AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **51** (2011), Pages 41–48

On the meanness of arbitrary path super subdivision of paths

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Abstract

Let $G(V, E)$ be a graph with p vertices and q edges. For every assignment $f : V(G) \to \{0, 1, 2, \dots, q\}$, consider an induced edge labeling $f^* : E(G) \to \{1, 2, \dots, q\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity,} \\ \frac{f(u)+f(v)+1}{2} & \text{otherwise} \end{cases}$$

for every edge $uv \in E(G)$. If $f^*(E) = \{1, 2, \dots, q\}$, then we say that f is a mean labeling of G. If a graph G admits a mean labeling, then G is called a mean graph. In this paper, we study the meanness of arbitrary path super subdivision of the graph P_n.

1 Introduction

Let $G(V, E)$ be a graph with p vertices and q edges. For notation and terminology we follow [1].

The concept of mean labeling was first introduced by Somasundaram and Ponraj [7]. They studied in [3, 4, 7, 8] the meanness of many standard graphs such as P_n, C_n, $K_n (n \leq 3)$, the ladder, the triangular snake, $K_{1,2}, K_{1,3}, K_{2,n}$, $K_2 + mK_1, K_n^c +$

$2K_2, S_m + K_1, C_m \cup P_n$ $(m \geq 3, n \geq 2)$, quadrilateral snake, comb, bistars $B(n)$, $B_{n+1,n}, B_{n+2,n}$, the corona of ladder, subdivision of central edge of $B_{n,n}$, subdivision of the star $K_{1,n}(n \leq 4)$, the friendship graph $C_3^{(2)}$, crown $C_n \odot K_1$, $C_n^{(2)}$, the dragon, arbitrary super subdivision of a path etc. In addition, they proved that the graphs $K_n(n > 3), K_{1,n}(n > 3)$, $B_{m,n}(m > n+2)$, $S(K_{1,n})(n > 4)$, $C_3^{(t)}(t > 2)$, and the wheel W_n are not mean graphs. Also some standard results are proved in [2, 5].

A vertex labeling of G is an assignment $f : V \rightarrow \{0, 1, 2, \ldots, q\}$. For a vertex labeling f, the induced edge labeling f^* is defined by

$f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ for any edge uv in G, that is,

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity} \\ \frac{f(u)+f(v)+1}{2} & \text{otherwise.} \end{cases}$$

Then the vertex labeling f is called a mean labeling of G if its induced edge labeling $f^* : E \rightarrow \{1, 2, \ldots, q\}$ is a bijection, that is, $f^*(E) = \{1, 2, \ldots, q\}$. If a graph G has a mean labeling, then we say that G is a mean graph.

Example 1.1. *A mean labeling of the Petersen graph is given in Figure 1.*

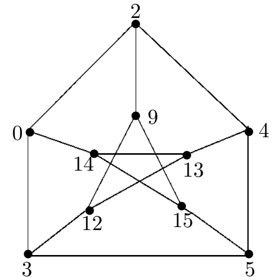

Figure 1. A mean labeling of the Petersen graph

In [6], Sethuraman and Selvaraju introduced the concept of arbitrary super subdivision of graphs. An arbitrary super subdivision of a graph G is obtained from G by replacing every edge uv of G by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the vertices x and y respectively, where $\{x, y\}$ is a partition of $K_{2,m}$.

Example 1.2.

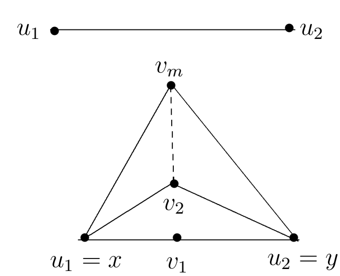

Figure 2. The graph P_2 and its arbitrary super subdivision

In [4], Ponraj and Somasundaram proved that any arbitrary super subdivision graph of the path is a mean graph. Motivated by the work in this paper, we study the meanness of another form of arbitrary path super subdivision graph of P_n. Here any arbitrary path super subdivision of a graph G is obtained from G by replacing every edge uv of G by $P_{r,m}$ ($r \geq 2$ and $m \geq 2$ may vary for each edge) where $P_{r,m}$ is obtained from m copies of the path of size r by identifying their corresponding end vertices.

Notation. We denote the arbitrary path super subdivision of the path $P_n = u_1 u_2 \dots u_n$ by APS(P_n).

2 Meanness of arbitrary path super subdivision graph of P_n

Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n. We replace each edge $u_s u_{s+1}$ ($1 \leq s \leq n-1$) of P_n by P_{r_s,m_s} for any $r_s \geq 2$ and $m_s \geq 2$, where P_{r_s,m_s} is obtained from m_s copies of the path of size r_s by identifying their corresponding end vertices. The resulting graph is an arbitrary path super subdivision graph of P_n.

Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n; each edge of the path $u_s u_{s+1}$ is replaced by P_{r_s,m_s}, $s = 1, 2, \dots, n-1$. Let $u_s = v^i_{s_0}, v^i_{s_1}, v^i_{s_2}, \dots, v^i_{s_{r_s}} = u_{s+1}$ be the vertices of the i^{th} copy of the path of size r_s where $i = 1, 2, \dots, m_s$, $s = 1, 2, \dots, n-1$. We observe that the graph APS(P_n) has $\sum_{s=1}^{n-1} (r_s - 1)m_s + n$ vertices and $\sum_{s=1}^{n-1} r_s \; m_s$ edges.

Example 2.1. *An APS(P_4) when the edge u_1u_2 is replaced by $P_{3,4}$, u_2u_3 is replaced by $P_{2,5}$ and u_3u_4 is replaced by $P_{4,3}$ is shown in Figure 3.*

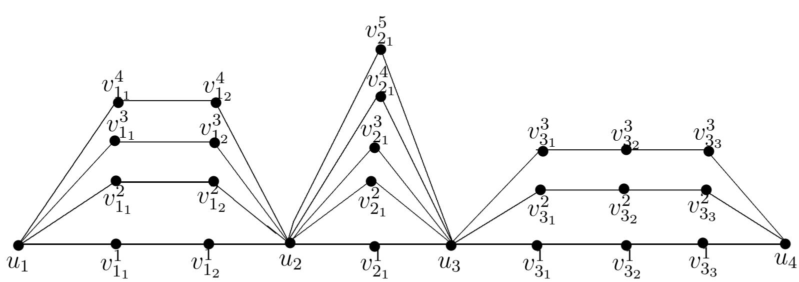

Figure 3. An APS(P_4)

Theorem 2.2. *APS(P_n) is a mean graph where each edge $u_s u_{s+1}$ of P_n is replaced by P_{r_s,m_s}, $m_s = 2k_s + 1$ for some $k_s, s = 1, 2, \dots, n-1$ and $r_s \geq 2$.*

Proof. Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n and let each edge of the path $u_s u_{s+1}$ ($1 \leq s \leq n-1$) be replaced by $P_{r_1,2k_1+1}, P_{r_2,2k_2+1}, \dots, P_{r_{n-1},2k_{n-1}+1}$ respectively. Let $u_s = v^i_{s_0}, v^i_{s_1}, v^i_{s_2}, \dots, v^i_{s_{r_s}} = u_{s+1}$ be the vertices of the i^{th} copy of the path of size r_s where $i = 1, 2, \dots, 2k_s + 1$, $s = 1, 2, \dots, n-1$. We observe that the number of vertices of the graph APS(P_n) is $\sum_{s=1}^{n-1}(r_s-1)m_s+n$ and the number of edges of the graph is $\sum_{s=1}^{n-1} r_s m_s$.

We define f as follows:

$$\begin{aligned}
f(u_1) &= 0\\
f(u_s) &= f(u_{s-1})+(m_{s-1})(r_{s-1}), 2\le s\le n\\
f(v^i_{s_{2j+1}}) &= f(u_s)+2m_s j+2i-1, 1\le s\le n-1\\
&\qquad i=1,2,\dots,2k_s+1\\
&\qquad j=\begin{cases}0,1,\dots,\frac{r_s-1}{2}-1 & \text{if } r_s \text{ is odd}\\ 0,1,\dots,\frac{r_s}{2}-1 & \text{if } r_s \text{ is even}\end{cases}\\
f(v^i_{s_{2j}}) &= \begin{cases} f(u_s)+(6k_s+2) & \\ +2m_s(j-1)-4(i-1), & 1\le i\le k_s+1\\ f(u_s)+6k_s & \\ +2m_s(j-1)-4(i-(k_s+2)), & k_s+2\le i\le 2k_s+1\end{cases}\\
&\qquad j=\begin{cases}1,2,\dots,\frac{r_s-1}{2} & \text{if } r_s \text{ is odd}\\ 1,2,\dots,\frac{r_s}{2}-1 & \text{if } r_s \text{ is even}, 1\le s\le n-1\end{cases}
\end{aligned}$$

The edge labels of APS(P_n) are given as follows:

When r_s is odd or even, the labels of the edges $v^1_{s_0}v^1_{s_1}$, $v^2_{s_0}v^2_{s_1},\dots,v^{m_s}_{s_0}v^{m_s}_{s_1}$ are $f(u_s)+1, f(u_s)+2,\dots,f(u_s)+m_s$ respectively and for each t, $1\le t\le r_s-2$, the labels of the edges

$$v^{k_s+1}_{s_t}v^{k_s+1}_{s_{t+1}}, v^{k_s}_{s_t}v^{k_s}_{s_{t+1}},\dots,v^1_{s_t}v^1_{s_{t+1}}, v^{2k_s+1}_{s_t}v^{2k_s+1}_{s_{t+1}}, v^{2k_s}_{s_t}v^{2k_s}_{s_{t+1}},\dots,v^{k_s+2}_{s_t}v^{k_s+2}_{s_{t+1}}$$

are $f(u_s)+tm_s+1, f(u_s)+tm_s+2,\dots,\ f(u_s)+(t+1)m_s$ respectively.

The labels of the edges

$$v^{k_s+1}_{s_{r_s-1}}v^{k_s+1}_{s_{r_s}}, v^{2k_s+1}_{s_{r_s-1}}v^{2k_s+1}_{s_{r_s}}, v^{k_s}_{s_{r_s-1}}v^{k_s}_{s_{r_s}}, v^{2k_s}_{s_{r_s-1}}v^{2k_s}_{s_{r_s}},\dots,v^2_{s_{r_s-1}}v^2_{s_{r_s}}, v^{k_s+2}_{s_{r_s-1}}v^{k_s+2}_{s_{r_s}}, v^1_{s_{r_s-1}}v^1_{s_{r_s}}$$

are $f(u_s)+(r_s-1)m_s+1, f(u_s)+(r_s-1)m_s+2,\dots,f(u_s)+r_sm_s$ respectively if r_s is odd and the labels of the edges

$$v^1_{s_{r_s-1}}v^1_{s_{r_s}}, v^2_{s_{r_s-1}}v^2_{s_{r_s}},\dots,v^{2k_s+1}_{s_{r_s-1}}v^{2k_s+1}_{s_{r_s}}$$

are $f(u_s)+(r_s-1)m_s+1$, $f(u_s)+(r_s-1)m_s+2,\dots,f(u_s)+r_sm_s$ respectively if r_s is even.

Hence, in each P_{r_s,m_s}, $1\le s\le n-1$, the set of parallel edges $\{v^i_{s_t}v^i_{s_{t+1}} : 1\le i\le m_s\}$ has distinct edge labels $f(u_s)+tm_s+1, f(u_s)+tm_s+2,\dots,f(u_s)+(t+1)m_s$, for each $0\le t\le r_s-1$.

Thus, the edge labels of APS(P_n) are $1,2,3,\dots,\sum_{s=1}^{n-1} r_sm_s$ and the edge labels are distinct. Hence APS(P_n) is a mean graph. □

Example 2.3. *Consider $APS(P_5)$ where the edge u_1u_2 is replaced by $P_{2,7}$, u_2u_3 is replaced by $P_{4,3}$, u_3u_4 is replaced by $P_{5,5}$ and u_4u_5 is replaced by $P_{3,7}$. A mean labeling of $APS(P_5)$ is shown in Figure 4.*

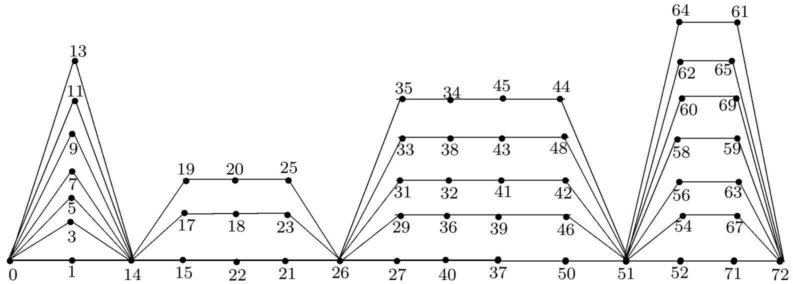

Figure 4. A mean labeling of an APS(P_5)

The above theorem has been proved only for odd integers m_s. We found it to be a hard problem to show a similar result for APS(P_n) when m_s is even and $r_s \geq 7$. In the next theorem we consider $r_s \in \{2, 3, \ldots, 6\}$ whenever m_s is even.

Theorem 2.4. *$APS(Pn)$ is a mean graph where each edge u_su_{s+1} of P_n is replaced by P_{r_s,m_s}, when m_s is odd; $r_s \geq 2$ and when m_s is even; $2 \leq r_s \leq 6$, for some $s = 1, 2, \ldots, n-1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and suppose the edge u_tu_{t+1} is replaced by P_{r_t,m_t} where $m_t = 2k_t$ for some k_t and we consider $2 \leq r_t \leq 6$. Now we give the mean labeling of the arbitrary path super subdivision of the edge u_tu_{t+1} as follows.

Let $u_t = v^i_{t_0}, v^i_{t_1}, v^i_{t_2}, \ldots, v^i_{t_{r_t}} = u_{t+1}$ be the vertices of the i^{th} copy of the path of length r_t where $i = 1, 2, \ldots, 2k_t$.

Case i. When $r_t = 2$ or $r_t = 3$ we define f as follows:

$$f(u_t) = \begin{cases} 0, & t = 1 \\ \sum_{s=1}^{t-1} r_s m_s, & 2 \leq t \leq n \end{cases}$$

$f(v^i_{t_{2j+1}}) = f(u_t) + 2i - 1, i = 1, 2, \ldots, 2k_t$ and $j = 0$ if $r_t = 2, 3$

$$f(v^i_{t_{2j}}) = \begin{cases} f(u_t) + 4k_t + 1 + 2(i-1), & 1 \leq i \leq k_t \\ f(u_t) + 2k_t + 2 + 2(i - (k_t+1)), & k_t + 1 \leq i \leq 2k_t, \\ & j = 1 \text{ if } r_t = 3. \end{cases}$$

It can be verified that the labels of the edges are $f(u_t)+1, f(u_t)+2, \ldots, f(u_{t+1})$ and all the edge labels are distinct.

Case ii. When $r_t = 4$ or 5 we define f as follows:

$$f(u_t) = \begin{cases} 0, & t = 1 \\ \sum_{s=1}^{t-1} r_s m_s, & 2 \leq t \leq n \end{cases}$$

$f(v^i_{t_{2j+1}}) = f(u_t) + 4k_tj + 2i - 1, i = 1, 2, \ldots, 2k_t$ and $j = 0, 1$ if $r_t = 4, 5$

$$f(v^i_{t_{2j}}) = \begin{cases} f(u_t) + 4k_t \\ +2 + (4k_t - 1)(j-1) + 2(i-1), & 1 \le i \le k_t \\ f(u_t) + 2k_t + (4k_t + 2)(j-1) \\ +2(i - (k_t + 1)), & k_t + 1 \le i \le 2k_t \\ & j = 1 \text{ when } r_t = 4 \text{ and} \\ & j = 1, 2 \text{ when } r_t = 5. \end{cases}$$

It is easy to check that the labels of the edges are $f(u_t)+1, f(u_t)+2, \ldots, f(u_{t+1})$ and all the edge labels are distinct.

Case iii. When $r_t = 6$ we define f as follows:

$$f(u_t) = \begin{cases} 0, & t = 1 \\ \sum_{s=1}^{t-1} r_s m_s, & 2 \le t \le n \end{cases}$$

$$f(v^i_{t_{2j+1}}) = \begin{cases} f(u_t) + 4k_t j + 2i - 1, & 1 \le i \le k_t, j = 0, 1 \\ f(u_t) + (4k_t + 1)j + 2i - 1, & k_t + 1 \le i \le 2k_t, j = 0, 1 \\ f(u_t) + (4k_t + 1)j + 2(i-1), & 1 \le i \le k_t, j = 2 \\ f(u_t) + (5k_t + 1)j - 1 \\ +2(i - (k_t + 1)), & k_t + 1 \le i \le 2k_t, j = 2 \end{cases}$$

$$f(v^i_{t_{2j}}) = \begin{cases} f(u_t) + (4k_t + 2) \\ +(4k_t - 1)(j-1) + 2(i-1), & 1 \le i \le k_t, j = 1, 2 \\ f(u_t) + 2k_t + (4k_t + 1)(j-1) \\ +2(i - (k_t + 1)), & k_t + 1 \le i \le 2k_t, j = 1, 2. \end{cases}$$

It can be easily verified that the labels of the edges are $f(u_t) + 1, f(u_t) + 2, \ldots,$ $f(u_{t+1})$ and all the edge labels are distinct.

For odd values of m_t, we give the labels for the vertices of arbitrary path super subdivision of $u_t u_{t+1}$ as in Theorem 2.2.

It is easy to check that the edge labels of APS(P_n) are $1, 2, 3, \ldots, \sum_{s=1}^{n-1} r_s m_s$ and the edge labels are distinct. Hence APS(P_n) is a mean graph. □

Example 2.5. *A mean labeling of APS(P_5) where the edge u_1u_2 is replaced by $P_{3,5}$, u_2u_3 is replaced by $P_{5,4}$, u_3u_4 is replaced by $P_{2,7}$, u_4u_5 is replaced by $P_{3,6}$, is shown in Figure 5.*

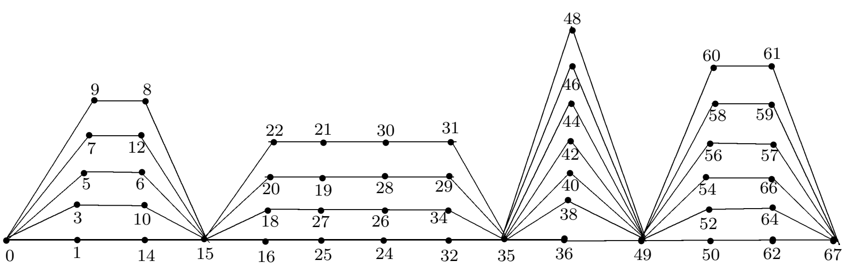

Figure 5. A mean labeling of an APS(P_5)

Corollary 2.6. *$APS(P_n)$ is a mean graph where each edge $u_s u_{s+1}$ of P_n is replaced by P_{r_s,m_s} when $m_s = 2k_s$ for some $k_s, 2 \leq r_s \leq 6, s = 1, 2, \ldots, n-1$.*

Proof. It follows from the labeling given in the part of the proof of Theorem 2.4 when m_t is even. ☐

Example 2.7. *Consider $APS(P_5)$ where the edge u_1u_2 is replaced by $P_{2,8}$, u_2u_3 is replaced by $P_{4,4}$, u_3u_4 is replaced by $P_{3,6}$ and u_4u_5 is replaced by $P_{6,4}$. A mean labeling of $APS(P_5)$ is shown in Figure 6.*

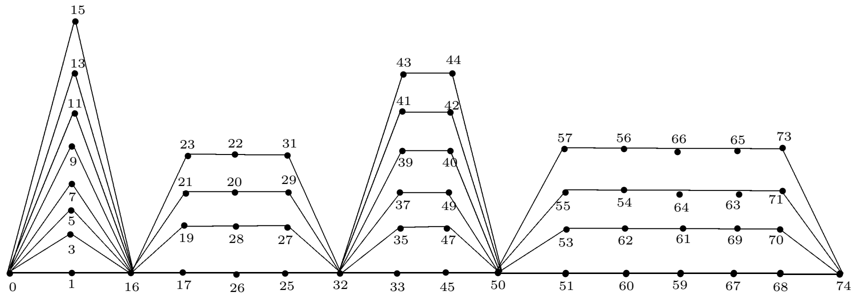

Figure 6. A mean labeling of an APS(P_5)

We propose the following open problem for further research.

Problem 2.8. *For what values of $r_s > 6$, is $APS(P_n)$ a mean graph where each edge $u_s u_{s+1}$ of P_n is replaced by P_{r_s,m_s} when m_s is even?*

Acknowledgements

The authors are thankful to the referees for their helpful suggestions which lead to a substantial improvement in the presentation of the paper.

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(Received 1 Apr 2010; revised 22 Nov 2010 and 12 May 2011)