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Latin squares with prescriptions and restrictions

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Abstract

We investigate the question of when it is possible to produce an $n \times n$ Latin square that abides by two types of specifications: the *prescription* that a certain symbol be used in a certain cell, and the *restriction* that a certain symbol must not be used in a certain cell. When only one or two distinct symbols are involved in the specifications, we solve the problem completely.

1 Introduction

An $n \times n$ *Latin square* is an $n \times n$ array on n symbols, usually taken to be $[n] = \{1, \ldots, n\}$, such that each symbol occurs exactly once in each row and each column. An $n \times n$ *partial Latin square* is an $n \times n$ array on n symbols, each of which occurs at most once in each row and at most once in each column. A partial Latin square may contain empty cells, and may or may not be *completable* to a full Latin square, by entering symbols into the empty cells.

An $r \times s$ *subarray* is the set of cells that is the intersection of a selection of r rows and a selection of s columns. Any $r \times s$ subarray can thus, by suitable permutations of the rows and columns, be brought together into an $r \times s$ connected rectangle of cells, even though the original cells may be scattered around the whole initial array.

The entries in our arrays will be sets of symbols, and we will allow cells to contain the empty set, in which case we shall say that they are empty. If each cell contains a set of size at most one the array is a *single entry* array, and if at least one cell contains a set of size at least two, the array is a *multiple entry* array. In this paper, apart from some concluding remarks, we shall be concerned only with single entry arrays.

Definition 1.1. An $n \times n$ array A on the symbol set $[n]$ is *avoidable* if there exists an $n \times n$ Latin square L on $[n]$ such that the entry in the cell $\ell_{i,j}$ of the Latin square is not present in the cell $a_{i,j}$ of the array, for all $i, j = 1, \ldots, n$.

Note that any empty cell in A is trivially avoided. The question of which $n \times n$ arrays are avoidable was posed by Häggkvist in 1989 [6]. Further results of Chetwynd and Rhodes [3], Cavenagh [2] and the present author [8], established that there are unavoidable partial Latin squares of orders 2 and 3, and that all partial Latin squares of order at least 4 are avoidable. An attempt at classifying all unavoidable arrays was undertaken by Markström and the present author in [7].

The question of which partial Latin squares are completable is well studied, but it seems natural to consider the possibility of there being both prescribed and forbidden symbols. To the present author's knowledge, these are the first stumbling steps on this path of investigation.

Definition 1.2. An $n \times n$ *specification array* A involving k symbols is an $n \times n$ array on $b_1, B_1, b_2, B_2, \ldots, b_k, B_k$. A lower case letter b_ℓ present in cell $a_{i,j}$ *prescribes* the use of the symbol b_ℓ in that cell, and the presence of an upper case B_ℓ *forbids* the use of symbol b_ℓ in that cell.

We say that a Latin square L *abides by* the specification array A if it has all the prescribed symbols where they should be, and no symbols where they should not be, according to the specifications codified in A. An array that defines specifications which can not be met we shall call *unabiding*, the opposite of which, of course, is *abiding*.

After a moment's reflection, we realize that it is not meaningful to allow multiple prescribed symbols b_k in any cell, for this would immediately render the array unabiding. Likewise, forbidding the use of some symbol in a cell where some other symbol has already been prescribed is superfluous, and if we should forbid the use of a symbol in a cell where we have already prescribed it, the array is trivially unabiding. In what follows, we shall therefore tacitly assume that no two prescribed symbols occupy the same cell, and that no forbidden symbol shares a cell with a prescribed symbol. We shall also tacitly assume that the prescribed symbols in any specification array form a partial Latin square, for otherwise, the array is again trivially unabiding.

It is meaningful to allow multiple forbidden symbols in any given cell. This line of investigation has been pursued in [4] and [5], where, however, no prescribed symbols were considered. As mentioned above, multiple forbidden symbols will only receive cursory treatment in the present paper.

2 Specification arrays involving one distinct symbol

We recall that a specification array A *involves* the distinct symbols $b_1, \ldots, b_k$, if A only contains $b_1, \ldots, b_k$ and $B_1, \ldots, B_k$. In this section, since only one symbol is involved, we shall drop the subscript, and use b for positions where b is prescribed, and B where b is forbidden. We shall use the term "symbol" both for b and B, although technically only b is one of the symbols, and B represents the fact that the symbol b is forbidden in that cell. We shall denote by $A[\sigma]$ the set of cells in the array A containing σ.

In what follows, we will often consider only certain parts of an array. For instance, we know immediately that any row or column where there is a prescribed b will hold no further b. When attempting to complete a partial diagonal on b, we need therefore only consider what happens in the subarray obtained from A by removing any row or column already containing the symbol b. The following definition is therefore useful: We denote by A_b the subarray obtained from A by removing all rows and columns containing the symbol b. By $A_{\neg b}$ we denote the subarray obtained from A by removing all rows and columns not containing b. Note that since we assume that $A[b]$ forms a partial diagonal, both A_b and $A_{\neg b}$ are square arrays. A *critical rectangle* in an $n \times n$ array, is an $r \times (n - r + 1)$ subarray, for some $1 \leq r \leq n$.

Any partial Latin square on one symbol is completable; the (partial) diagonal on the only symbol can trivially be completed, and by Knig's colouring theorem, any partial Latin square consisting of a number of complete diagonals is completable. If the prescribed symbols do not form a partial Latin square, there is of course no hope of A being abiding. Characterising the unavoidable arrays on one symbol is simply a matter of applying Hall's theorem on distinct representatives. We rephrase a result from [7] in terms of abiding arrays:

Proposition 2.1. *An $n \times n$ specification array A on the symbol B is unabiding if and only if there is a critical rectangle $R \subset A[B]$.*

As observed above, if a b and a B occupy the same cell, the array is unabiding, and if the symbols b do not form a partial Latin square, or the condition from Proposition 2.1 on the placement of the symbols B is violated, again the array is unabiding.

We shall say that an abiding specification array A *forces* the use of symbol b in a cell c not already containing a b if every Latin square L that abides by A has symbol b in cell c. We shall also say that an unabiding specification array forces the use of symbol b in a cell c not already containing a b, if A_b contains a critical rectangle $R_b \subset A[B] \cup c \cup \bigcup_{\sigma \neq b} A[\sigma]$.

We will also need the following lemmata, likewise rephrased from [7], which state when an avoidable array with only one distinct forbidden symbol forces the use of this symbol in a specific cell.

Lemma 2.2. *Let A be an $n \times n$ abiding array on the symbol B. Suppose that A forces the use of symbol b in each of the cells in a set S. Then for each cell $c \in S$, A contains a critical rectangle R such that $c \in R$ and $R \setminus c \subset A[B]$.*

Lemma 2.3. *Let A be an $n \times n$ abiding array on the symbol B. Suppose that any Latin square that abides by A must use the symbol b in at least one of the cells in the set of cells S. Then there is a nonempty subset $T \subset S$, such that A contains a critical rectangle R with $T \subset R$ and $R \setminus T \subset A[B]$.*

When considering specification arrays involving morethan one symbol, Lemma 2.2 is not quite enough. We therefore prove the following extension of it.

Lemma 2.4. *An abiding array A on symbols b and B forces the use of the symbol b in a cell c that is empty in A if and only if A_b contains a critical rectangle R_0 such that $c \in R_0$ and $R_0 \setminus c \subset A[B]$.*

Proof. We need only consider A_b, for no further b will be placed outside of A_b. By Lemma 2.2, we are forced to place a b in $c \in A_b$ if and only if there is a critical rectangle R_0 in A_b such that $R_0 \setminus c \subset A[B]$. □

We have now laid the groundwork for proving the characterisation of unabiding specification arrays involving one symbol. Figure 1 gives some small examples, and it should be clear from these how to generalize to arbitrary order.

b			
		B	B
		B	B

b			
	b		
		B	B

Figure 1: Examples of minimal unabiding specification arrays involving one symbol

Theorem 2.5. *A specification array A on b and B is abiding if and only if the subsquare A_b does not contain a critical rectangle R such that $R \subset A[B]$.*

Proof. Necessity. We must show that if the condition is violated, then A is unabiding. If the subsquare A_b, in which we must complete the partial diagonal specified by the symbols b, holds a critical rectangle R, then Proposition 2.1 clearly states that the required completion of the diagonal can not be carried out, and hence A is unabiding.

Sufficiency. The prescribed symbols b form a partial diagonal T. In the array A_b, we can find another partial diagonal T_0 by Proposition 2.1, since there is no critical rectangle $R \subset A[B]$ in A_b. T and T_0 together form a complete diagonal, and completing this to a full Latin square is trivial, as there are no specifications for the $n - 1$ other symbols. □

3 Specification arrays involving two distinct symbols

As noted in the previous section, an array with two complete, non-intersecting diagonals on prescribed symbols b and d respectively is always completable. Unavoidable arrays with two distinct forbidden symbols, but no prescribed symbols, were completely characterised in [7]:

Theorem 3.1. *Let A be an $n \times n$ unabiding specification array on symbols B and D, that does not constitute an unabiding array when either symbol is completely removed.*

Then A contains two critical rectangles; one $r \times (n-r+1)$ rectangle R_B and one $(n-r+1) \times r$ rectangle R_D as follows: $R_B \cap R_D = c$, the single cell c is empty, and $R_i \setminus c \subset A[i]$ for $i = B, D$.

If A contains subarrays as described in Theorem 3.1, then A is unabiding, since in cell c both the symbol b and the symbol d are forced, so the theorem is in effect 'if-and-only-if'. Note that Theorem 3.1 does not allow multiple entries in any cell. The situation where multiple forbidden symbols are allowed seems considerably more complicated. In what follows we shall restrict ourselves to arrays with no two forbidden symbols in any cell, so by the remarks in Section 1 each cell in A will hold at most one specification.

Before generalising further, we will investigate what happens when the two symbols only appear as prescriptions or only appear as restrictions. We shall start with the case when the two symbols are only used as prescriptions. Two examples of unabiding arrays of this type are given in Figure 2.

b	d		
	b	d	
d		b	

d			
			b
	b		
		b	

Figure 2: Examples of minimal unabiding specification arrays on symbols b and d

Proposition 3.2. *Let A be an $n \times n$ array with symbols b and d prescribed. Then A is abiding if and only if the following does not hold:*

A contains a partial diagonal T of length $n-1$ on one of the symbols, say b, together with one or both of the following, where $c = A_b$ is the cell that would complete T to a full diagonal:

(I). Cell c contains the symbol d.

(II). $A_{\neg b}$ contains a diagonal T_d of length $n-1$ on the symbol d.

Proof. Necessity. In both case (I) and case (II), the symbols b force the use of a b in cell c. In case (I), the cell is already occupied, and in case (II), the symbols d in T_d force the use of a d in cell c, resulting in an irresolvable conflict.

Sufficiency. Suppose first that the neither the number of symbols b nor d is $n-1$. If either of the two partial diagonals on b and d should already be complete, we are finished with that diagonal. Suppose that the number of symbols b is strictly less than $n-1$. Then A_b is at least 2×2, and since there is at most one d in any row or column, the partial diagonal on b can be completed. The same argument holds for d, so we are finished in this case.

Now suppose we have exactly $n-1$ symbols b. Since (I) does not hold, the partial diagonal on b can be completed. If the number of symbols d is not exactly $n-1$, we are finished, for then we can complete the partial diagonal on d, if it is not already complete.

If the number of symbols d should also happen to be exactly $n-1$, and we cannot complete the partial diagonal they form, we see by switching names on b and d that we have either case (I) or case (II), contrary to assumption. □

Next, in Proposition 3.3, we treat the case where one symbol is only used for prescribing, and the other symbol only for forbidding. Note that this proposition properly contains Proposition 2.1. In Figure 3 we give two examples of unabiding arrays of this kind, from which the general construction should be clear.

b	D	D	
D	b	D	

b			
	b		
	D	b	D
	D	D	

Figure 3: Examples of unabiding arrays on symbols b and D

Proposition 3.3. *Let A be an $n \times n$ specification array on symbols b and D. Then A is abiding if and only if none of the following holds:*

(I). The array A contains a critical rectangle $R \subset A[D] \cup A[b]$.

(II). Symbol b occurs $n-1$ times, and A contains a critical rectangle $R \subset A[D] \cup A[b] \cup A_b$.

Proof. Necessity. If (I) holds, then by Proposition 2.1, we can find no place to put a full diagonal on the symbol d. If (II) holds, we must place a b in cell $c = A_b$, and subsequently, again by Proposition 2.1, we will not find space for a full diagonal on the symbol d.

Sufficiency. Suppose that neither (I) nor (II) holds. If there are exactly n symbols b, they form a full diagonal, and since (I) does not hold, by Proposition 2.1, we can also find space enough for a full diagonal on the symbol d, and we are finished.

If there are exactly $n-1$ symbols b, then we complete this partial diagonal to a full diagonal in the only possible way (namely filling cell $c = A_b$ with a b), and since (II) does not hold, there is still room for a diagonal on symbol d.

Finally, if there are at most $n-2$ symbols b present in the partial diagonal T_b, we start by choosing a diagonal in which to put symbols d, which is possible by Proposition 2.1, since (I) does not hold. We must still complete the partial diagonal on b, so we investigate the subarray A_b, which is at least 2×2. In A_b, at most one cell in each row and column now holds a d, so we can find a partial diagonal $T_0 \subset A_b$, containing no symbols d, in which to put symbols b. Now $T_0 \cup T_b$ forms a full diagonal on b, and we are finished. □

We now investigate the case when one symbol, say d, takes both roles (prescription and restriction) and the other symbol b takes only the prescriptive role. If the number of either of these should be zero, we refer back to Theorem 2.5 or the previous propositions in this section. Examples of unabiding specification arrays of this type are given in Figure 4.

b	d		
	b		
		b	D
		D	

d			
		b	D
		D	b

Figure 4: Examples of minimal unabiding specification arrays on symbols b, d and D

Theorem 3.4. *Let A be a specification array on symbols b, d and D, that is abiding when any one of these symbols is disregarded in its entirety. Then A is abiding if and only if none of the following holds:*

(I). There are $(n-1)$ symbols b, and A_d contains a critical rectangle R such that $c = A_b$ is in R, $R \setminus c \subset A[D] \cup A[b]$, and the cell c is empty.

(II). A_d contains a critical rectangle $R \subset A[D] \cup A[b]$.

Proof. Necessity. In case (I), we see by Lemma 2.4 that symbol d is forced in cell c, but clearly, symbol b is also forced in cell c, so A is unabiding. In case (II), there is not enough room for a diagonal on symbol d, again by Lemma 2.4.

Sufficiency. If there are exactly n symbols b, they form a full diagonal, and consequently, we only have to worry about completing the diagonal on d. In doing this, we treat any b as a position where we are forbidden to place a d. Therefore, by Theorem 2.5, A is abiding if and only if case (II) does not occur.

If there are strictly less than $(n-1)$ symbols b, then we may always complete the diagonal on b after completing the diagonal on d, so again, we need only worry about completing the diagonal on d. By Theorem 2.5 this is possible if and only if (II) does not occur.

Finally, if there are exactly $(n-1)$ symbols b, it may be that we are forced to place a d in cell $c = A_b$, creating an unresolvable conflict. By Lemma 2.4, this will happen if and only if we have case (I). Ensuring that the diagonal on d is completable in this case, amounts, by Theorem 2.5, to making sure that case (II) does not occur. ☐

The following theorem, which concludes the characterisation of unabiding arrays involving two symbols, presupposes that there are both symbols B and symbols D present. If either of them should be missing, we resort instead to Theorem 3.4.

Theorem 3.5. *Let A be a specification array on symbols b, B, d and D, that is abiding when any one of these symbols is disregarded in its entirety. Let k be the* *number of symbols b, and ℓ be the number of symbols d in A, where either of these numbers might be zero, whereas the number of symbols B and symbols D are both assumed to be nonzero. Then A is abiding if and only if the following does not hold:*

For some r and s, there is in $A_b \cap A_d$ an empty cell c with the following properties:

The $(n-k) \times (n-k)$ subsquare A_b contains an $r \times ((n-k)-r+1)$ critical rectangle R_0 containing c, such that $R_0 \setminus c \subset A[B] \cup A[d]$, and the $(n-\ell) \times (n-\ell)$ subsquare A_d contains an $s \times ((n-\ell)-s+1)$ critical rectangle R_1 containing c such that $R_1 \setminus c \subset A[b] \cup A[D]$.

Proof. Necessity. If there indeed exists such a cell c, we are forced to use both a b and a d there, so A is unabiding.

Sufficiency. We assume that A is unabiding, and must show that there exists a cell c that satisfies the conditions stated.

Let $M_b \subset A_b$ be the set of cells where $A[B] \cup A[d]$ forces us to place a b when completing the partial diagonal on b, and let $M_d \subset A_d$ be the set of cells where $A[b] \cup A[D]$ forces us to place a d. By Lemma 2.2, M_b consists of single cells c, for each of which it holds that there is a critical rectangle R_c in A_b such that $c \in R_c$, and $R_c \setminus c \subset A[B] \cup A[d]$. An analogous statement holds for M_d. We will show that $M_b \cap M_d \neq \emptyset$, and since each cell $c \in M_b \cap M_d$ satisfies the conditions stated in the theorem, we will be finished. The first step will be to show that both M_b and M_d are nonempty.

Since the number of symbols D is non-zero and, by assumption, A is abiding when all D are disregarded, it is possible to complete the partial diagonal on symbol b. If it were possible to now complete the partial diagonal on d, the rest of the Latin square could easily be completed, since A holds no specifications for symbols other than b and d.

Since A is unabiding, no matter how we completed the diagonal on b, it will then be impossible to complete the partial diagonal on d. By Proposition 2.1 it is possible to complete the diagonal on d if and only if there for no t exists a $t \times ((n-k)-t+1)$ critical rectangle in A_d where we are forbidden to place a d. There are two distinct reasons for which we may be forbidden to place a d in a given cell. Either the cell holds a D, or there is a b in it, either present already in A, or placed there by us. We draw the conclusion that for any possible completed diagonal T_b on b, there is a non-empty set of cells $S \subset A_d \cap T_b$ such that $(S \cup A[D]) \cap A_d$ contains an $r \times ((n-k)-r+1)$ critical rectangle $R_S \subset S \cup A[D]$ for some r.

Suppose now that there is some possible completed diagonal T_b on the symbol b, where the added symbols b lie in the cells S, and the contribution of S to each such critical rectangle $R_S \subset A_d$ is at least two cells. Our focus will be those cells of R_S that were previously empty, and now hold added symbols b. The reason for this focus is that these cells cause problems when trying to complete the partial diagonal on d, but it is possible to move them around, by choosing a different diagonal T_b on b.

We claim that we could then reform the diagonal T_b by choosing a new suitable set S', so that the added symbols b are not used in the empty cells of the $r\times((n-k)-r+1)$ critical rectangle $R_S \subset (S \cup A[D]) \cap A_d$. In fact, we can separately for each such critical rectangle R_S pick out two cells from $S \cap R_S$ and reform S to S' so that R_S is no longer contained in $A[D] \cup S'$. This procedure is illustrated in Figure 5. We read "X/y" as X being forbidden, and y being placed in that cell. The symbol $\emptyset$ indicates either the absence of any restriction X or the absence of any added symbol y.

	$\emptyset/b$	$\dots$	$D/\emptyset$	
	$\dots$		$\dots$	
	$D/\emptyset$	$\dots$	$\emptyset/b$	

$\rightarrow$

	$\emptyset/\emptyset$	$\dots$	D/b	
	$\dots$		$\dots$	
	D/b	$\dots$	$\emptyset/\emptyset$	

Figure 5: Reforming S to S'

Thus, if there is a completed diagonal T_b whose contribution to each such critical rectangle is at least 2 cells, then A is abiding, which is a contradiction.

We see therefore that there are single cells in $A_b \cap A_d$, where the symbol d *must* be used when completing the diagonal on d. Thus M_d is not empty. By reversing the roles of b and d in the above argument, we find that M_b is likewise nonempty. More specifically, we see that M_b and M_d both have nonempty intersection with $A_b \cap A_d$. Also, if M_b and M_d intersect, they must do so in $A_b \cap A_d$, since $M_b \subset A_b$ and $M_d \subset A_d$.

We now prove that if A is unabiding, then each possible completed diagonal T_b on b intersects M_d. Suppose for the sake of contradiction, that there exists a possible completed diagonal T_d on d, that completely avoids $T_b \cup A[D]$.

Then T_b and T_d are both completed diagonals, and extending them to a full Latin square is always possible, since no other symbols are involved in the specifications. This obviously contradicts the fact that A is unabiding. Therefore, there must exist a critical rectangle R_b in A_d, contained in $T_b \cup A[D]$.

Suppose now, for the sake of contradiction, that $T_b \cap M_d = \emptyset$. Then T_b contributes at least two cells to each such critical rectangle R_b. We can then again reform $S \subset T_b$ exactly as in Figure 5.

We have now established that any possible completed diagonal on b intersects M_d, and, by symmetry, that any possible completed diagonal on d intersects M_b. We have yet to show that M_b intersects M_d.

Since each possible diagonal on b intersects M_d, by Lemma 2.3 there exists a non-empty set of cells $T \subset M_d$ such that A_b contains an $r \times ((n-k) - r + 1)$ critical rectangle $R_T \subset A[B] \cup A[d] \cup T$.

If $|T| = 1$, the symbol b is forced in T, so $T \subset M_b$, and thus $M_b \cap M_d \neq \emptyset$, and we are finished. Suppose, for the sake of contradiction that $|T| \geq 2$. Then we would not be forced to use d in the cells of T, contradicting the fact that $T \subset M_d$. To see why this is so, we choose arbitrarily two cells from T. Figure 6 shows how the symbols d claimed to be forced in these two cells of T can be moved.

$B/\emptyset$	$\dots$	$\emptyset/d$
$\dots$		$\dots$
$\emptyset/d$	$\dots$	$B/\emptyset$

$\rightarrow$

B/d	$\dots$	$\emptyset/\emptyset$
$\dots$		$\dots$
$\emptyset/\emptyset$	$\dots$	B/d

Figure 6: Moving d from two cells of T

Thus $|T| = 1$ and therefore $T \subset M_b$, so that $\emptyset \neq T \subset M_b \cap M_d$. $\square$

Examples of unabiding arrays involving two distinct symbols where each symbol is used both for prescriptions and restrictions are given in Figure 7.

b			
	D	D	
	D		B
		B	d

b			d
	b	D	
	D		B

Figure 7: Examples of minimal unabiding arrays involving two symbols

4 Concluding remarks

A natural extension of the present results would be to consider what happens when we allow both b and d to be forbidden in some cells. This, however, is considerably more complicated. As an indication of this, we present in Figure 8 two examples of unabiding arrays with entries from the set $\{B, D, \{B, D\}\}$, taken from [7]. These two arrays were found using extensive computer search, and we know of no corresponding examples of order 5 or greater. Furthermore, Casselgren [1] has shown that the problem of deciding whether an array on $\{B, D, \{B, D\}\}$ is abiding is $\mathcal{NP}$-complete.

B, D		
	B	D
	D	B

B, D	B, D		
B, D		B	D
	B, D	D	B

Figure 8: Examples of multiple entry unabiding arrays on symbols B and D

Another obvious path of investigation is keeping the condition that each cell hold only one forbidden symbol, but allowing three or more symbols to be involved in the specifications. Again, this seems like a difficult problem. In Figure 9 two other unabiding arrays from [7], this time with three distinct symbols forbidden, are presented to illustrate the potentially rich flora of unabiding arrays in this category.

B	D	E
E		B
D	B	

B	B	D	E
B	B	E	D
		D	D
		E	E

Figure 9: Examples of unabiding arrays on three symbols

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