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# A construction of 3-existentially closed graphs using quadrances

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**Abstract**

A graph is $n$-e.c. ($n$-existentially closed) if for every pair of subsets $A, B$ of vertex set $V$ of the graph such that $A \cap B = \emptyset$ and $|A| + |B| = n$, there is a vertex $z$ not in $A \cup B$ joined to each vertex of $A$ and no vertex of $B$. Few explicit families of $n$-e.c. are known for $n > 2$. In this short note, we give a new construction of 3-e.c. graphs using the notion of quadrance in the finite Euclidean space $\mathbb{Z}_p^d$.

## 1 Introduction

For a positive integer $n$, a graph is *$n$-existentially closed* or $n$-e.c. if we can extend all $n$-subsets of vertices in all possible ways. Precisely, if for every pair of subsets $A, B$ of vertex set $V$ of the graph such that $A \cap B = \emptyset$ and $|A| + |B| = n$, there is a vertex $z$ not in $A \cup B$ joined to each vertex of $A$ and no vertex of $B$. From the results of Erdős and Rényi [2], almost all finite graphs are $n$-e.c. Despite this result, until recently, only a few explicit examples of $n$-e.c. graphs are known for $n > 2$ (see [1] for a comprehensive survey on the constructions of $n$-e.c. graphs). In this short note, we give a new construction of 3-e.c. graphs using the notion of quadrance in the finite Euclidean space $\mathbb{Z}_p^d$.

Suppose that $p$ is an odd prime, and that $\mathbb{Z}_p = \{0, \dots, p-1\}$ is the prime field with $p$ elements. We will construct a 3-e.c. graph with the vertex set $\mathbb{Z}_p^d$ for some large $d$. The following definition of quadrance is taken from [4].

**Definition 1.1** *The quadrance between the points $X = (x_1, \dots, x_d)$ and $Y = (y_1, \dots, y_d)$ in $\mathbb{Z}_p^d$ is the number*

$$Q(X, Y) := (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 \in \mathbb{Z}_p.$$

Let $V_1 = \{0, 1, 2, \dots, (p-1)/2\}$. We define the graph $G_{p,d}$ as follows. The vertices of the graph $G_{p,d}$ are the points of $\mathbb{Z}_p^d$. There is an edge between two vertices $X$ and $Y$ if and only if $Q(X,Y) \in V_1$. We claim that $G_{p,d}$ is 3-e.c. for $p \geqslant 7$ and $d \geqslant 5$.

**Theorem 1.2** *Suppose that $p \geqslant 7$ is an odd prime and let $d \geqslant 5$ be an integer. Then the graph $G_{p,d}$ is 3-e.c.*

Note that these quadrance graphs are just Cayley graphs of $\mathbb{Z}_p^d$.

## 2 The 3-e.c. property of the graph $G_{p,d}$

We now give a proof of Theorem 1.2. Let $V_2 = \{(p+1)/2 \dots, p-1\} = \mathbb{Z}_p \backslash V_1$. It suffices to show that for any three distinct points $A = (a_1, \dots, a_d)$, $B = (b_1, \dots, b_d)$, $C = (c_1, \dots, c_d)$ in $\mathbb{Z}_p^d$ and $i, j, k \in \{1, 2\}$, there is a point $X = (x_1, \dots, x_d) \in \mathbb{Z}_p^d$, $X \neq A, B, C$ such that $Q(X, A) \in V_i$, $Q(X, B) \in V_j$ and $Q(X, C) \in V_k$. Therefore, we only need to show that there exist $u \in V_i, v \in V_j$, and $w \in V_k$ such that the following system has at least four solutions (in this case, one of these solutions is different from $A$, $B$, and $C$),

$$(x_1 - a_1)^2 + \cdots + (x_d - a_d)^2 = u \tag{2.1}$$
$$(x_1 - b_1)^2 + \cdots + (x_d - b_d)^2 = v \tag{2.2}$$
$$(x_1 - c_1)^2 + \cdots + (x_d - c_d)^2 = w. \tag{2.3}$$

For any $X = (x_1, \dots, x_d) \in \mathbb{Z}_p^d$, define

$$\|X\| = x_1^2 + \cdots + x_d^2.$$

By eliminating $x_i^2$'s from (2.2) and (2.3), we get an equivalent system of equations

$$Q(X, A) = u \tag{2.4}$$
$$\langle X, B - A \rangle = (u - v + \|B\| - \|A\|)/2 \tag{2.5}$$
$$\langle X, C - A \rangle = (u - w + \|C\| - \|A\|)/2. \tag{2.6}$$

We first show that the system of two equations (2.5) and (2.6) has a solution $X_0$ for some choices of $u \in V_i$, $v \in V_j$, and $w \in V_k$. We consider two cases.

Case 1. Suppose that $B - A$ and $C - A$ are linearly independent. For any $u \in V_i$, $v \in V_j$, and $w \in V_k$, it is clear that there is a solution $X_0$ to the system of two equations (2.5) and (2.6).

Case 2. Suppose that $B - A$ and $C - A$ are linearly dependent. Since $C - A \neq B - A \neq 0$, we have $C - A = t(B - A)$ for some $t \neq 0, 1$. The two equations (2.5) and (2.6) have a common solution if we can choose $u \in V_i$, $v \in V_j$, and $w \in V_k$ such that

$$u - w + \|C\| - \|A\| = t(u - v + \|B\| - \|A\|),$$

or equivalently,

$$w = tv + a,$$

where $a = \|C\| + (t-1)\|A\| - t\|B\| - (t-1)u$. In other words, we need to show that $\{tv : v \in V_j\} \cap \{w - a : w \in V_k\} \neq \emptyset$. We have two subcases.

- Suppose that $t \neq 0, \pm 1$. We label $\mathbb{Z}_p$ around the circle. The set $\{w-a : w \in V_k\}$ is a block of $(p\pm 1)/2$ consecutive points. Going $|V_k| = (p\pm 1)/2$ steps of length $2 < |t| \leqslant (p-1)/2$ around the circle, we cannot avoid any block of $(p \pm 1)/2$ consecutive points. Hence, for any fixed $u \in V_i$, we can choose $v \in V_j$ and $w \in V_k$ such that $w = tv + a$.

- Suppose that $t = -1$. The set $\{w + v : w \in V_k, v \in V_j\}$ contains at least $p - 2$ elements. Since $|A_i| \geqslant (p-1)/2 \geqslant 3$, we can choose $u$ such that $a \in \{w + v : w \in V_k, v \in V_j\}$.

Therefore, we always can choose $u \in V_i$, $v \in V_j$, and $w \in V_k$ such that the two equations (2.5) and (2.6) have a common solution $X_0$.

Take a basis of solutions of the system

$$\begin{aligned} \langle X, B - A\rangle &= 0 \\ \langle X, C - A\rangle &= 0, \end{aligned}$$

and the solution $X_0$. Substitute them into (2.4), we get a single quadratic equation of $d-2$ variables. Since $d - 2 \geqslant 3$, this quadratic equation has at least $p$ $(\geq 4)$ solutions. Theorem 1.2 follows immediately.

## 3 Remarks and Further Questions

Note that the construction is well defined over $\mathbb{Z}_m$ for any $m \in \mathbb{N}$ and it gives 3-e.c. graphs as well. The proof goes without any essential changes when $p$ is not a prime.

Moreover, the proof of Theorem 1.2 only works for $d \geq 5$. It is plausible to conjecture that the graphs are 3-e.c. for $d \geq 2$. Another interesting question is to consider other constructions with difference choices of $V_1 \subset \mathbb{Z}_p$. When $d = 2$, let $V = \{a^2 : a \in \mathbb{Z}_p^*\}$. We define the graph $G_{V,p}$ as follows. The vertices of the graph $G_{V,p}$ are the points of $\mathbb{Z}_p^2$. There is an edge between two vertices $X$ and $Y$ if and only if $Q(X, Y) \in V$. We know that $G_{V,p}$ is isomorphic to the Paley graph $P_p$ (see, for example, [3]). It is well-known that $P_p$ is $n$-e.c. for any $n$, given that $p$ is sufficiently large, and so $G_{V,p}$ is also. However we do not know of any results for the remaining cases.

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## References

[1] A. Bonato, The search for n-e.c. graphs, *Contributions to Discrete Mathematics*, to appear.

[2] P. Erdős and A. Rényi, Asymmetric graphs, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 295–315.

[3] L. A. Vinh, Quadrance graphs, *Austral. Math. Soc. Gazette* **33** (5) (2006), 330–332.

[4] N. J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005.

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