-1}{2}$.

1 Introduction

Let G be a simple graph with vertex set $V(G)$. The *order* of G is $|G| := |V(G)|$. A vertex of degree one is called a *leaf*. The set of leaves of G is denoted by $L(G)$. If x is a vertex of G, then $N_G(x)$ is the set of vertices adjacent to x and $N_G[x] = N_G(x) \cup \{x\}$.

More generally, we define $N_G(X) = \bigcup_{x\in X} N_G(x)$ and $N_G[X] = N_G(X) \cup X$ for a subset X of $V(G)$.

For a positive integer k, a set of vertices S in a graph G is said to be a *k-dominating set* if each vertex of G not contained in S has at least k neighbors in S. The cardinality of a smallest k-dominating set of G is called the *k-domination number*, and it is denoted by $\gamma_k(G)$. By definition, a dominating set coincides with a 1-dominating set, and $\gamma_1(G)$ is the domination number $\gamma(G)$ of G.

A subset $I \subseteq V(G)$ of the vertex set of a graph G is called *independent* if the subgraph induced by I is edgeless. The number $\alpha(G)$ represents the cardinality of a maximum independent set of G.

For each vertex x in a graph G, we introduce a new vertex x' and join x and x' by an edge. The resulting graph is called the *corona* of G. A graph G is said to be a *corona graph* if it is the corona of some graph J and it is denoted by $K_1 \circ J$. If G is the corona graph of a graph J, then, for each vertex $x \in V(J)$, $l_G(x)$ represents the leaf of G whose support vertex is x.

For graph-theoretic notation not explained in this paper, we refer the reader to [2].

A well-known upper bound for the domination number of a graph was given by Ore in 1962.

Theorem 1.1 ([7]) *If G is a graph with no isolated vertices, then $\gamma(G) \le |G|/2$.*

In 1982, Payan and Xuong, and independently in 1985, Fink, Jacobson, Kinch and Roberts, characterized the graphs achieving equality in Ore's bound.

Theorem 1.2 ([8], [3]) *Let G be a connected graph. Then $\gamma(G) = |G|/2$ if and only if G is the corona graph of a connected graph J or G is isomorphic to the cycle C_4.*

In 1998, Randerath and Volkmann [9], and independently in 2000, Xu, Cockayne, Haynes, Hedetniemi and Zhou [10], characterized the odd order graphs G for which $\gamma(G) = \lfloor n(G)/2 \rfloor$. In the next theorem, we note just the part of this characterization which we will use in the next section.

Theorem 1.3 ([9], [10]) *Let G be a nontrivial connected bipartite graph of odd order. Then $\gamma(G) = \lfloor |G|/2 \rfloor$ if and only if one of the following holds:*

(i) G consists of two cycles with a common vertex;

(i) G is isomorphic to the complete graph $K_{2,3}$:

(iii) $|N_G(L(G))| = |L(G)| - 1$ and $G - N_G[L(G)] = \emptyset$;

(iv) $|N_G(L(G))| = |L(G)|$ and $G - N_G[L(G)]$ is an isolated vertex;

(v) $|N_G(L(G))| = |L(G)|$ *and* $G - N_G[L(G)]$ *is a star of order three such that the center of the star has degree two in* G*;*

(vi) $|N_G(L(G))| = |L(G)|$ *and* $G - N_G[L(G)]$ *is a bipartite graph* G_1 *with* $|G_1| = 5$, $\gamma(G_1) - \delta(G_1) = 2$, *and the graph* G_1', *induced by the vertices of* G_1, *which are not adjacent to a vertex of* $N(L(G), G)$, *is a* C_4*;*

(vii) $|N_G(L(G))| = |L(G)|$ *and* $G - N_G[L(G)]$ *is a bipartite graph* H_1 *with one leaf* u, *which is also a cut vertex of* G, *and* $H_1 - u = C_4$.

In [1], Blidia, Chellali and Favaron studied the relationship between the 2-domination number and the independence number of a tree. In particular, they proved that the ratio $\gamma_2(T)/\alpha(T)$ for a tree T is contained in a small interval.

Theorem 1.4 ([1]) *For any tree,* $\alpha(T) \leq \gamma_2(T) \leq \frac{3\alpha(T)}{2}$.

They also proved that both the upper and lower bounds are sharp. As a generalization of the second inequality in Theorem 1.4, we recently proved the next result.

Theorem 1.5 ([4]) *If* G *is a connected bipartite graph of order at least* 3, *then*

$$\gamma_2(G) \leq \frac{|G| + |L(G)|}{2} \leq \frac{3\alpha(G)}{2}.$$

Furthermore, $\gamma_2(G) = \frac{3\alpha(G)}{2}$ *if and only if* G *is the corona of the corona of a connected bipartite graph or* G *is the corona of the cycle* C_4.

Corollary 1.6 ([4]) *If* T *is a tree of order at least* 3, *then* $\gamma_2(G) \leq \frac{3}{2}\alpha(G)$ *with equality if and only if* T *is the corona of the corona of a tree.*

If $\gamma_2(G) = \frac{3\alpha(G)}{2}$, then $\alpha(G)$ is even. In this note, we present a characterization of the bipartite graphs G with $\alpha(G)$ odd and $\gamma_2(G) = \frac{3\alpha(G)-1}{2}$.

2 Characterization of bipartite graphs with $\gamma_2(G) = \left\lfloor \frac{3\alpha(G)}{2} \right\rfloor$

Theorem 2.1 *Let* G *be a connected bipartite graph of order at least* 3 *such that* $\alpha = \alpha(G)$ *is odd. Then* $\gamma_2(G) = \frac{3\alpha(G)-1}{2}$ *if and only if*

(a) $G \cong K_1 \circ (K_1 \circ J) + \{x, xy\}$, *where* $y \in V(J)$ *and* x *is a new vertex.*

(b) $G \cong K_1 \circ H$, *where* H *is a member of the family described in Theorem 1.3.*

(c) $G \cong K_1 \circ H - \{a, b\}$, *where* a *and* b *are leaves of* $K_1 \circ H$ *with adjacent support vertices* u *and* v *such that* $d_H(u)$, $d_H(v) \geq 2$ *and one of the following holds:*

(i) $H \cong K_1 \circ J$, *where* J *is a connected bipartite graph with* $u, v \in V(J)$,

(ii) $H \cong C_4$,

(iii) $H \cong (K_1 \circ J) + \{uv\}$, *where* J *is a bipartite graph and* $u, v \in L(K_1 \circ J)$,

(iv) $H \cong (K_1 \circ J) + \{x, y, uv, xu', yv', xy\}$, *where* J *is a bipartite graph* $u, v \in L(K_1 \circ J)$, $l_{K_1 \circ J}(u') = u$, $l_{K_1 \circ J}(v') = v$ *and* x *and* y *are new vertices,*

(v) $H \cong K_1 \circ J - \{l_{K_1 \circ J}(u), l_{K_1 \circ J}(v)\}$, *where* J *is a bipartite graph with* $u, v \in V(J)$ *and where* $d_H(u) = 2$,

(vi) $H \cong (K_1 \circ J) + \{l_{K_1 \circ J}(u)x\}$, *where* J *is a connected bipartite graph with* $u, v \in V(J)$ *and* x *is a vertex in* $L(K_1 \circ J) \cap N_{K_1 \circ J}(N_{K_1 \circ J}(u) - \{v\})$,

(vii) $H \cong (K_1 \circ J) + \{l_{K_1 \circ J}(u)x, l_{K_1 \circ J}(v)y\}$, *where* J *is a connected bipartite graph with* $u, v \in V(J)$ *and* x *is a vertex in* $L(K_1 \circ J) \cap N_{K_1 \circ J}(N_{K_1 \circ J}(u) - \{v\})$ *and* y *a vertex in* $L(K_1 \circ J) \cap N_{K_1 \circ J}(N_{K_1 \circ J}(v) - \{u\})$.

Proof. Let $L = L(G)$. According to Theorem 1.5, we have

$$\gamma_2(G) \leq \frac{|G| + |L|}{2} \leq \frac{3\alpha}{2}. \tag{1}$$

Since G is a bipartite graph of order at least 3, we observe that $|G| \leq 2\alpha$ and $|L| \leq \alpha$. Combining this with (1), the hypothesis $\gamma_2(G) = (3\alpha - 1)/2$ implies that (a) $|G| = 2\alpha - 1$ and $|L| = \alpha$, (b) $|G| = 2\alpha$ and $|L| = \alpha - 1$, or (c) $|G| = 2\alpha$ and $|L| = \alpha$.

(a) Assume that $|G| = 2\alpha - 1$ and $|L| = \alpha$. If $\gamma_2(G) = (3\alpha - 1)/2$, then $\gamma_2(G) = (3|G| + 1)/4$ and thus $|G| = 4q + 1$ and $\gamma_2(G) = 3q + 1$ for an integer $q \geq 1$. Since $|L| = \alpha$ and $|G| = 2\alpha - 1$, it follows that each vertex $x \in V(G) - L$ is adjacent to at least one leaf, and exactly one vertex of $V(G) - L$ is adjacent to two leaves of G. If $H = G - L$, then H is a connected bipartite graph of order $2q$. If D is a $\gamma(H)$-set, then $D \cup L$ is a 2-dominating set of G. Therefore Theorem 1.1 implies that

$$3q + 1 = \gamma_2(G) \leq |L| + |D| \leq |L| + \frac{|H|}{2} = |L| + \frac{|G - L|}{2} = 3q + 1$$

and so $\gamma(H) = |D| = |H|/2$. In view of Theorem 1.2, the graph H is a corona graph of a connected bipartite graph or H is isomorphic to the cycle C_4 of length four.

If $H = C_4$, then G does not have the desired properties. Now let H be a corona graph with $L(H) = \{u_1, u_2, \ldots, u_q\}$ and $V(H) - L(H) = \{v_1, v_2, \ldots, v_q\}$ such that u_i is adjacent to v_i for $1 \leq i \leq q$. If, say, u_q is adjacent to two leaves of G, then we arrive at the contradiction

$$3q + 1 = \gamma_2(G) \leq |L| + |\{v_1, v_2, \ldots, v_{q-1}\}| = 3q.$$

In the remaining case that v_i is adjacent to two leaves of G, we obtain the desired result $\gamma_2(G) = 3q + 1$ and G has the form of (a).

(b) Assume that $|G| = 2\alpha$ and $|L| = \alpha$. If $\gamma_2(G) = (3\alpha - 1)/2$, then $\gamma_2(G) = (3|G| - 2)/4$ and thus $|G| = 4q + 2$ and $\gamma_2(G) = 3q + 1$ for an integer $q \geq 1$. Since $|L| = \alpha$ and $|G| = 2\alpha$, it follows that each vertex $x \in V(G) - L$ is adjacent to exactly one leaf of G, and hence G is a corona graph of a connected bipartite graph H of order $|H| = 2q + 1$. If D is a $\gamma(H)$-set, then $D \cup L$ is a 2-dominating set of G. Therefore Theorem 1.1 implies that

$$3q + 1 = \gamma_2(G) \leq |L| + |D| \leq |L| \leq \left\lfloor \alpha + \frac{|H|}{2} \right\rfloor = 3q + 1$$

and so $\gamma(H) = |D| = (|H| - 1)/2$. In view of Theorem 1.3, the graph H is a member of the family described in Theorem 1.3 (i)–(vii). Conversely, if H is a member of the family described in Theorem 1.3 (i)–(vii), then it is straightforward to verify that G has the desired properties.

(c) Assume that $|G| = 2\alpha$ and $|L| = \alpha - 1$. If $\gamma_2(G) = 3(\alpha - 1)/2$, then $\gamma_2(G) = (3|G| - 2)/4$ and thus $|G| = 4q + 2$, $\gamma_2(G) = 3q + 1$, $\alpha = 2q + 1$ and $|L| = 2q$ for an integer $q \geq 1$.

First we show that no vertex of $H = G - L$ is adjacent to two or more leaves of G. Suppose to the contrary that $u \in V(G) - L$ is adjacent to $r \geq 2$ leaves. If $R \subset V(G) - L$ is the set of vertices not adjacent to any leaf, then $|L| = \alpha - 1 = 2q$ implies that $|R| \geq 3$. Thus $\alpha = 2q + 1$ implies that $G[R]$ is a complete graph, a contradiction to the hypothesis that G is a bipartite graph.

Now let $u, v \in V(G) - L$ be precisely the two vertices which are not adjacent to a leaf of G. Since $\alpha = |L| + 1$, we observe that u and v are adjacent and $d_H(u), d_H(v) \geq 2$. Since H is a connected bipartite graph of order $|H| = 2q + 2$, Theorem 1.1 implies that $\gamma(H) \leq q + 1$. If $\gamma(H) \leq q - 1$, then we easily obtain the contradiction

$$3q + 1 = \gamma_2(G) \leq |L| + q = 3q.$$

Assume that $\gamma(H) = q + 1 = |H|/2$. According to Theorem 1.2, the graph H is a corona graph of a connected bipartite graph J or H is isomorphic to the cycle C_4 of length four. Since $d_H(u), d_H(v) \geq 2$, if $H \cong K_1 \circ J$, we deduce that $u, v \in V(J)$. Hence G is of the form of (c)(i) or (c)(ii). Conversely, if H is as in (c)(i) or (c)(ii), then G has the desired properties.

Finally, assume that $\gamma(H) = q = (|H| - 2)/2$. Let $\hat{H} = H - N_H[\{u, v\}]$, let I be the set of isolated vertices in $\hat{H}$ and let $Q = \hat{H} - I$. Define $I_u = I \cap N_H(N_H(u))$ and $I_v = I \cap N_H(N_H(v))$, and let D be a minimum dominating set of the graph Q. Since G is bipartite and $uv \in E(G)$, it is clear that $I_u \cap I_v = \emptyset$. Since H is connected, each component of $\hat{H}$ has vertices adjacent to some vertex in $N = N(\{u, v\}) - \{u, v\}$; in particular, the vertices from I all have at least one neighbor in N. Now we distinguish three cases.

Case 1. Assume that $I = \emptyset$. Then $\hat{H} = Q$ and $L \cup D \cup \{u, v\}$ is a 2-dominating set of G and thus, with Theorem 1.1, we obtain

$$3q+1=\gamma_2(G)\le |L|+|D|+2\le 2q+\frac{|Q|}{2}+2=2q+\frac{|H|}{2}=3q+1,$$

which implies that $\gamma(Q)=|Q|/2$ and $|N_H[\{u,v\}]|=4$. Hence, since Q has no isolated vertices, according to Theorems 1.1 and 1.2, each component of Q is a corona graph or a cycle of length four. Let $\{u'\}=N_H(v)-\{v\}$ and $\{v'\}=N_H(v)-\{u\}$. Suppose that there is a component C of Q which is a C_4, say $C=x_1x_2x_3x_4x_1$. Since G is connected, one of the vertices x_i has a neighbor in $\{u',v'\}$. Without loss of generality, suppose that $x_1v'\in E(G)$. Now, if D' is a minimum dominating set of $Q-C$, then $L\cup D'\cup\{u,v',x_3\}$ is a 2-dominating set of G with at most $2q+|Q-C|/2+3=3q$ vertices, a contradiction. Therefore, every component of Q is a corona graph, that is, $Q\cong K_1\circ J'$ for a bipartite graph J'. Now we will determine which vertices of Q can be adjacent to u' or to v'. If u' and v' only have neighbors in $V(J')$, then G is of the form of (c)(iii) with $J=J'$. Thus, suppose first that u' (v') is neighbor of a leaf z of a component C of Q with $|C|\ge 4$. If z' is the support vertex of z in Q, then $L\cup(V(J')-\{z'\})\cup\{u',v\}$ ($L\cup(V(J')-\{z'\})\cup\{v',u\}$) is a 2-dominating set of G with $3q$ vertices, which is a contradiction. Suppose now that there are two trivial components C_1 and C_2 of J' with $V(C_i)=\{x_i\}$ for $i=1,2$ and such that u' is a neighbor of x_1 and x_2 and v' is a neighbor of $l_Q(x_1)$ and of $l_Q(x_2)$ in G. Then the set $L\cup(V(J')-\{x_1,x_2\})\cup\{u,u',v'\}$ is a 2-dominating set of G with $3q$ vertices, which is not possible. Hence there is at most one trivial component C of J' such that, if $V(C)=\{x\}$, then u' is a neighbor of x and v' is a neighbor of $y=l_Q(x)$. In this case we find that G has the structure as in (c)(iv) with $J=H[V(J')\cup\{u',v'\}]$.

Case 2. Assume that $I\ne\emptyset$.

Subcase 2.1. Suppose that $|N|<|I|$. Then $L\cup N\cup\{v\}\cup D$ is a 2-dominating set of G and thus

$$\begin{aligned}3q+1=\gamma_2(G) &\le |L|+|N\cup\{v\}|+|D|\\ &< 2q+\frac{|N\cup\{u,v\}\cup I|}{2}+\frac{|Q|}{2}\\ &= 2q+\frac{|H|}{2}=3q+1,\end{aligned}$$

which is a contradiction.

Subcase 2.2. Suppose that $|N|=|I|$. Assume first that both $d_G(u)$ and $d_G(v)$ are at least 3. Then $L\cup N\cup D$ is a 2-dominating set of G with at most

$$2q+\frac{|N\cup I|}{2}+\frac{|Q|}{2}=2q+\frac{|H|-2}{2}=3q$$

vertices, which contradicts the hypothesis taken for this case. Thus, assume, without loss of generality, that $d_G(u)=2$. Now the set $L\cup N\cup\{v\}\cup D$ is a 2-dominating set of G and thus

$$\begin{aligned}3q+1=\gamma_2(G) &\leq |L|+|N\cup\{v\}|+|D| \\ &\leq 2q+\frac{|N\cup\{u,v\}\cup I|}{2}+\frac{|Q|}{2} \\ &= 2q+\frac{|H|}{2}=3q+1,\end{aligned}$$

which implies that $\gamma(Q) = |Q|/2$. Again, the components of Q have to be either corona graphs or cycles of length 4. As in Case 1, the possibilities that a component of Q is a cycle of length 4 and that a vertex from N is adjacent to a leaf of a corona component C of Q with $|C| \geq 4$ can be eliminated analogously. Hence, we can regard Q as the corona of a (not necessarily connected) bipartite graph J'. Now suppose that there is a component C of Q with $V(C) = \{x, y\}$ and that there are vertices $u' \in N_G(u) - \{v\} = N \cap N_G(u)$ and $v' \in N_G(v) - \{u\} = N \cap N_G(v)$ such that u' is adjacent to x and v' is adjacent to y. Then the set $L \cup N \cup (V(J) - \{x\})$ is a 2-dominating set of G with $3q$ vertices and we have a contradiction. Thus we can say, without loss of generality, that the vertices of N only have neighbors from $V(J') \cup I$ and thus, if $J = J'+I$, then H is the corona of the graph J without the leaves whose support vertices are u, and v, i.e. H is as in (c)(v).

Subcase 2.3. Suppose that $|N| = |I| + 1$. Then there is a vertex $x \in I$ such that $|N(x) \cap N| \geq 2$. If y is a vertex from $N(x) \cap N$, then $L \cup (N - \{y\}) \cup \{u\} \cup D$ is a 2-dominating set of G and thus

$$\begin{aligned}3q+1=\gamma_2(G) &\leq |L|+|N-\{y\}|+1+|D| \\ &\leq 2q+\frac{|N\cup I|-1}{2}+1+\frac{|Q|}{2} \\ &= 2q+\frac{|H|-1}{2}=3q+\frac{1}{2},\end{aligned}$$

which implies that this case is not possible.

Subcase 2.4. Suppose that $|N| = |I| + 2$. Assume first that $|N - N_H(I)| = 2$. Then we have $|N_H(I)| = |I|$. If there were vertices $u' \in N_G(u)$ and $v' \in N_G(v)$ such that $N = N_H(I)\cup\{u', v'\}$, then $N_H(I)\cup\{u, v\}\cup D$ would be a dominating set of H with at most $\frac{|H|}{2}$ vertices, a contradiction to the assumption that $\gamma(H) = q$. Hence we may assume that $N - N_H(I) \subseteq N_G(u)$ and thus $L \cup N_H(I) \cup \{u\} \cup D$ is a 2-dominating set of G and therefore we obtain the following contradiction:

$$\begin{aligned}3q+1=\gamma_2(G) &\leq |L|+|N_H(I)|+1+|D| \\ &\leq 2q+\frac{|N_H(I)\cup I|}{2}+1+\frac{|Q|}{2} \\ &= 2q+\frac{|H|-2}{2}=3q.\end{aligned}$$

It follows that $|N - N_H(I)| \leq 1$. Let S be a subset of $N_H(I)$ with $|S| = |I|$ such that every vertex in I has a neighbor in S. Then $L \cup S \cup \{u, v\} \cup D$ is a 2-dominating set of G and we obtain

$$\begin{aligned} 3q+1=\gamma_2(G) &\leq |L|+|S|+2+|D| \\ &\leq 2q+\frac{|S\cup I|}{2}+2+\frac{|Q|}{2} \\ &= 2q+\frac{|H|}{2}=3q+1. \end{aligned}$$

Therefore, we again have that $\gamma(Q)=|Q|/2$ and thus the components of Q are either corona graphs or cycles of length 4. Similarly as in the former cases, we obtain contradictions for the cases in which a component of Q is a cycle of length 4 and where a vertex from $N_G(\{u,v\})-\{u,v\}$ is adjacent to a leaf of a corona component C of Q with $|C|\geq 4$. Also, as in Case 2, it is not possible that two vertices from $N_G(\{u,v\})-\{u,v\}$ are each adjacent to one of the vertices of a component C of Q with $|C|=2$. With similar arguments as before, and using the fact that G does not contain cycles of odd length, it is straightforward to verify that there can only be added either an edge joining u' and a vertex in $N_G(u)-\{v\}$, or else an edge joining v' and a vertex in $N_G(v)-\{u\}$, or both. It follows that H is the corona of a graph $J=J'$ together with one or two of the edges mentioned here. These are exactly the graphs described in (c)(vi) and (c)(vii).

Subcase 2.5. Suppose that $|N|>|I|+2$. Let S be a subset of $N_H(I)$ with $|S|=|I|$ and such that every vertex from I has a neighbor in S. Then $L\cup S\cup\{u,v\}\cup D$ is a 2-dominating set of G and, since $|N-S|\geq 3$, we obtain the contradiction

$$\begin{aligned} 3q+1=\gamma_2(G) &\leq |L|+|S|+2+|D| \\ &\leq 2q+\frac{|N\cup I|-3}{2}+2+\frac{|Q|}{2} \\ &= 2q+\frac{|H|-1}{2}=3q+\frac{1}{2}. \end{aligned}$$

Hence this case cannot occur.

Conversely, if G has structure as in (c)(i)–(vii), it is straightforward to verify that $\gamma_2(G)=\frac{3\alpha(G)-1}{2}$. □

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