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# 5-cycle decompositions from paired 3- and 4-cycle decompositions

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**Abstract**

Let $(V, T)$ be a 3-fold triple system and $(V, C)$ a 4-fold 4-cycle system on the same set $V$. This choice of indices 3 and 4 ensures that each system contains the same number of cycles: $|T| = |C|$. We pair up the cycles, $\{t, c\}$, where $t \in T$ and $c \in C$, in such a way that $t$ and $c$ share one edge. If $t = (x, y, z)$ and $c = (x, y, u, v)$, so $t$ and $c$ share the edge $\{x, y\}$, then we retain the 5-cycle $(z, x, v, u, y)$ and remove the repeated edge $\{x, y\}$.

Doing this for all the pairs $\{t, c\}$, we rearrange all the shared edges, common to $t$ and $c$, into further 5-cycles, so that the result is a 7-fold 5-cycle system on $V$. The necessary conditions are that the order $|V|$ is 1 or 5 (mod 10); these conditions are shown to be sufficient for such a "metamorphosis" from pairs of 3- and 4-cycles into 5-cycles.

## 1 Introduction

Various papers on metamorphosis of designs have appeared in recent years. In [2], Gionfriddo and Lindner take a twofold triple system, pair the triples so that each pair is on four vertices with a repeated edge, remove the repeated edges while retaining the remaining 4-cycle, and rearranging the removed double edges into further copies of 4-cycles. Thus a type of metamorphosis from a twofold triple system into a twofold 4-cycle system is obtained. See Figure 1.

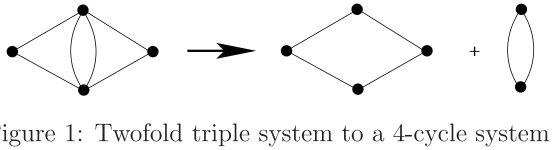

Figure 1: Twofold triple system to a 4-cycle system.

In [3], Yazıcı takes a twofold 4-cycle system, pairs the 4-cycles so that each pair shares a common edge, and then removes this double edge. The resulting 6-cycle is retained, and the removed double edges are rearranged into further 6-cycles so that the result is a twofold 6-cycle system. See Figure 2.

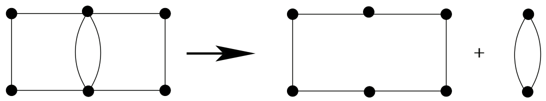


Figure 2: Twofold 4-cycle system to a 6-cycle system.

In this paper, we want to combine 3-cycles and 4-cycles. A $\lambda_1$-fold triple system of order $n$ contains $\lambda_1 n(n-1)/6$ triples, and a $\lambda_2$-fold 4-cycle system of order $n$ contains $\lambda_2 n(n-1)/8$ cycles. In order for these numbers to be equal, so that we can pair triples and 4-cycles having a common edge, we require $4\lambda_1 = 3\lambda_2$. So we take a 3-fold triple system and a 4-fold 4-cycle system; such systems of order $n$ both contain $n(n-1)/2$ cycles. See Figure 3.

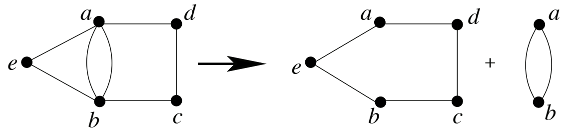


Figure 3: 3-cycle and a 4-cycle into a 5-cycle and a double edge.

In order to use up all the removed double edges in further 5-cycles, we need the total number of 3-cycles and 4-cycles to be a multiple of 5. Consequently, we require the order $n$ to satisfy $n(n-1)/2 \equiv 0 \pmod 5$. Moreover, for a 3-fold triple system to exist, we need the order $n$ to be odd, and so we require $n \equiv 1$ or $5 \pmod{10}$.

So our aim in the rest of this note is to construct, from a 3-fold triple system of order $n$ and a 4-fold 4-cycle system of order $n$ on the same vertex set, a 7-fold 5-cycle system of order $n$, by pairing triples and 4-cycles as shown in Figure 3, removing the double edge, retaining the 5-cycle so formed, and then rearranging the double edges removed from such pairs of 3- and 4-cycles into further 5-cycles.

## 2 Constructions

If we take the triple $(a, b, e)$ and the 4-cycle $(a, b, c, d)$, then we shall use the notation $(e, b, a; a, b, c, d)$ or else $(e, a, b; b, a, d, c)$ to denote these paired cycles. Then the 5-cycle obtained by removing the double edge $\{a, b\}$ is clearly seen to be $(e, b, c, d, a)$ or $(e, a, d, c, b)$. (See Figure 3.)

We begin with some necessary examples.

**Example 2.1 Order 5:** With vertex set $\{1,2,3,4,5\}$, we take the ten configurations:

$(4,1,2;2,1,3,5)$, $(5,1,3;3,1,4,2)$, $(1,2,3;3,2,5,4)$, $(4,3,2;2,3,5,1)$,
$(1,3,4;4,3,5,2)$, $(3,2,5;5,2,4,1)$, $(2,4,5;5,4,1,3)$, $(3,4,5;5,4,2,1)$,
$(2,5,1;1,5,4,3)$, $(5,4,1;1,4,3,2)$.

The double edges, when removed, form four more 5-cycles: $(1,2,3,4,5)$ and $(1,3,2,5,4)$, each twice. □

**Example 2.2 Order 11:** We take the vertex set $\mathbb{Z}_{11}$, and the following five starter configurations modulo 11:

$(2,0,1;1,0,9,7)$, $(8,1,4;4,1,9,3)$, $(7,4,2;2,4,0,3)$, $(4,2,7;7,2,5,1)$,
$(1,7,0;0,7,6,2)$.

The double edges form 22 more 5-cycles: $(0,1,4,2,7)$ mod 11, each taken twice. □

**Example 2.3 A configuration on $K_{5,5,5}$:** We take the vertex set

$$\{0,3,6,9,12\} \cup \{1,4,7,10,13\} \cup \{2,5,8,11,14\}.$$

The following five starter configurations (mod 15) provide a decomposition, and the double edges form 30 more 5-cycles: $(0,1,3,7,14)$ mod 15, each twice.

$(5,0,1;1,0,4,2)$, $(8,1,3;3,1,0,5)$, $(5,3,7;7,3,13,2)$,
$(0,7,14;14,7,2,6)$, $(10,14,0;0,14,6,13)$.

□

**Example 2.4 Order 21:** We have ten starter configurations mod 21, and use the vertex set $\mathbb{Z}_{21}$. The double edges from the following ten starters yield 84 further 5-cycles from each of the following two starters (taken twice): $(0,1,4,10,12)$, $(4,9,0,11,10)$.

$(10,0,1;1,0,11,3)$, $(7,1,4;4,1,17,8)$, $(8,4,10;10,4,18,17)$,
$(17,10,12;12,10,0,15)$, $(8,0,12;12,0,9,2)$, $(14,4,9;9,4,0,17)$,
$(8,0,9;9,0,7,6)$, $(18,0,11;11,0,13,8)$, $(3,10,11;11,10,15,17)$,
$(6,4,10;10,4,12,14)$.

□

**Example 2.5 Order 25:**

Let the vertex set of $K_{25}$ be $\{(i,j) \mid i,j \in \mathbb{Z}_5\}$. On each of the sets $\{(i,j) \mid i \in \mathbb{Z}_5\}$, for each $j \in \mathbb{Z}_5$, we place a copy of the decomposition of order 5 given in Example 2.1. We then take a transversal design with block size 5 and groups of size 5, the groups being $\{(i,j) \mid i \in \mathbb{Z}_5\}$, for each $j \in \mathbb{Z}_5$. Then replace each block of the transversal design by a further copy of Example 2.1. The result is a suitable decomposition of $K_{25}$. □

**Example 2.6 Order 55:**

We take the vertex set $X = \{(i,j) \mid (i,j) \in (\mathbb{Z}_{11}, \mathbb{Z}_5)\}$. On each of the five sets $Y_j = \{(i,j) \mid i \in \mathbb{Z}_{11}\}$, for each value of $j$ in $\mathbb{Z}_5$, we take a copy of a configuration of order 11 (see Example 2.2).

Then we use a transversal design with five groups of size 11 on the set $X$; the groups are of course the five sets $Y_j$, $j \in \mathbb{Z}_5$. (This transversal design exists since there exist three MOLS(11); see [1], Section III.3). On each block of the transversal design we place a copy of a configuration of order 5 (Example 2.1). This gives a suitable design of order 55. □

Subsequently, when we use a group divisible design in our constructions, the existence of such a GDD can be verified in [1], Section IV.4.

**The construction, order 1 (mod 10):**

For order $10x + 1$, we take the vertex set $(\mathbb{Z}_{2x} \times \mathbb{Z}_5) \cup \{\infty\}$.

When $x \equiv 0$ or 1 (mod 3) and $x \geqslant 3$, there exists a 3-GDD of type $2^x$. We take such a 3-GDD on the set $\mathbb{Z}_{2x}$. Then for each group $\{a, b\}$ in the GDD, we place a copy of Example 2.2 (of order 11) on the set $\{(a,j),(b,j) \mid j \in \mathbb{Z}_5\} \cup \{\infty\}$. Next, for each block $\{p, q, r\}$ in the GDD, we place a copy of Example 2.3 on $K_{5,5,5}$ with the vertex set $\{(p,j) \mid j \in \mathbb{Z}_5\} \cup \{(q,j) \mid j \in \mathbb{Z}_5\} \cup \{(r,j) \mid j \in \mathbb{Z}_5\}$.

When $x \equiv 2$ (mod 3), we use a 3-GDD of type $4^1 2^{x-2}$ on $\mathbb{Z}_{2x}$, which exists for $x \geqslant 5$. Then if $\{a, b, c, d\}$ is the group of size 4, place a copy of Example 2.4 on the vertex set $\{(i,j) \mid i \in \{a,b,c,d\},\ j \in \mathbb{Z}_5\} \cup \{\infty\}$. Also for each group $\{u, v\}$ of size 2, place a copy of Example 2.2 (order 11) on the vertex set $\{(u,j),(v,j) \mid j \in \mathbb{Z}_5\} \cup \{\infty\}$. Finally, for each block $\{p, q, r\}$ in the GDD, place a copy of Example 2.3 on $K_{5,5,5}$ with vertex set $\{(p,j) \mid j \in \mathbb{Z}_5\} \cup \{(q,j) \mid j \in \mathbb{Z}_5\} \cup \{(r,j) \mid j \in \mathbb{Z}_5\}$.

The case of order 21 (when $x = 2$) appears in Example 2.4.

This completes the construction for order 1 modulo 10.

**The construction, order 5 (mod 10)**:

With order $10x + 5$, we take the vertex set $\mathbb{Z}_{2x+1} \times \mathbb{Z}_5$.

If $2x+1 \equiv 1$ or 3 (mod 6), we take a Steiner triple system of order $2x+1$ on $\mathbb{Z}_{2x+1}$; then on each set $\{(i,j) \mid j \in \mathbb{Z}_5\}$, for each $i$ in $\mathbb{Z}_{2x+1}$, we place a copy of the design of order 5 in Example 2.1, and for each block $\{p, q, r\}$ in the Steiner triple system of order $2x + 1$, we place a copy of the decomposition of $K_{5,5,5}$ given in Example 2.3 on the vertex set $\{(p,j) \mid j \in \mathbb{Z}_5\} \cup \{(q,j) \mid j \in \mathbb{Z}_5\} \cup \{(r,j) \mid j \in \mathbb{Z}_5\}$.

If $2x + 1 \equiv 5$ (mod 6), with the same vertex set $\mathbb{Z}_{2x+1} \times \mathbb{Z}_5$, we use a 3-GDD of type $5^1 3^{2u}$ where here $u = (x - 2)/3$, and $2x + 1 \geqslant 17$. If the group of size 5 is $\{a, b, c, d, e\}$, then we place a copy of Example 2.5 on the vertex set $\{(i,j) \mid i \in \{a,b,c,d,e\},\ j \in \mathbb{Z}_5\}$. For each group of size 3, say $\{p, q, r\}$, we place a copy of a system of order 15 (when $x = 1$ above), on the set $\{(p,j),(q,j),(r,j) \mid j \in \mathbb{Z}_5\}$. Then for each block, say $\{d, e, f\}$, of size 3 in the 3-GDD, we place a copy of Example 2.3 on the vertex set $\{(d,j) \mid j \in \mathbb{Z}_5\} \cup \{(e,j) \mid j \in \mathbb{Z}_5\} \cup \{(f,j) \mid j \in \mathbb{Z}_5\}$.

The case $2x+1=5$, or order 25, appears in Example 2.5, while the case $2x+1=11$, or order 55, appears in Example 2.6.

This completes the order 5 (mod 10) case.

## 3 Concluding comments

We now have the following result.

**Theorem 3.1** *Let $V$ be a vertex set of order 1 or 5 (mod 10). Then there exists a three-fold triple system $(V,T)$ and a four-fold 4-cycle system $(V,C)$ for which it is possible to pair up all the 3-cycles in $T$ and 4-cycles in $C$ as $\{\{t,c\} \mid t \in T,\ c \in C\}$, so that*

*(a) each pair $\{t,c\}$, $t \in T$, $c \in C$, shares one common edge;*

*(b) removal of each of these common edges results in a 5-cycle;*

*(c) all these common edges can be rearranged into further 5-cycles.*

*The resulting collection of 5-cycles forms a seven-fold 5-cycle system, $(V,F)$.*

We remark that if we were to take a twofold decomposition of $K_n$ into copies of the left-hand graph in Figure 3 (instead of taking decompositions of $\lambda K_n$, for different $\lambda$, into triples, and into 4-cycles, so that the number of triples and number of 4-cycles are equal for pairing up), then the necessary conditions include $n(n-1) \equiv 0$ (mod 7) and $n$ odd (to ensure even degree, since the configuration is even). Then in order to ensure that the number of double edges is a multiple of 5, since these are to be formed into 5-cycles, we also need $n(n-1) \equiv 0$ (mod 5). Consequently, the necessary requirements are that the order should be $n \equiv 1, 15, 21, 35$ (mod 70). This is a harder problem, and remains to be solved.

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## References

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