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C_4-factorizations with two associate classes, λ_1 is odd

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Abstract

Let $K = K(a, p; \lambda_1, \lambda_2)$ be the multigraph with: the number of vertices in each part equal to a; the number of parts equal to p; the number of edges joining any two vertices of the same part equal to λ_1; and the number of edges joining any two vertices of different parts equal to λ_2. This graph was of interest to Bose and Shimamoto in their study of group divisible designs with two associate classes. Necessary and sufficient conditions for the existence of z-cycle decompositions of this graph have been found when $z \in \{3, 4\}$. The existence of C_4-factorizations of K has been settled when a is even, and when $a \equiv 1 (\text{mod } 4)$ and λ_1 is even. In this paper, necessary and sufficient conditions for the existence of a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$ are found when $a \equiv 1 (\text{mod } 4)$ and λ_1 is odd with one possible exception.

1 Introduction

In this paper, graphs usually contain multiple edges. In particular, if G is a simple graph then for any $\lambda \geq 1$, let λG denote the multigraph formed by replacing each edge in G with λ edges. Throughout this paper we allow sets to contain repeated elements. Let C_z denote a cycle of length z.

Let $K = K(a, p; \lambda_1, \lambda_2)$ denote the graph formed from p vertex-disjoint copies of the multigraph $\lambda_1 K_a$ by joining each pair of vertices in different copies with λ_2 edges (so naturally, λ_1, λ_2 are non-negative integers). The vertex set, $V(K(a, p; \lambda_1, \lambda_2))$, is always chosen to be $\mathbb{Z}_a \times \mathbb{Z}_p$, with parts $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_p$; naturally, each part induces a copy of $\lambda_1 K_a$. We say the vertex (i, j) is on *level* i and in *part* j. An edge is said to be a *mixed edge* if it joins vertices in different parts, and is said to be a *pure edge* (in part j) if it joins two vertices in the j^{th} part.

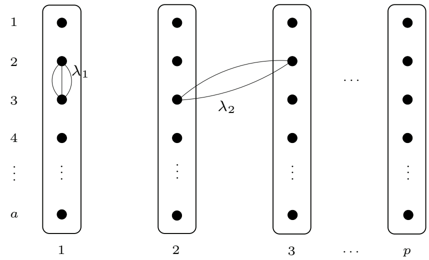


Figure 1: $K(a, p; \lambda_1, \lambda_2)$

A 2-*factor* of a graph G is a spanning 2-*regular* subgraph of G. A 2-*factorization* of G is a set of edge-disjoint 2-factors, the edges of which partition $E(G)$. A C_z-factor of G is a 2-factor that consists of cycles of length z. A C_z-factorization of G is a 2-factorization such that each 2-factor is a C_z-factor.

A set of z-cycles is said to be a *near* C_z-factor of G if it contains $\left\lfloor \frac{|V(G)|}{z} \right\rfloor$ z-cycles, which are vertex disjoint. The vertices in $V(G)$ that are in none of these z-cycles are called the *deficient* vertices of the near C_z-factor. A near C_z-factorization of G is a set of edge-disjoint near C_z-factors, the edges of which partition $E(G)$. A G-decomposition of a graph H is a partition of $E(H)$, each element of which induces a copy of G. Note that C_z-factorizations are also known as *resolvable* C_z-*decompositions*.

There has been considerable interest over the past 20 years in C_z-decompositions of various graphs, such as complete graphs and complete mutipartite graphs. In the resolvable case, these results are collectively known as addressing the Oberwolfach problem. More recently, the existence problem for C_z-decompositions of $K(a, p; \lambda_1, \lambda_2)$ for $z \in \{3, 4\}$ has been solved [4, 5, 6]. Such decompositions are known as C_z-*group-divisible designs with two associate classes*, following the notation of Bose and Shimamoto who considered the existence problem for K_z-group divisible designs. The reason for this name is that the structure can be thought of as partitioning ap symbols, or vertices, into p sets of size a in such a way that symbols that are in the same set in the partition occur together in λ_1 blocks, and are known as *first associates*, whereas symbols that are in different sets in the partition occur together in λ_2 blocks, and are known as *second associates*.

C_z-factorizations of G have also been of interest [6]. Recently the existence of a

C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ has been completely settled when a is even [1] and when $a \equiv 1 (\mathrm{mod}\ 4)$ with λ_1 even [8]. The case where $a \equiv 1 (\mathrm{mod}\ 4)$ with λ_1 odd has proven to be considerably more difficult. In this paper, we settle this case with one possible exception.

It turns out that every C_4-factor must contain at least p mixed edges. So a C_4-factor is said to be *efficient* if it contains exactly p mixed edges, and otherwise it is said to be *inefficient*. If a C_4-factor consists entirely of mixed edges, we say it is a *mixed* C_4-factor. When λ_1 is even, it is possible for all C_4-factors to be efficient; indeed, this is necessary when λ_1 is maximal, see Lemma 2.2. However, when λ_1 is odd, there must be some C_4-factors that are inefficient, and it is this property that makes the λ_1 odd case so difficult.

Example 1 The following examples of C_4-factors of $K(5,4;4,2)$ give good insight into the constructions used in Section 3 (see Figure 2).

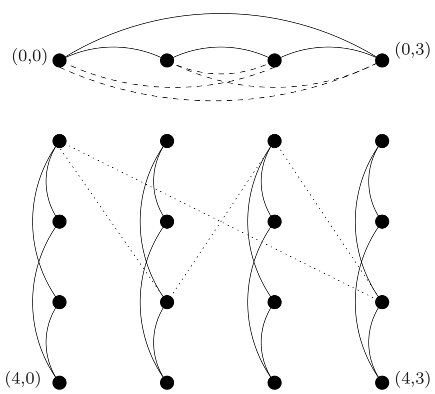


Figure 2: Example C_4-factors

For each $r \in \mathbb{Z}_5$, let $\pi_r^-(k) = \{(r+1,k),(r+2,k),(r+4,k),(r+3,k)\}$ be a near C_4-factor (i.e. includes all except one of the vertices) in the k^{th} part. Then $\bigcup_{0\le k\le 3} \pi_r^-(k) \cup \{(r,0),(r,1),(r,2),(r,3)\}$ is a C_4-factor of K (see the solid edges for the case when $r=0$). Notice that $\bigcup_{0\le k\le 3} \pi_r^-(k) \cup \{(r,0),(r,2),(r,1),(r,3)\}$ is also a C_4-factor that could be used if λ_1 is large (see the dashed mixed edges). Finally, observe that mixed edges can easily be used in C_4-factors of the form $P(s,j) = \{((i,0),(i+j,1),(i,2),(i+j,3)) | i \in \mathbb{Z}_5\}$ (see the dotted lines for one component when $j=2$).

2 Preliminary Results

We begin by finding some necessary conditions in the next two lemmas.

Lemma 2.1 *[8] Let a be odd. If there exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$, then:*

1. *$p \equiv 0(mod\, 4)$, and*
2. *$\lambda_2 > 0$ and is even.*

Lemma 2.2 *[8] Let $a \equiv 1(mod\, 4)$. If there exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$, then $\lambda_1 \leq a(p-1)\lambda_2$.*

Lemma 2.3 *[3] Suppose $a \equiv 1(mod\, 4)$. Then near C_4-factorizations of λK_a exist for all even λ.*

Lemma 2.4 *[7] Suppose $p \equiv 0(mod\, 4)$. Then C_4-factorizations of λK_p exist for all even λ.*

The following result provides C_4-factorizations of $K(a, p; \lambda_1, \lambda_2)$ with $a \equiv 1 \pmod 4$ when λ_1 is even.

Theorem 2.1 *[8] Suppose $a \equiv 1(mod\, 4)$, and λ_1 is even. There exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$ if and only if $p \equiv 0(mod\, 4)$; $\lambda_2 > 0$ and is even; and $\lambda_1 \leq a(p-1)\lambda_2$.*

The following result provides C_4-factorizations of $K(a, p; \lambda_1, \lambda_2)$ with $a \equiv 1$ (mod 4) for small values of λ_1 when λ_1 is odd.

Theorem 2.2 *[8] Suppose $a \equiv 1$ (mod 4), $p \equiv 0$ (mod 4), and $\lambda_2 > 0$ and is even. There exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$ when λ_1 is odd with $\lambda_1 \leq a(p-1)\lambda_2 - a$.*

Here, we provide some useful structures that will be used in Section 3.

The graph $G \sim H$ defined below is also an important ingredient in the proof of Theorem 3.1.

Lemma 2.5 *[8] Let $G \sim H$ be the graph with:*

1. *$V(G \sim H) = V(G) \times V(H)$, and*
2. *$E(G \sim H) = \{\{(g_1, h_1), (g_2, h_2)\} \mid \{g_1, g_2\} \in E(G)$ and $h_1 = h_2\}$ $\cup \{\{(g_1, h_1), (g_2, h_2)\} \mid g_1, g_2 \in V(G)$ and $\{h_1, h_2\} \in E(H)\}$.*

Let F be a C_4-factor of K_p and let l be even with $0 \leq l \leq 2a$. There exists a C_4-factorization of $lK_a \sim F$.

We need a special C_4-factorization of λK_p.

Lemma 2.6 *[7] Suppose λ is even. There exists a C_4-factorization of λK_p:*

$$\{F_{0,0}, F_{0,1}, F_{0,2}\} \cup \{F_i \mid 1 \leq i \leq \lambda(p-1)-3\}$$

on the vertex set $\mathbb{Z}_{4x}$ in which:

1. $F_{0,0} = \{(4i, 4i+1, 4i+2, 4i+3) \mid i \in \mathbb{Z}_x\}$,
2. $F_{0,1} = \{(4i, 4i+2, 4i+1, 4i+3) \mid i \in \mathbb{Z}_x\}$, *and*
3. $F_{0,2} = \{(4i, 4i+1, 4i+3, 4i+2) \mid i \in \mathbb{Z}_x\}$.

In Theorem 3.1 of the paper, we construct structures known as frames. Let $M(b,n)$ be the complete multipartite graph with b parts $B_0, \dots, B_{b-1}$ of size n. A 4-cycle system of $M(b,n)$ is said to be a *frame* if the 4-cycles can be partitioned into sets $S_1, \dots, S_z$ such that for $1 \leq j \leq z$, S_j is a 2-factor of $M(b,n) \setminus B_i$ for some $i \in \mathbb{Z}_b$.

Lemma 2.7 *There exists a frame of $M(b,4)$ for all $b \geq 3$.*

Proof There are several constructions based on the parity of b. We begin with the case where b is odd.

Case 1: b is odd

Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_b$, and for each $d \in \mathbb{Z}_b$ let F'_d be the near 1-factor in F' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one edge in F'_d.

Let $K(B_x, B_y)$ be the complete simple bipartite graph on the parts $B_x = \{x\} \times \mathbb{Z}_4$ and $B_y = \{y\} \times \mathbb{Z}_4$, $0 \leq x < y \leq b-1$. For each $\{x,y\} \in E(F'_d)$, define a C_4-factorization of $K(B_x, B_y)$, consisting of two C_4-factors:

$$\pi_{x,y}(0) = \{((x,0),(y,0),(x,2),(y,2)), ((x,1),(y,1),(x,3),(y,3))\}$$
$$\pi_{x,y}(1) = \{((x,0),(y,1),(x,2),(y,3)), ((x,1),(y,2),(x,3),(y,0))\}$$

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{\{x,y\} \in E(F'_d)} K(B_x, B_y),$$

which has a C_4-factorization, P_d, consisting of the two C_4-factors:

$$M_d(j) = \bigcup_{\{x,y\} \in E(F'_d)} \pi_{x,y}(j) \text{ for each } j \in \mathbb{Z}_2.$$

Notice that

$$M(b,4) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d\in\mathbb{Z}_b\\ j\in\mathbb{Z}_2}} M_d(j).$$

Notice also that each $M_d(j)$ is a 2-factor of $M(b,4)\setminus(\{d\}\times\mathbb{Z}_4)$ so the 4-cycles in

$$P(b)=\bigcup_{d\in\mathbb{Z}_b} P_d,$$

form a frame of $M(b,4)$.

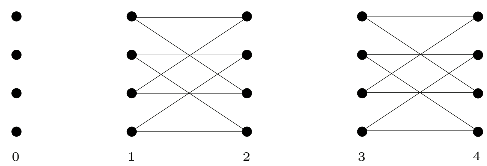


Figure 3: Example of a 2-factor, $M_0(j)$.

Case 2: b is even

There are two cases based on whether $b\equiv 0$ or $2 \pmod 4$.

Define $C=\{(c_0(i),c_1(i),\ldots,c_{b-2}(i)) \mid i\in\mathbb{Z}_{b-1},\ c_{b-2}(i)=\infty,\ c_j(i)=i+(-1)^{j+1}\lceil j/2\rceil \text{ for } 0\le j\le b-3\}\cup\{(0,1,\ldots,b-2)\}$ to be a $(b-1)$-cycle system of $2K_b$ on the vertex set $V=\mathbb{Z}_{b-1}\cup\{\infty\}$. Let $c'=(0,1,\ldots,b-2)$. For each $d\in V$, let C_d be the cycle with deficiency d.

Case 2.1: $b\equiv 0 \pmod 4$

For each $c=(c_0,c_1,\ldots,c_{b-2})\in C\setminus\{c'\}$, say $c=C_d$, and for each $j\in\mathbb{Z}_2$, define a C_4-factor, $P(c,j)$, of $(V\times\mathbb{Z}_4)\setminus(\{d\}\times\mathbb{Z}_4)$ as follows (with the first subscripts reduced modulo $(b-1)$ and the second subscripts reduced modulo 4):

$P(c,j)=\{((c_i,2j+(1+(-1)^{i+1})),(c_{i+1},2j+(1+(-1)^{i+1})),(c_i,1+2j+(1+(-1)^{i+1})),(c_{i+1},1+2j+(1+(-1)^{i+1}))) \mid -1\le i\le \frac{b}{2}-2\}\cup\{((c_i,2j+2),(c_{i+1},2j),(c_i,2j+3),(c_{i+1},2j+1)) \mid \frac{b}{2}-1\le i\le b-3\}$.

Also, for each $j\in\mathbb{Z}_2$, define a C_4-factor, $P(c',j)$, of $(V\times\mathbb{Z}_4)\setminus(\{\infty\}\times\mathbb{Z}_4)$ as follows (with the first subscripts reduced modulo $(b-1)$ and the second subscripts reduced modulo 4):

$$P(c',j)=\{((i,2j),(i+1,2j+2),(i,2j+1),(i+1,2j+3)) \mid i\in\mathbb{Z}_{b-1}\}.$$

Notice that for each $c\in C$ and each $j\in\mathbb{Z}_2$, $P(c,j)$ is a 2-factor of $M(b,4)\setminus(\{d\}\times\mathbb{Z}_4)$ where $c=C_d$ so the 4-cycles in

$$\bigcup_{\substack{c\in C\\ j\in\mathbb{Z}_2}} P(c,j)$$

form a frame of $M(b,4)$.

Case 2.2: $b \equiv 2$ (**mod** 4)

In the case where $b \equiv 2 \pmod 4$, there are two constructions for the 2-factors. For each $c = (c_0, c_1, \ldots, c_{b-2}) \in C \setminus \{c'\}$, say $c = C_d$, define two C_4-factors, $P(c,0)$ and $P(c,1)$, of $(V \times \mathbb{Z}_4) \setminus (\{d\} \times \mathbb{Z}_4)$ as follows (with the first subscripts reduced modulo $(b-1)$ and the second subscripts reduced modulo 4):

1. $P(c,0) = \{((c_i, 1+(-1)^{i+1}), (c_{i+1}, 1+(-1)^{i+1}), (c_i, 2+(-1)^{i+1}), (c_{i+1}, 2+(-1)^{i+1})) \mid -2 \leq i \leq \frac{b}{2}-2\} \cup \{((c_i,0),(c_{i+1},2),(c_i,1),(c_{i+1},3)) \mid \frac{b}{2}-1 \leq i \leq b-3\}$, and
2. $P(c,1) = \{((c_i, 3+(-1)^{i+1}), (c_{i+1}, 3+(-1)^{i+1}), (c_i, 4+(-1)^{i+1}), (c_{i+1}, 4+(-1)^{i+1})) \mid 0 \leq i \leq \frac{b}{2}-2\} \cup \{((c_i,2),(c_{i+1},0),(c_i,3),(c_{i+1},1)) \mid \frac{b}{2}-1 \leq i \leq b-2\}$

Also, for each $j \in \mathbb{Z}_2$, define a C_4-factor, $P(c',j)$, of $(\mathbb{Z}_b \times \mathbb{Z}_4) \setminus (\{\infty\} \times \mathbb{Z}_4)$ as follows (with the first subscripts reduced modulo $(b-1)$ and the second subscripts reduced modulo 4):

$$P(c',j) = \{((i,2j),(i+1,2j+2),(i,2j+1),(i+1,2j+3)) \mid i \in \mathbb{Z}_{b-1}\}.$$

Notice that for each $c \in C$ and each $j \in \mathbb{Z}_2$, $P(c,j)$ is a 2-factor of $M(b,4) \setminus (\{d\} \times \mathbb{Z}_4)$ where $c = C_d$ so the 4-cycles in

$$\bigcup_{\substack{c \in C \\ j \in \mathbb{Z}_2}} P(c,j)$$

form a frame of $M(b,4)$. ■

3 The Main Result: λ_1 is Odd and Large

In Theorem 2.2, we are able to produce C_4-factorizations of K for λ_1 values up to $a(p-1)\lambda_2 - a$. However, the upper bound when λ_1 is odd is $a(p-1)\lambda_2 - 1$. So there exists a gap $a(p-1)\lambda_2 - a < \lambda_1 \leq a(p-1)\lambda_2 - 1$ that must be filled. We now turn our attention to the construction that produces the factorization of K for the largest values of λ_1. The one exception that we encounter in the proof is when $a = 9$; it is later shown that the techniques used in this proof do not allow us to produce the C_4-factorization of $K = K(9, p; \lambda_1, \lambda_2)$ with $a(p-1)\lambda_2 - a < \lambda_1 \leq a(p-1)\lambda_2 - 1$.

Theorem 3.1 *Suppose $a \equiv 1 (mod\ 4), a \neq 9$, and λ_1 is odd. There exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if:*

1. $p \equiv 0 (mod\ 4)$,

2. $\lambda_2 > 0$ *and is even, and*

3. $\lambda_1 \le a(p-1)\lambda_2 - 1$.

Proof The necessity of these conditions is proved in Lemmas 2.1 and 2.2, so now assume that Conditions (1–3) are true.

If $\lambda_1 \le a(p-1)\lambda_2 - a$, then by Theorem 2.2 there exists a C_4-factorization of $K = K(a, p; \lambda_1, \lambda_2)$. In this proof, we provide a construction that finds the required C_4-factorization whenever $\lambda_1 \ge (a-2)$. Notice that the result will follow because $a(p-1)\lambda_2 - a \ge (a-2)$ since $\lambda_2 \ge 2$ and $p \ge 4$. So now it suffices to assume that $(a-2) \le \lambda_1 \le a(p-1)\lambda_2 - 1$. We now construct a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$ in these cases.

We begin by showing that if the theorem is true when $p = 4$, then it is true for all $p \ge 8$ with $p \equiv 0 \pmod 4$. Let $\lambda_1 = l_0 + l_1 + \cdots + l_{(\lambda_2(p-1)/2)-3}$ where:

(a) $0 < l_0 \le 6a - 1$ and is odd, and

(b) $0 < l_j \le 2a$ and is even for $1 \le j \le (\lambda_2(p-1)/2) - 3$.

We begin by using Lemma 2.6 with $\lambda = \lambda_2$. Notice that $\bigcup_{i \in \mathbb{Z}_3} F_{0,i}$ is the union of $p/4$ disjoint copies of $2K_4$. For each $j \in \mathbb{Z}_{p/4}$, let $(\mathbb{Z}_a \times \{4j, 4j+1, 4j+2, 4j+3\}, T_i)$ be a C_4-factorization of $K(a, 4; l_0, 2)$, which we are currently assuming exists since it satisfies Condition 3. Then clearly taking the union for all $j \in \mathbb{Z}_{p/4}$ of these C_4-factorizations produces a C_4-factorization, $(\mathbb{Z}_a \times \mathbb{Z}_p, T_0')$, of $l_0 K_a \sim \cup_{i \in \mathbb{Z}_3} F_{0,i}$.

For each $i \in \mathbb{Z}_{\lambda_2(p-1)/2-3} \setminus \{0\}$, let $(\mathbb{Z}_a \times \mathbb{Z}_p, T_i')$ be a C_4-factorization of $l_i K_a \sim F_i$, which exists by Lemma 2.5. Then

$$(\mathbb{Z}_a \times \mathbb{Z}_p, \cup_{i \in \mathbb{Z}_{\lambda_2(p-1)/2-3}} T_i')$$

is a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$.

So we now assume that $p = 4$. As stated before, since λ_1 is odd, some of the C_4-factors must be inefficient; these are produced first. We begin by considering the subgraph of K with $\lambda_1 = 3$. Let $b = \frac{1}{4}(a-1)$. Partition the vertices in $\mathbb{Z}_a \setminus \{0\}$ into sets $B = \{B_0, \ldots, B_{b-1}\}$, each of size 4, where $B_i = \{4i+1, 4i+2, 4i+3, 4i+4\}$ for each $i \in \mathbb{Z}_b$.

Let $M(B) = M(b, 4)$ be the complete simple multipartite graph with parts being the sets in B. In order to complete the factorization, we need a frame of $M(B)$, which exists by Lemma 2.7. In the frame of $M(B)$ constructed in Lemma 2.7, notice that for each $d \in \mathbb{Z}_b$, there are exactly two C_4-factors, say $M_{d,k}$ for $k \in \mathbb{Z}_2$, on the vertex set $\mathbb{Z}_a \setminus (B_d \cup \{0\})$. To produce the inefficient C_4-factors of K, we will use each $M_{d,k}$ twice.

Remark In order to produce a frame of $M(b, 4)$ using Lemma 2.7, b must be greater than or equal to three. When $a = 9$, $b = 2$, and there is no frame of $M(2, 4)$; therefore, we must currently exclude $a = 9$ from the theorem. However, when $a = 9$, we have previously shown in Theorem 2.1 that if $\lambda_1 \leq a(p-1)\lambda_2 - a$, then there exists a C_4-factorization of K.

Using the frames of $M(B)$, we can produce the minimum number of inefficient C_4-factors in K required by the necessary condition. All other inefficient C_4-factors in our constructions contain only mixed edges, and occur only when $\lambda_1 < a(p-1)\lambda_2 - 1$. If a C_4-factor of K contains only mixed edges, we call it a *mixed*-C_4-factor.

Recall that we are now assuming that $p = 4$. For each $i \in \mathbb{Z}_b$ and $j \in \mathbb{Z}_4$, let $B_{i,j} = B_i \times \{j\}$. For each $j \in \mathbb{Z}_4$, let $M(j)$ be the complete multipartite graph with parts $\{B_{i,j} \mid i \in \mathbb{Z}_b\}$, and let $M_{d,k}(j)$ be the natural isomorphic copy of the C_4-factor $M_{d,k}$ on the vertex set $(\mathbb{Z}_a \setminus (B_d \cup \{0\})) \times \{j\}$.

For each $d \in \mathbb{Z}_b$ and $k \in \mathbb{Z}_2$, we form four inefficient C_4-factors of $K' = K(a, 4; 3, 2)$ on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$ as follows (reducing sums in the second subscript modulo 4):

$$\begin{aligned}
\pi_{2k+1}(d) \;=\;& \{((4d+1, j+k), (0, j+k), (4d+4, j+k), (0, j+k+1)) \\
& \qquad \mid j \in \{0, 2\}\} \\
& \cup ((4d+2, k), (4d+3, k), (4d+2, k+2), (4d+3, k+2)) \\
& \cup \{((4d+1, j+k), (4d+3, j+k), (4d+2, j+k), (4d+4, j+k)) \\
& \qquad \mid j \in \{1, 3\}\} \\
& \cup \{M_{d,k}(j) \mid j \in \mathbb{Z}_4\},
\end{aligned}$$

and

$$\begin{aligned}
\pi_{2k+2}(d) \;=\;& \{((4d+1, j+k), (0, j+k), (4d+4, j+k), (0, j+k-1) \\
& \qquad \mid j \in \{0, 2\}\} \\
& \cup ((4d+2, k), (4d+3, k), (4d+2, k+2), (4d+3, k+2)) \\
& \cup \{((4d+1, j+k), (4d+2, j+k), (4d+4, j+k), (4d+3, j+k)) \\
& \qquad \mid j \in \{k+1, k+3\}\} \\
& \cup \{M_{d,k}(j) \mid j \in \mathbb{Z}_4\}.
\end{aligned}$$

Let $P^* = \{\pi_{2j+1}(i), \pi_{2j+2}(i) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_2\}$ be the set of these $4b$ inefficient C_4-factors. Let $E(P^*)$ be the set of edges of the sets of 4-cycles in P^*. Let K^* be the graph induced by the edge-set $E(P^*)$; then K^* is a subgraph of K'. For each $j \in \mathbb{Z}_4$, let $W(j)$ be the pure edges in $K' \setminus E(K^*)$ induced by the vertex set $\mathbb{Z}_a \times \{j\}$ (see Figure 5). In K^*, clearly each vertex has degree $8b$ since its edges can be partitioned into $4b$ C_4-factors. More specifically, for each $j \in \mathbb{Z}_4$ the pure degree of v in K^* is

$$d(v) = \begin{cases} 4b = a-1 & \text{if } v = (0, j), \text{ and} \\ 8b - 2 = 2(a-1) & \text{otherwise} \end{cases} \qquad (1)$$

and the mixed degree of v is

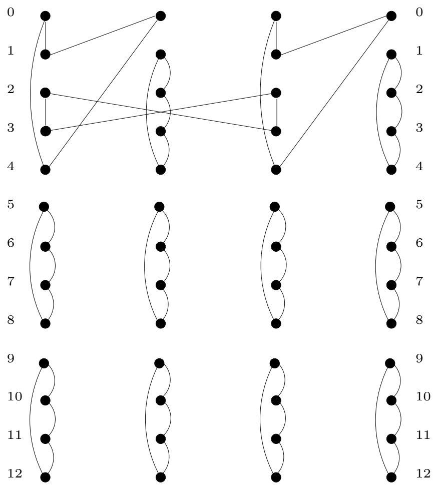


Figure 4: Example of an inefficient C_4-factor.

$$d(v) = \begin{cases} 4b = a-1 & \text{if } v = (0,j), \text{ and} \\ 2 & \text{otherwise.} \end{cases} \qquad (2)$$

We would like to supplement P^* with some efficient C_4-factors that equalize the pure and mixed degrees of all the vertices in K^* to $(a-1)(a-2)$ and $a-1=4b$ respectively while using precisely the mixed edges of the *broken* differences; that is, broken in the sense that some edges of these differences are already used in $E(P^*)$. Let A be the multiset of mixed edges of differences $\{4i+1, 4i+4 \mid i \in \mathbb{Z}_b\}$ in which each mixed edge of those differences occurs twice. So all the mixed edges in $E(P^*)$ are in A.

To equalize the pure and mixed degree of the vertices will also require using the remaining edges in $K' \setminus E(K^*)$ and an additional $(a-5)K_a$ on the vertex set $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_4$; the following three paragraphs indicate why one might expect this approach to be possible. It is worth reiterating now that the number of inefficient C_4-factors we have already constructed was carefully chosen so that if λ_1 is as large as condition (3) allows, then all remaining C_4-factors must be efficient. It is also worth noting that this is why we require $\lambda_1 \geq (a-5)+3 = (a-2)$ in this proof.

Notice that for each $j \in \mathbb{Z}_4$, the remaining pure edges in the j^{th} part in $K \setminus E(K^*)$, namely the edges in $W(j)$, consist of the $8b(b-1)$ edges of the complete multipartite graph $M(B)$ and the $16b$ edges in the 4-cycles $C(i,j) = \{((0,j),(4i+1,j),(4i+4,j),(4i+3,j)), ((0,j),(4i+2,j),(4i+4,j),(4i+3,j)), ((0,j),(4i+2,j),(4i+1,j),(4i+4,j)), ((0,j),(4i+2,j),(4i+1,j),(4i+3,j))\}$ for each $i \in \mathbb{Z}_b$.

In order to raise the mixed degree of the $a-1$ vertices $v \in \{(1,j),\dots,(a-1,j) \mid j \in \mathbb{Z}_4\}$ from 2 (see (2)) to $a-1=4b$ in the most efficient fashion (that is, to be used in efficient C_4-factors), each such v must be in $\frac{1}{2}(a-3)$ C_4-factors in

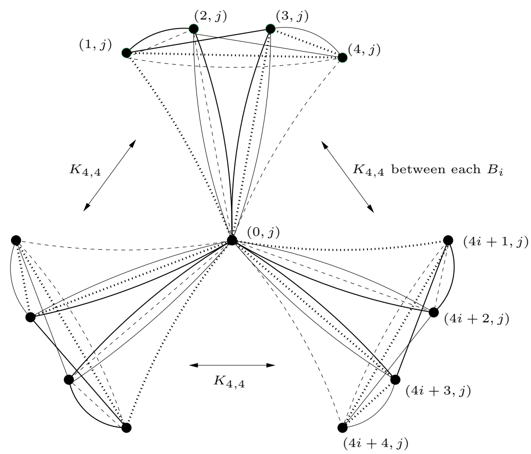


Figure 5: $W(j)$

which v is the only vertex in the j^{th} part that is in a mixed edge 4-cycle. (Notice that $(0, j)$ is excluded since it already has mixed degree $a - 1 = 4b$.) So the number of C_4-factors needed to accomplish this is $\frac{1}{2}(a-1)(a-3)$. Clearly if we proceed in this way then the mixed degree of $v = (0, j)$ is not raised for all $j \in \mathbb{Z}_4$ since each C_4-factor is required to be efficient.

Since in each of these C_4-factors $v = (0, j)$ must be incident with only pure edges, $(0, j)$ must be incident with $(a-1)(a-3)$ pure edges. In $W(j)$, $(0, j)$ is incident with $8b = 2(a-1)$ pure edges. So $(a-1)(a-5)$ more pure edges incident with $(0, j)$ are needed. This can be achieved by adding $(a-5)K_a$ with vertex set $\mathbb{Z}_a \times \{j\}$ to $W(j)$. So let $W^+(j) = W(j) \cup (a-5)K_a$.

As a check on this construction, one might ask the following questions. With how many pure edges must $v \in \{(1, j), \ldots, (a-1, j) \mid j \in \mathbb{Z}_4\}$ be incident in order to complete the C_4-factors that raise the mixed degree of v to $a - 1$? Among the previously described $\frac{1}{2}(a-1)(a-3)$ C_4-factors, each of the $a-1$ choices for v must be incident with no pure edges in exactly $\frac{1}{2}(a-3)$ of the C_4-factors, implying v must be incident with $a-3$ fewer pure edges than $(0, j)$. With how many pure edges are vertices $v \in \{(1, j), \ldots, (a-1, j) \mid j \in \mathbb{Z}_4\}$ incident in $W^+(j)$? They are each incident with $(a-2)(a-3)$ pure edges, which is exactly $a-3$ fewer than $(a-1)(a-3)$.

We next partition the edges in $\bigcup_{j \in \mathbb{Z}_4} W^+(j)$ together with a set, M^+, of $\frac{1}{2}(a-1)(a-3)$ mixed 4-cycles into efficient C_4-factors. The edges in $\bigcup_{j \in \mathbb{Z}_4} W^+(j)$ are partitioned into sets that induce pure near C_4-factors in Lemma 3.1. The interested reader can skip to there now, but it also can be saved for later reading.

The precise set, M^+, of $\frac{1}{2}(a-1)(a-3)$ mixed 4-cycles used are described now. Notice that the only requirements we need to enforce are that each vertex (i, j) with $i \in \mathbb{Z}_a \setminus \{0\}$ and $j \in \mathbb{Z}_4$ occurs in exactly $\frac{1}{2}(a-3)$ of these mixed 4-cycles, and that

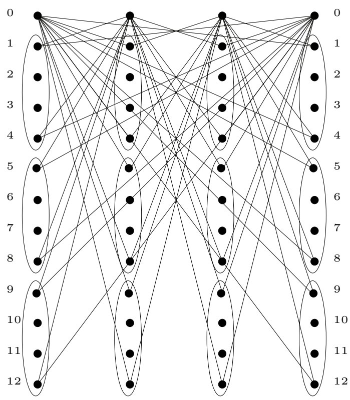


Figure 6: C with $a = 13$

the edges they contain all come from the edges of differences broken when forming P^*. Then each of the $\frac{1}{2}(a-3)$ mixed 4-cycles on the vertex set say $\{(i(j), j) \mid j \in \mathbb{Z}_4\}$ can be added to a pure near C_4-factor with deficiency $(i(j), j)$ for each $j \in \mathbb{Z}_4$ to form a C_4-factor of K.

The mixed edges of the broken differences that have already been used in the $4b$ inefficient C_4-factors in P^*, and hence contained in A, can be described by the following two sets:

(1) $E(C)$, where $C = \{((0, j), (4i+1, j+1), (0, j+2), (4i+1, j+3)), ((0, j), (4i+4, j+1), (0, j+2), (4i+4, j+3)) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_2\}$, and

(2) $D = \{\{(4i+2, j), (4i+3, j+2)\}, \{(4i+2, j), (4i+3, j+2)\}, \{(4i+3, j), (4i+2, j+2)\}, \{(4i+3, j), (4i+2, j+2)\} \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_2\}$.

Then in the subgraph induced by $E(C)$,

$$d(v) = \begin{cases} 4b & \text{if } v = (0, j), \\ 2 & \text{if } v \in \{(4i+1, j), (4i+4, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and in the subgraph induced by D

$$d(v) = \begin{cases} 2 & \text{if } v \in \{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We now define three sets of 4-cycles, R_1, R_2, and L such that each mixed edge in A will be used exactly once in $E(C) \cup D \cup E(R_1) \cup E(R_2) \cup E(L)$. The edges in these three sets of cycles will be recombined with the near C_4-factors of $W^+(j)$

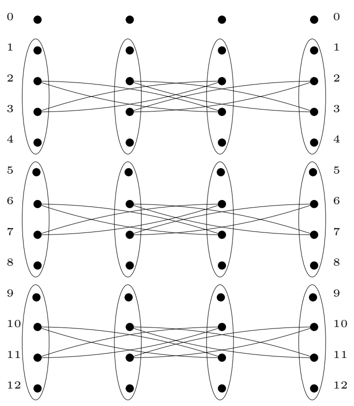


Figure 7: D with $a = 13$

into 4-cycles that can be partitioned into $4b$ mixed C_4-factors and $\frac{1}{2}(a-3)$ efficient C_4-factors of K.

In forming R_1, we need to avoid the edges in C already used in the inefficent C_4-factors; this is done by disallowing values of i and j such that $i + j = 0$.

(1) $R_1 = \{((i,0),(i+j,1),(i,2),(i+j,3)) \mid i \in \mathbb{Z}_a \setminus \{0\}, j \in \{4x+1, 4x+4 \mid x \in \mathbb{Z}_b\}, i+j \neq 0\}$.

In forming R_2, we need to avoid the edges in D already used in the inefficent C_4-factors; this is reflected in the values of (i,j) disallowed in the following set.

(2) $R_2 = \{((i,0),(i+j,2),(i,1),(i+j,3)),((i,0),(i+j,2),(i,3),(i+j,1)) \mid i \in \mathbb{Z}_a, j \in \{-1,1\}, (i,j) \notin \{(4x+2,1),(4x+3,-1) \mid x \in \mathbb{Z}_b\}\} \cup \{((4x+2,j),(4x+3,j+1),(4x+2,j+2),(4x+3,j+3)) \mid x \in \mathbb{Z}_b, j \in \mathbb{Z}_2\}$, and

(3) $L = \{((i,0),(i+j,2),(i,1),(i+j,3)),((i,0),(i+j,2),(i,3),(i+j,1)) \mid i \in \mathbb{Z}_a, j \in \{4x+1, 4x+4 \mid x \in \mathbb{Z}_b\} \setminus \{-1,1\}\}$.

Each of the $2b-2$ values of j in L produce two mixed C_4-factors of K, so this forms $4b-4$ of the $4b$ mixed C_4-factors claimed to exist. It is worth noting again that the edges in $E(C) \cup D \cup E(R_1) \cup E(R_2) \cup E(L)$ use all the mixed edges in A exactly once as Table 1 indicates.

The edges of $R = R_1 \cup R_2$ will now be partitioned in a different way into two sets. First notice the degree of each vertex in the subgraph induced by the edges of $E(R_1)$ and $E(R_2)$.

In the subgraph induced by $E(R_1)$,

Difference	Incident with 0	Between levels $4x+2$ and $4x+3$	Between parts j and $j+1$	Otherwise
$-1, 1$	C, R_2	R_1, R_2	R_1, R_2	R_1, R_2
$4x+1, x \neq 0$	C, L	No such edges exist	R, L	L, L
$4x+4, x \neq b$	C, L	No such edges exist	R, L	L, L

Table 1: Locations of Mixed Edges in A

$$d(v) = \begin{cases} 0 & \text{if } v = (0, j), \\ a-3 & \text{if } v \in \{(4i+1, j), (4i+4, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ a-1 & \text{if } v \in \{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \end{cases}$$

and in the subgraph induced by $E(R_2)$,

$$d(v) = \begin{cases} 6 & \text{if } v \in \{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ 8 & \text{otherwise.} \end{cases}$$

We remove a 2-regular subgraph on the vertex set $\{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}$ of the subgraph induced by $E(R_1)$ and add it to the subgraph induced by $E(R_2)$ in such a way that we have

(1) a graph R^* on the vertex set $(\mathbb{Z}_a \setminus \{0\}) \times \mathbb{Z}_4$ that is $(a-3)$-regular and whose edges can be partitioned into $\frac{1}{2}(a-3)$ 4-cycles, and

(2) a graph R_2^* on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$ that is 8-regular and whose edges can be partitioned into 4-cycles, which can be partitioned into four mixed C_4-factors.

Let R_1^* be the graph induced by $E(R_1)$. To form R^* from R_1^*, first remove the mixed edges that occur in the following subset of 4-cycles in R_1: $R_{1,1} = \{((4i+2,0),(4x+2,1),(4i+2,2),(4x+2,3)),((4i+3,0),(4x+3,1),(4i+3,2),(4x+3,3)) \mid i, x \in \mathbb{Z}_b, i \neq x\}$. The degree of each vertex in the subgraph, $R_{1,1}^*$, induced by $E(R_{1,1})$ is $2(b-1) = 2(\frac{1}{4}(a-1)-1) = \frac{1}{2}(a-1)-2$. Notice that each 4-cycle in $R_{1,1}$ is a 4-cycle in R_1.

Observe the degree of each vertex in $R_1^* - R_{1,1}^*$ is

$$d(v) = \begin{cases} 0 & \text{if } v = (0, j), \\ a-3 & \text{if } v \in \{(4i+1, j), (4i+4, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ a-1-2(b-1) & \text{if } v \in \{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}. \end{cases}$$

Therefore, we must add back to $R_1^* - R_{1,1}^*$ a $2(b-2)$-regular subgraph, T_1^*, of $R_{1,1}^*$ on the vertex set $v \in \{(4i+2, j), (4i+3, j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}$ to complete the formation of R^*.

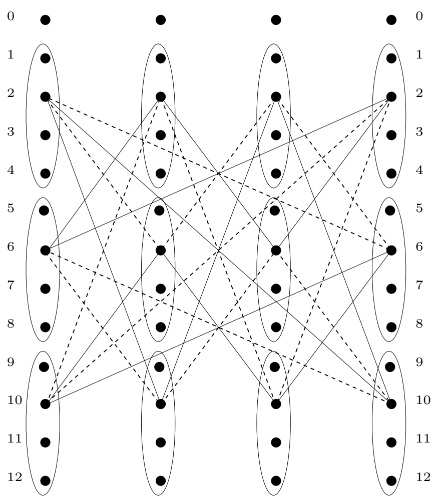


Figure 8: The 4-cycles of $R_{1,1}$ incident with vertices $\{4i+2 \mid i \in \mathbb{Z}_b\}$; the other half of $R_{1,1}$ is formed by moving each cycle "down" one level in each part.

Let $[x]_b$ denote the integer $m \in \mathbb{Z}_b$ with $m \equiv x \pmod{b}$. We now form two sets of 4-cycles, T_1 and T_2, whose edges partition $E(R_{1,1})$. The set of 4-cycles in T_1 is constructed based on the parity of b:

Case 1: b is odd

$T_1 = \{((4[i-k]_b+2,0),(4i+2,1),(4[i+k]_b+2,2),(4i+2,3)),((4[i-k]_b+3,0),(4i+3,1),(4[i+k]_b+3,2),(4i+3,3)) \mid i \in \mathbb{Z}_b,\ k \in 2,\dots,b-1\}$.

Case 2: b is even

$T_1 = \{((4[i+k]_b+2,0),(4i+2,1),(4[i+k+1]_b+2,2),(4i+2,3)),((4[i+k]_b+3,0),(4i+3,1),(4[i+k+1]_b+3,2),(4i+3,3)) \mid i \in \mathbb{Z}_b,\ k \in 2,\dots,b-1,\ k \text{ is even}\} \cup \{((4[i+k-2]_b+2,0),(4i+2,1),(4[i+k-1]_b+2,2),(4i+2,3)),((4[i+k-2]_b+3,0),(4i+3,1),(4[i+k-1]_b+3,2),(4i+3,3)) \mid i \in \mathbb{Z}_b,\ k \in 2,\dots,b-1,\ k \text{ is odd}\}$.

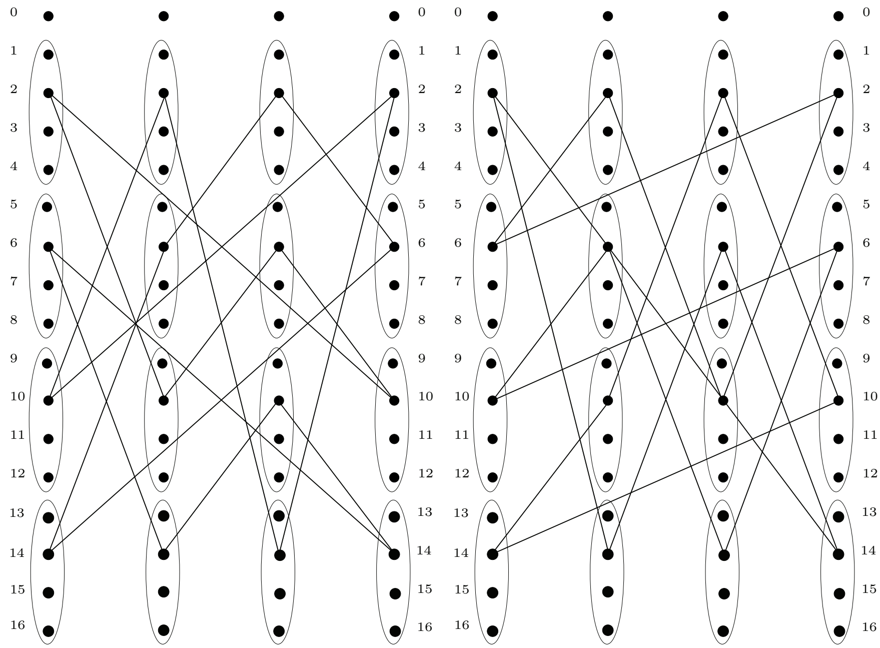
The set T_2 does not depend on the parity of b:

$T_2 = \{((4i+2,1),(4[i-k]_b+2,0),(4i+2,3),(4[i+k]_b+2,2)),((4i+3,1),(4[i-k]_b+3,0),(4i+3,3),(4[i+k]_b+3,2)) \mid i \in \mathbb{Z}_b,\ k=1\}$.

Notice that the subgraph, T_1^*, induced by $E(T_1)$ is $2(b-2)$-regular on the vertices $\{4i+2, 4i+3 \mid i \in \mathbb{Z}_b\}$. Add the edges of T_1 to the graph $R_1^* - R_{1,1}^*$ to form R^*. Then in R^* each vertex v has degree:

$$d(v) = \begin{cases} 0 & \text{if } v = (0,j), \\ a-3 & \text{if } v \in \{(4i+1,j),(4i+4,j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}, \text{ and} \\ a-3 & \text{if } v \in \{(4i+2,j),(4i+3,j) \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_4\}. \end{cases}$$

So let $R^* = R_1^* - R_{1,1}^* + T_1^*$. The edges of R^* consist of the edges of the 4-cycles



(a) T_1 with $k = 2$. (b) T_1 with $k = 3$

Figure 9: The 4-cycles of T_1 incident with vertices $\{4i + 2 \mid i \in \mathbb{Z}_b\}$; the other half of T_1 is formed by moving each cycle "down" one level in each part.

in $R_1 \setminus R_{1,1}$ and the edges in the 4-cycles of T_1; therefore, the edges of R^* can be partitioned into 4-cycles. Let $M^+ = (R_1 \setminus R_{1,1}) \cup T_1$ be the set of these 4-cycles that partition the edges of R^*.

Notice that the graph, T_2^* induced by $E(T_2)$ is 2-regular on the vertices $\{4i + 2, 4i+3 \mid i \in \mathbb{Z}_b\}$. Let R_2^* be the graph induced by $E(R_2) \cup E(T_2)$, which is 8-regular on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$. R_2^* can be partitioned into the following four C_4-factors:

1. Begin with $R_{2,1} = \{((i, 0), (i + 1, 2), (i, 1), (i + 1, 3)) \mid i \in \mathbb{Z}_a\}$, which is a C_4-factor. The edges in cycles in $R_{2,1}$ that are not edges in cycles in R_2 are precisely the edges in $S_1^- = \{\{(4x+2, 0), (4x+3, 2)\}, \{(4x+2, 1), (4x+3, 3)\} \mid x \in \mathbb{Z}_b\}$. Remove the edges in S_1^- from the cycles in $R_{2,1}$ and replace them with the edges in the subset of T_2: $S_1^+ = \{\{(4x + 2, 1), (4[x - 1]_b + 2, 0)\}, \{(4x + 3, 3), (4[x + 1]_b + 3, 2)\} \mid x \in \mathbb{Z}_b\}$. Then the edges of $\pi_1 = (R_{2,1} \setminus S_1^-) \cup S_1^+$ induce a mixed C_4-factor.

2. Begin with $R_{2,2} = \{((i, 0), (i + 1, 2), (i, 3), (i + 1, 1)) \mid i \in \mathbb{Z}_a\}$, which is a C_4-factor. The edges in cycles in $R_{2,2}$ that are not edges in cycles in R_2 are precisely the edges in $S_2^- = \{\{(4x+2, 0), (4x+3, 2)\}, \{(4x+2, 3), (4x+3, 1)\} \mid x \in \mathbb{Z}_b\}$.

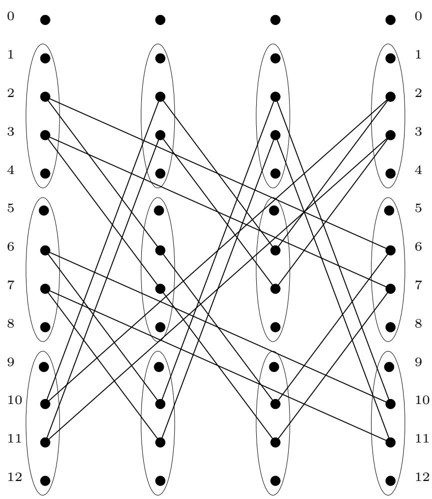


Figure 10: T_2 with $a = 13$

Remove the edges in S_2^- from the cycles in $R_{2,2}$ and replace them with the edges in the subset of T_2: $S_2^+ = \{\{(4x+2,0),(4[x+1]_b+2,3)\},\{(4x+3,1),(4[x+1]_b+3,2)\} \mid x \in \mathbb{Z}_b\}$. Then the edges of $\pi_2 = (R_{2,2} \setminus S_2^-) \cup S_2^+$ induce a mixed C_4-factor.

3. Begin with $R_{2,3} = \{((i,0),(i-1,2),(i,1),(i-1,3)) \mid i \in \mathbb{Z}_a\}$, which is a C_4-factor. The edges in cycles in $R_{2,3}$ that are not edges in cycles in R_2 are precisely the edges in $S_3^- = \{\{(4x+2,2),(4x+3,0)\},\{(4x+2,3),(4x+3,1)\} \mid x \in \mathbb{Z}_b\}$. Remove the edges in S_3^- from the cycles in $R_{2,3}$ and replace them with the edges in the subset of T_2: $S_3^+ = \{\{(4x+2,3),(4[x+1]_b+2,2)\},\{(4x+3,0),(4[x+1]_b+3,1)\} \mid x \in \mathbb{Z}_b\}$. Then the edges of $\pi_3 = (R_{2,3} \setminus S_3^-) \cup S_3^+$ induce a mixed C_4-factor.

4. Begin with $R_{2,4} = \{((i,0),(i-1,2),(i,3),(i-1,3)) \mid i \in \mathbb{Z}_a\}$, which is a C_4-factor. The edges in cycles in $R_{2,4}$ that are not edges in cycles in R_2 are precisely the edges in $S_4^- = \{\{(4x+2,1),(4x+3,3)\},\{(4x+2,2),(4x+3,0)\} \mid x \in \mathbb{Z}_b\}$. Remove the edges in S_4^- from the cycles in $R_{2,4}$ and replace them with the edges in the subset of T_2: $S_4^+ = \{\{(4x+2,1),(4[x+1]_b+2,2)\},\{(4x+3,0),(4[x+1]_b+3,3)\} \mid x \in \mathbb{Z}_b\}$. Then the edges of $\pi_4 = (R_{2,4} \setminus S_4^-) \cup S_4^+$ induce a mixed C_4-factor.

We can now supplement the C_4-factors of P^* in order to equalize the pure and mixed degrees of the vertices on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$ while using mixed edges in A. Notice that: by Lemma 3.1, each vertex (i,j) with $i \neq 0$ is deficient in $\frac{a-3}{2}$ pure near C_4-factors; and since R^* is $(a-3)$-regular, each such vertex is in $\frac{a-3}{2}$ mixed 4-cycles in M^+. For each mixed 4-cycle $m \in M^+$, let $\pi^+(m)$ be the efficient C_4-factor on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$ comprised of a *near* C_4-factor of $W^+(j)$, with deficiency being the vertex in m that is in $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_4$, and the mixed 4-cycle $m \in M^+$. Let $P^+ = \{\pi^+(m) \mid m \in M^+\}$ be set of efficient C_4-factors induced by the graph with edge-set $E(W^+(j)) + E(M^+)$ for each $j \in \mathbb{Z}_4$.

Notice that now the subgraph induced by $E(P^*) + E(P^+)$ of K is $16b^2$-regular on the vertex set $\mathbb{Z}_a \times \mathbb{Z}_4$ and hence can be partitioned into C_4-factors of K. This gives a C_4-factorization, P, of $K(a,4;(a-2),0) + (E(C) \cup D \cup E(R_1^*))$. So it remains to partition the edges of $K(a,4;\lambda_1-(a-2),0) + K(a,4;0,\lambda_2) - (E(C) \cup D \cup E(R_1^*))$ into C_4-factors.

Since $\lambda_1 - (a-2)$ is even, it turns out that we can adapt the construction in Theorem 2.1. By Condition 3 with $p = 4$, $\lambda_1 \le 3a\lambda_2 - 1$, so $\frac{\lambda_1-(a-2)}{2} \le \frac{3a\lambda_2}{2} - \frac{(a-1)}{2}$. Once we produce $\frac{3a\lambda_2}{2} - \frac{(a-1)}{2}$ mixed C_4-factors from the remaining mixed edges, then we produce the needed C_4-factorization.

Let A' be the subset of all the mixed edges formed by removing 2 copies of each of the edges joining vertices x levels apart for each $x \in \{4i+1, 4i+4 \mid i \in \mathbb{Z}_b\}$. The mixed edges of A' may be partitioned into mixed C_4-factors, $P(s,j)$, of K as defined in Theorem 2.1. The number of such C_4-factors is $\frac{|A'|}{ap} = 3\left(\frac{a+1}{2} + \frac{a(\lambda_2-2)}{2}\right)$. The edges of L and R_2^* can be partitioned into $a-5+4 = a-1$ mixed C_4-factors. So combining both of these produces $3\left(\frac{a+1}{2} + \frac{a(\lambda_2-2)}{2}\right) + (a-1) = \frac{3a\lambda_2}{2} - \frac{(a-1)}{2}$ mixed C_4-factors, say $P(m)$ for $m \in \mathbb{Z}_{\left(\frac{3a\lambda_2}{2} - \frac{(a-1)}{2}\right)}$ as required.

Now we can partition the edges of $K(a,4;\lambda_1-(a-2),0)+K(a,4;0,\lambda_2)-(E(C) \cup D \cup E(R^*))$ into C_4-factors as follows. Since $\lambda_1-(a-2)$ is even, we can produce a *near* C_4-factorization, $C_j = \{c_j(1), \ldots, c_j(\frac{a}{2}(\lambda_1-(a-2)))\}$ on the vertex set $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_4$ consisting of $\frac{a}{2}(\lambda_1-(a-2))$ *near* C_4-factors. By Condition 3, $\lambda_1 \le 3a\lambda_2 - 1$, so $\frac{\lambda_1-(a-2)}{2} \le \frac{3a\lambda_2}{2} - \frac{(a-1)}{2}$, so for $1 \le i \le \frac{\lambda_1-(a-2)}{2}$ and for each cycle $c \in P(i)$, we can extend c to a C_4-factor by adding it to four near C_4-factors, one from each C_j, $j \in \mathbb{Z}_4$ that are each vertex disjoint from c.

Thus, we have a C_4-factorization of $K(a,4;\lambda_1-(a-2),0)+K(a,4;0,\lambda_2)-(E(C) \cup D \cup E(R^*))$; therefore, we have a C_4-factorization of $K(a,4;\lambda_1,\lambda_2)$. ∎

The following lemma is used in the proof of Theorem 3.1, so all notation is adopted from there. Although the parameter j could be omitted in this lemma, it retained so the notation here matches exactly with the notation of Theorem 3.1.

Lemma 3.1 *Let $a \ge 13$ be odd, $j \in \mathbb{Z}_4$. Then $W^+(j) = W(j) \cup (a-5)K_a$ can be decomposed into $\frac{1}{2}(a-1)(a-3)$ near C_4-factors such that each $v \in \{(1,j), \ldots, (a-1,j)\}$ is deficient exactly $\frac{1}{2}(a-3)$ times and $v = (0,j)$ is never deficient.*

Proof Notice that for each $i \in \mathbb{Z}_b$ and $j \in \mathbb{Z}_4$, the 4-cycles in $C(i,j)$ exhaust all the edges in $B_{i,j}$ and all the edges joining vertices in $B_{i,j}$ to $(0,j)$ in the graph $W(j)$.

Let $C_1(i,j)$ be a 4-cycle system of $(a-5)K_5$ defined on the vertex set $B_{i,j} \cup \{0\}$ that contains a set, $C_0(i,j)$, of $\frac{1}{2}(a-5)$ copies of the 4-cycle $((4i+1,j),(4i+2,j),(4i+$

$3, j), (4i + 4, j))$. This can be done by taking $\frac{1}{2}(a - 5)$ copies of a 4-cycle system of $2K_5$, in which case: each 4-cycle in $C_1(i, j) \setminus C_0(i, j)$ contains the vertex $(0, j)$.

We must use $(a - 4)$ copies of a frame of $M(B)$ to complete the decomposition; let $M_{d,k}$ be defined as before.

For each $i \in \mathbb{Z}_b$, pair all but two of the 4-cycles in $(C_1(i, j) \setminus C_0(i, j)) \cup C(i, j)$ with the 4-cycles in a C_4-factor, $M_{d,k}$, to form a near C_4-factor of $W^+(j)$ (See Figure 11). This is possible since $|C_1(i, j)| - |C_0(i, j)| + |C(i, j)| - 2 = \frac{5}{2}(a - 5) - \frac{1}{2}(a - 5) + 4 - 2 = 2(a - 4)$, which is the number of C_4-factors on the vertex set $(\mathbb{Z}_a \setminus (B_i \cup \{0\})) \times \{j\}$ in $(a - 4)$ copies of the frame of $M(B)$.

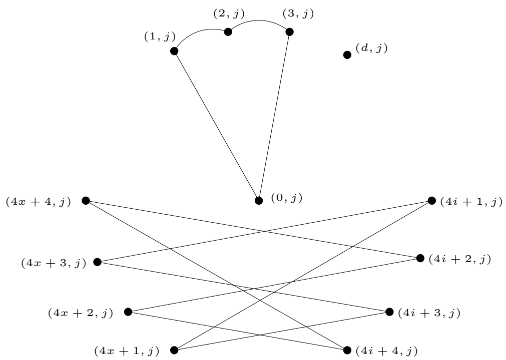


Figure 11: Near C_4-factor of $W^+(j)$

For each $i \in \mathbb{Z}_b$, form 2 C_4-factors of $W^+(j)$ consiting of the following 4-cycles (See Figure 12):

(a) one of the two remaining 4-cycles in $(C_1(i, j) \setminus C_0(i, j)) \cup C(i, j)$, and

(b) for each $d \in \mathbb{Z}_b \setminus \{i\}$, one of the 4-cycles in $C_0(d, j)$.

Notice that the number of 4-cycles used in (b) in each block $B_{i,j}$ is $2(b - 1) = |C_0(d, j)|$. The total number of C_4-factors produced this way is $2b = \frac{1}{2}(a - 1)$.

Notice that $\frac{1}{2}(a - 1)(a - 4) + \frac{1}{2}(a - 1) = \frac{1}{2}(a - 1)(a - 3)$ as required, and that each vertex is deficient exactly once in the 4-cycles in the $2K_5$-4-cycle-decompostion so that each vertex in a 4-cycle in $C_0(i, j)$ is deficient exactly $\frac{1}{2}(a - 5)$ times. Also, each vertex is deficient once in $C(i, j)$; therefore, in total, each vertex is deficient $\frac{1}{2}(a - 5) + 1 = \frac{1}{2}(a - 3)$ times, and $v = 0$ is never deficient. ■

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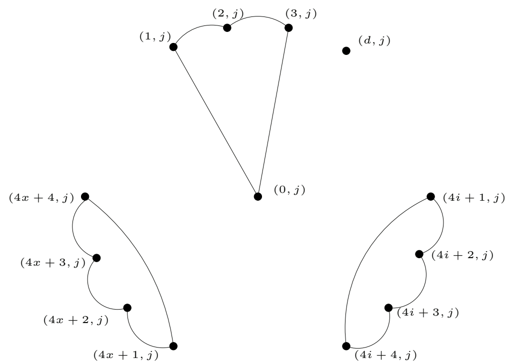

Figure 12: Near C_4-factor of $W^+(j)$

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