AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **50** (2011), Pages 233–242

Sufficient conditions for maximally edge-connected graphs and arc-connected digraphs*

Shiying Wang† Guozhen Zhang Xiuli Wang

School of Mathematical Sciences
Shanxi University
Taiyuan, Shanxi, 030006
People's Republic of China

Abstract

A connected graph with edge connectivity λ and minimum degree δ is called maximally edge-connected if $\lambda = \delta$. A strongly connected digraph with arc connectivity λ and minimum degree δ is called maximally arc-connected if $\lambda = \delta$. In this paper, some new sufficient conditions are presented for maximally edge-connected graphs and arc-connected digraphs.

We only consider finite graphs (digraphs) without loops and multiple edges (arcs). Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For a vertex v of D, denote $N^+(v) = \{u \in V(D) : vu \in A(D)\}$, $N^-(v) = \{w \in V(D) : wv \in A(D)\}$, $d^+(v) = |N^+(v)|$, $d^-(v) = |N^-(v)|$ and $\delta = \delta(D) = \min\{d^+(v), d^-(v) : v \in V(D)\}$. For $uv \in A(D)$, uv is called isolated, if $N^+(u) = \{v\}, N^-(u) = \emptyset, N^+(v) = \emptyset$ and $N^-(v) = \{u\}$. For two disjoint vertex sets X and Y of D, let $[X, Y]$ be the set of arcs from X to Y and $D[X]$ be the subdigraph induced by X. A digraph is strongly connected if for every pair u, v of vertices there exists a directed path from u to v in D. For a strongly connected digraph D, a set of arcs $W \subseteq A(D)$ is an arc cut if $D - W$ is not strongly connected. A k-arc cut is an arc cut of order k and the arc connectivity $\lambda = \lambda(D)$ of D is the minimum value of k. D is called maximally arc-connected if $\lambda = \delta$. For Similar definitions and notations for graphs we refer the reader to [2].

The study of maximally edge-connected graphs and arc-connected digraphs has special relevance to the design of reliable networks. This is due to the fact that the higher the edge connectivity, the more reliable the network. Sufficient conditions for

* This work is supported by the National Natural Science Foundation of China (61070229).
† Corresponding author.

maximally edge-connected graphs and arc-connected digraphs were given by several authors. Let G be a connected graph with $|V(G)| = n$. Chartrand [3] gave a sufficient condition $\delta \geq \lfloor n/2 \rfloor$ for $\lambda = \delta$. Lesniak [12] weakened the condition $\delta \geq \lfloor n/2 \rfloor$ to $d(u) + d(v) > n$ for all pairs of nonadjacent vertices u and v in G. Goldsmith and White [8] proved that it is sufficient to have $\lfloor n/2 \rfloor$ disjoint pairs of vertices u_i, v_i with $d(u_i) + d(v_i) \geq n$. Bollobás [1] gave a degree sequence condition that includes the condition of Goldsmith and White for odd n. Xu [17] gave an analogue for digraphs to the result by Goldsmith and White. Dankelmann and Volkmann [4] generalized Bollobás' and Xu's results and gave analogous degree conditions for bipartite graphs. The more conditions for $\lambda = \delta$ were given in [5, 6, 7, 9, 10, 11, 13, 14, 15, 16].

Let u and v be two vertices of a graph (digraph) D. The distance $d_D(u, v) = d(u, v)$ from u to v is the length of a shortest (directed) path from u to v in D. Let X and Y be two vertex sets of D. The distance $d_D(X, Y)$ from X to Y is given by $d_D(X, Y) = d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$. A pair of vertex sets X and Y of D with distance $d_D(X, Y) = k$ is called k-distance maximal, if there exist no vertex sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \neq X$ or $Y_1 \neq Y$ such that $d_D(X_1, Y_1) = k$. Let D be a bipartite graph (digraph) with the bipartition $V(D) = V' \cup V''$. For $X \subseteq V(D)$, denote $X' = X \cap V'$ and $X'' = X \cap V''$. A pair of vertex sets X and Y of D with $d_D(X', Y') = k$ and $d_D(X'', Y'') = k$ is called (k, k)-distance maximal, if there exist no vertex sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \neq X$ or $Y_1 \neq Y$ such that $d_D(X_1', Y_1') = d_D(X_1'', Y_1'') = k$. In 2003, Hellwig and Volkmann presented sufficient conditions for maximally arc-connected digraphs in term of the isolated vertex in [10].

Theorem 1 ([10]). *Let D be a strongly connected digraph with arc connectivity λ and minimum degree δ. If for all 3-distance maximal pairs of vertex sets X and Y there exists an isolated vertex in $D[X \cup Y]$, then $\lambda = \delta$.*

Theorem 2 ([10]). *Let D be a strongly connected bipartite digraph with arc connectivity λ and minimum degree δ. If for all (4,4)-distance maximal pairs of vertex sets X and Y there exists an isolated vertex in $D[X \cup Y]$, then $\lambda = \delta$.*

In this paper, inspired by Theorems 1 and 2, we present some new sufficient conditions for maximally edge-connected graphs and arc-connected digraphs in term of the isolated arc.

Let D be a strongly connected digraph with arc connectivity λ. By the definition of the arc connectivity λ, there exists an arc cut $[S, T]$ with $|[S, T]| = \lambda$, where $S \subseteq V(D)$ and $T = V(D) \backslash S$. Let $A \subseteq S$ and $B \subseteq T$ be the sets of vertices incident with at least an arc of $[S, T]$. Then $|A|, |B| \leqslant \lambda$. Define $A_0 = S \backslash A$ and $B_0 = T \backslash B$. In the following, we give an important lemma.

Lemma 1. *If there exist two disjoint, nonempty sets $X, Y \subseteq V(D)$ such that $A_0 \subseteq X, B_0 \subseteq Y$ and there exists an isolated arc uv in $D[X \cup Y]$, then $\lambda = \delta$.*

Proof. Suppose, to the contrary, that $\lambda < \delta$. We investigate four cases.

Case 1. $u \in A_0$.

By the definition of A_0, we have $v \in A_0$ or $v \in A$. If $v \in A_0$, then, by the definitions of the isolated arc and A_0, we obtain $N^+(u) \subseteq A \cup \{v\}$ and $N^+(v) \subseteq A$. Therefore

$$\begin{aligned} \delta &> |A| \geqslant |(N^+(u) \cup N^+(v)) \backslash \{v\}| \\ &= |N^+(u) \backslash \{v\}| + |N^+(v)| - |N^+(u) \cap N^+(v)| \\ &\geqslant d^+(u) - 1 + d^+(v) - \min\{d^+(u) - 1, d^+(v)\} \\ &= \max\{d^+(u) - 1, d^+(v)\} \geqslant \delta, \end{aligned}$$

a contradiction. If $v \in A$, then, by the definitions of the isolated arc, A_0 and A, we obtain $N^+(u) \subseteq A$ and $N^+(v) \subseteq A \cup B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^+(u) + d^+(v) \\ &= |N^+(u) \cap A| + |N^+(v) \cap B| + |N^+(v) \cap A| \\ &\leqslant |A| + |N^+(v) \cap B| + \sum_{x \in N^+(v) \cap A} |N^+(x) \cap B| \\ &\leqslant |A| + \sum_{x \in A} |N^+(x) \cap B| < \delta + \lambda < 2\delta, \end{aligned}$$

a contradiction.

Case 2. $u \in A$.

By the definition of A, we have $v \in A_0$, $v \in A$, or $v \in B$. If $v \in A_0$, then, by the definitions of the isolated arc, A_0 and A, we obtain $N^+(u) \subseteq A \cup B \cup \{v\}$ and $N^+(v) \subseteq A \backslash \{u\}$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^+(u) + d^+(v) \\ &= |N^+(u) \cap B| + |N^+(u) \cap A| + 1 + |N^+(v) \cap A \backslash \{u\}| \\ &\leqslant |N^+(u) \cap B| + \left(\sum_{x \in N^+(u) \cap A} |N^+(x) \cap B| \right) + 1 + |A| - 1 \\ &\leqslant \left(\sum_{x \in A} |N^+(x) \cap B| \right) + |A| < \lambda + \delta < 2\delta, \end{aligned}$$

a contradiction. If $v \in A$, then, by the definitions of the isolated arc and A, we obtain $N^+(u) \subseteq A \cup B$ and $N^+(v) \subseteq A \cup B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^+(u) + d^+(v) \\ &= |N^+(u) \cap B| + |N^+(u) \cap A| + |N^+(v) \cap B| + |N^+(v) \cap A| \\ &\leqslant |N^+(u) \cap B| + \left(\sum_{x \in N^+(u) \cap A} |N^+(x) \cap B| \right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad |N^+(v)\cap B| + \sum_{y\in N^+(v)\cap A} |N^+(y)\cap B| \\
&\leqslant \quad 2\sum_{x\in A} |N^+(x)\cap B| = 2\lambda < 2\delta,
\end{aligned}
$$

a contradiction. If $v \in B$, then, by the definitions of the isolated arc, A and B, we obtain $N^+(u) \subseteq A\cup B$ and $N^-(v) \subseteq A \cup B$. Therefore

$$
\begin{aligned}
2\delta &\leqslant d^+(u) + d^-(v) \\
&= |N^+(u)\cap B| + |N^+(u)\cap A| + |N^-(v)\cap B| + |N^-(v)\cap A| \\
&\leqslant |N^+(u)\cap B| + \left(\sum_{x\in N^+(u)\cap A} |N^+(x)\cap B|\right) \\
&+ \left(\sum_{y\in N^-(v)\cap B} |N^-(y)\cap A|\right) + |N^-(v)\cap A| \\
&\leqslant \sum_{x\in A} |N^+(x)\cap B| + \sum_{y\in B} |N^-(y)\cap A| = 2\lambda < 2\delta,
\end{aligned}
$$

a contradiction.

Case 3. $u \in B_0$.

By the definition of B_0, we have $v \in B_0$, $v\in B$, $v \in A_0$, or $v \in A$. If $v \in B_0$, then, by the definitions of the isolated arc and B_0, we obtain $N^-(u) \subseteq B$ and $N^-(v) \subseteq B \cup \{u\}$. Therefore

$$
\begin{aligned}
\delta &> |B| \geqslant |(N^-(u)\cup N^-(v))\backslash\{u\}| \\
&= |N^-(u)| + |N^-(v)\backslash\{u\}| - |N^-(u)\cap N^-(v)| \\
&\geqslant d^-(v) + d^-(u) - 1 - \min\{d^-(u), d^-(v)-1\} \\
&= \max\{d^-(u), d^-(v)-1\} \geqslant \delta,
\end{aligned}
$$

a conrtadiction. If $v \in B$, then, by the definitions of the isolated arc, B and B_0, we obtain $N^-(u) \subseteq B$ and $N^-(v) \subseteq A\cup B$. Therefore

$$
\begin{aligned}
2\delta &\leqslant d^-(u) + d^-(v) \\
&= |N^-(u)\cap B| + |N^-(v)\cap A| + |N^-(v)\cap B| \\
&\leqslant |B| + |N^-(v)\cap A| + \sum_{x\in N^-(v)\cap B} |N^-(x)\cap A| \\
&\leqslant |B| + \sum_{x\in B} |N^-(x)\cap A| < \delta + \lambda < 2\delta,
\end{aligned}
$$

a contradiction. If $v \in A_0$, then, by the definitions of the isolated arc, A_0 and B_0, we obtain $N^-(u) \subseteq B$ and $N^+(v) \subseteq A$. Therefore

$$
\begin{aligned}
2\delta &\leqslant d^-(u) + d^+(v) \\
&= |N^-(u)\cap B| + |N^+(v)\cap A| \\
&\leqslant |B| + |A| < 2\delta,
\end{aligned}
$$

a contradiction. If $v \in A$, then, by the definitions of the isolated arc, A and B_0, we obtain $N^-(u) \subseteq B$ and $N^+(v) \subseteq A \cup B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^-(u) + d^+(v) \\ &= |N^-(u) \cap B| + |N^+(v) \cap B| + |N^+(v) \cap A| \\ &\leqslant |B| + |N^+(v) \cap B| + \sum_{x \in N^+(v) \cap A} |N^+(x) \cap B| \\ &\leqslant |B| + \sum_{x \in A} |N^+(x) \cap B| < \delta + \lambda < 2\delta, \end{aligned}$$

a contradiction.

Case 4. $u \in B$.

By the definition of B, we have $v \in B_0$, $v \in B$, $v \in A_0$, or $v \in A$, If $v \in B_0$, then, by the definitions of the isolated arc, B_0 and B, we obtain $N^-(u) \subseteq A \cup B$ and $N^-(v) \subseteq B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^-(u) + d^-(v) \\ &= |N^-(u) \cap B| + |N^-(u) \cap A| + |N^-(v) \cap B| \\ &\leqslant \left(\sum_{x \in N^-(u) \cap B} |N^-(x) \cap A| \right) + |N^-(u) \cap A| + |B| \\ &\leqslant \left(\sum_{x \in B} |N^-(x) \cap A| \right) + |B| < \lambda + \delta < 2\delta, \end{aligned}$$

a contradiction. If $v \in B$, then, by the definitions of the isolated arc and B, we obtain $N^-(u) \subseteq A \cup B$ and $N^-(v) \subseteq A \cup B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^-(u) + d^-(v) \\ &= |N^-(u) \cap B| + |N^-(u) \cap A| + |N^-(v) \cap B| + |N^-(v) \cap A| \\ &\leqslant \left(\sum_{x \in N^-(u) \cap B} |N^-(x) \cap A| \right) + |N^-(u) \cap A| \\ &+ \left(\sum_{y \in N^-(v) \cap B} |N^-(y) \cap A| \right) + |N^-(v) \cap A| \\ &\leqslant 2 \sum_{x \in B} |N^-(x) \cap A| = 2\lambda < 2\delta, \end{aligned}$$

a contradiction. If $v \in A_0$, then, by the definitions of the isolated arc, A_0 and B, we obtain $N^-(u) \subseteq A \cup B$ and $N^+(v) \subseteq A$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^-(u) + d^+(v) \\ &= |N^-(u) \cap B| + |N^-(u) \cap A| + |N^+(v) \cap A| \end{aligned}$$

$$\begin{aligned} &\leqslant \left(\sum_{x\in N^-(u)\cap B}|N^-(x)\cap A|\right)+|N^-(u)\cap A|+|A| \\ &\leqslant \left(\sum_{x\in B}|N^-(x)\cap A|\right)+|A|<\lambda+\delta<2\delta, \end{aligned}$$

a contradiction. If $v\in A$, then, by the definitions of the isolated arc, A and B, we obtain $N^-(u)\subseteq A\cup B$ and $N^+(v)\subseteq A\cup B$. Therefore

$$\begin{aligned} 2\delta &\leqslant d^-(u)+d^+(v) \\ &= |N^-(u)\cap B|+|N^-(u)\cap A|+|N^+(v)\cap B|+|N^+(v)\cap A| \\ &\leqslant \left(\sum_{x\in N^-(u)\cap B}|N^-(x)\cap A|\right)+|N^-(u)\cap A|+ \\ &+ |N^+(v)\cap B|+\sum_{y\in N^+(v)\cap A}|N^+(y)\cap B| \\ &\leqslant \sum_{x\in B}|N^-(x)\cap A|+\sum_{y\in A}|N^+(y)\cap B|=2\lambda<2\delta, \end{aligned}$$

a contradiction. We conclude that $\lambda=\delta$.

Since all possible cases have been discussed, the proof of Lemma 1 is complete.□

Theorem 3. *Let D be a strongly connected digraph with arc connectivity λ and minimum degree δ. If for all 3-distance maximal pairs of vertex sets X and Y there exists an isolated arc in $D[X\cup Y]$, then $\lambda=\delta$.*

Proof. Suppose, to the contrary, that $\lambda<\delta$. By the definition of the arc connectivity λ, there exists an arc cut $[S,T]$ with $|[S,T]|=\lambda$, where $S\subseteq V(D)$ and $T=V(D)\backslash S$. Let $A\subseteq S$ and $B\subseteq T$ be the sets of vertices incident with at least an arc of $[S,T]$. Then $|A|,|B|\leqslant\lambda<\delta$. Define $A_0=S\backslash A$ and $B_0=T\backslash B$. We investigate two cases.

Case 1. $A_0=\emptyset$.

For an arbitrary vertex $w\in A$, by the definition of A, we have $N^+(w)\subseteq A\cup B$. Therefore

$$\begin{aligned} \delta &\leqslant d^+(w)=|N^+(w)\cap B|+|N^+(w)\cap A| \\ &\leqslant |N^+(w)\cap B|+\sum_{x\in N^+(w)\cap A}|N^+(x)\cap B| \\ &\leqslant \sum_{x\in A}|N^+(x)\cap B|=\lambda<\delta, \end{aligned}$$

a contradiction.

Case 2. $A_0 \neq \emptyset$.

Suppose that $B_0 = \emptyset$. For an arbitrary $w \in B$, by the definition of B, we have $N^-(w) \subseteq A \cup B$. Therefore

$$\begin{aligned} \delta &\leqslant d^-(w) = |N^-(w) \cap A| + |N^-(w) \cap B| \\ &\leqslant |N^-(w) \cap A| + \sum_{x \in N^-(w) \cap B} |N^-(x) \cap A| \\ &\leqslant \sum_{x \in B} |N^-(x) \cap A| = \lambda < \delta, \end{aligned}$$

a contradiction.

In the following suppose that $B_0 \neq \emptyset$. Clearly, the distance from A_0 to B_0 in D is finite and at least 3. Choose a 3-distance maximal pair of vertex sets X and Y with $A_0 \subseteq X$ and $B_0 \subseteq Y$. By the hypothesis, there exists an isolated arc in $D[X \cup Y]$. By Lemma 1 we obtain $\lambda = \delta$, contrary to $\lambda < \delta$. Therefore we conclude that $\lambda = \delta$. The proof of Theorem 3 is complete. □

Theorem 4. *Let D be a strongly connected digraph with arc connectivity λ, and let $[S, T]$ be an arc cut of D with $|[S, T]| = \lambda$, where $S \subseteq V(D)$ and $T = V(D) \backslash S$. If there exists at most a vertex $u \in S$ such that $|N^+(u) \cap T| = 0$, or there exists at most a vertex $v \in T$ such that $|N^-(v) \cap S| = 0$, then $\lambda = \delta$.*

Proof. Suppose that there exists at most a vertex $u \in S$ such that $|N^+(u) \cap T| = 0$. If there is no such vertex u, choose an arbitrary vertex in S as u. Then

$$\begin{aligned} \delta &\leqslant d^+(u) = |[\{u\}, S \backslash \{u\}]| + |[\{u\}, T]| \\ &\leqslant |S \backslash \{u\}| + |[\{u\}, T]| \\ &\leqslant \sum_{x \in S \backslash \{u\}} |N^+(x) \cap T| + |[\{u\}, T]| \\ &= |[S \backslash \{u\}, T]| + |[\{u\}, T]| = |[S, T]| = \lambda. \end{aligned}$$

Combining this with the fact that $\lambda \leqslant \delta$, we have $\lambda = \delta$. The other case can be proved analogously. The proof of Theorem 4 is complete. □

Corollary 1. *Let G be a connected graph with edge connectivity λ, and let $[S, T]$ be an edge cut of G with $|[S, T]| = \lambda$, where $S \subseteq V(G)$ and $T = V(G) \backslash S$. If there exists at most a vertex $u \in S$ such that $|N(u) \cap T| = 0$, or there exists at most a vertex $v \in T$ such that $|N(v) \cap S| = 0$, then $\lambda = \delta$.*

Proof. Replacing each edge of G by two oppositely oriented arcs with the same ends, we obtain a digraph D. Clearly, D is strongly connected, $\lambda(G) = \lambda(D)$ and $\delta(G) = \delta(D)$. By Theorem 4, we have $\delta(D) = \lambda(D)$. Therefore $\delta(G) = \lambda(G)$. The proof of Corollary 1 is complete. □

Theorem 5. *Let D be a strongly connected bipartite digraph with arc connectivity λ and minimum degree δ. If for all (4,4)-distance maximal pairs of vertex sets X and Y there exists an isolated arc in $D[X \cup Y]$, then $\lambda = \delta$.*

Proof. Suppose, to the contrary, that $\lambda < \delta$. By the definition of the arc connectivity λ, there exists an arc cut $[S, T]$ with $|[S, T]| = \lambda$, where $S \subseteq V(D)$ and $T = V(D) \backslash S$. Let $A \subseteq S$ and $B \subseteq T$ be the sets of vertices incident with at least an arc of $[S, T]$. Then $|A|, |B| \leqslant \lambda < \delta$. Define $A_0 = S \backslash A$ and $B_0 = T \backslash B$. Let (V', V'') be a bipartition of D. Denote $A'_0 = A_0 \cap V'$, $A' = A \cap V'$, $A''_0 = A_0 \cap V''$, $A'' = A \cap V''$, $B'_0 = B_0 \cap V'$, $B' = B \cap V'$, $B''_0 = B_0 \cap V''$ and $B'' = B \cap V''$.

Next, we show that $A'_0, A''_0, B'_0, B''_0 \neq \emptyset$. We only show that $A'_0 \neq \emptyset$. The proofs of $A''_0, B'_0, B''_0 \neq \emptyset$ are similar. Suppose, to the contrary, that $A'_0 = \emptyset$. Then $A''_0 = A_0$. Next, we show that $A''_0 = A' = A'' = \emptyset$.

Suppose that $A''_0 \neq \emptyset$. For any $w \in A''_0 = A_0$, by the definitions of A_0 and the bipartite digraph, we have $N^+(w) \subseteq A'$. Therefore $\delta \leqslant |N^+(w)| = |N^+(w) \cap A'| \leqslant |A'| \leqslant |A| \leqslant \lambda < \delta$, a contradiction. Thus $A''_0 = \emptyset$.

Suppose that $A' \neq \emptyset$. For any $w \in A' \subseteq A$, by the definitions of A and the bipartite digraph, we have $N^+(w) \subseteq A'' \cup B''$. Therefore

$$\begin{aligned}
\delta &\leqslant |N^+(w)| = |N^+(w) \cap B''| + |N^+(w) \cap A''| \\
&\leqslant |N^+(w) \cap B''| + \sum_{x \in N^+(w) \cap A''} |N^+(x) \cap B'| \\
&\leqslant \sum_{x \in A} |N^+(x) \cap B| = \lambda < \delta,
\end{aligned}$$

a contradiction. Therefore $A' = \emptyset$. Then $A'' = A$.

Suppose that $A'' \neq \emptyset$. For any $w \in A'' = A$, by the definitions of A and the bipartite digraph, we have $N^+(w) \subseteq B'$. Therefore $\delta \leqslant |N^+(w)| = |N^+(w) \cap B'| \leqslant |B'| \leqslant |B| \leqslant \lambda < \delta$, a contradiction. Therefore $A'' = \emptyset$.

Now, we have $A'_0 = A''_0 = A' = A'' = \emptyset$, that is, $S = \emptyset$. But this contradicts the fact that $[S, T]$ is an arc cut of D. Therefore $A'_0 \neq \emptyset$. Similarly, $A''_0, B'_0, B''_0 \neq \emptyset$.

Since $A'_0, A''_0, B'_0, B''_0 \neq \emptyset$, we have $d_D(A'_0, B'_0) \geqslant 4$ and $d_D(A''_0, B''_0) \geqslant 4$. Choose a (4,4)-distance maximal pair of vertex sets X and Y with $A_0 \subseteq X$ and $B_0 \subseteq Y$. By the hypothesis, there exists an isolated arc in $D[X \cup Y]$. By Lemma 1 we obtain $\lambda = \delta$, contrary to $\lambda < \delta$. Therefore we conclude that $\lambda = \delta$. The proof of Theorem 5 is complete. □

Theorem 6. *Let G be a connected graph with edge connectivity λ and minimum degree δ. If for all (4, 4)-distance maximal pairs of vertex sets X and Y there exists an isolated edge in $D[X \cup Y]$, then $\lambda = \delta$.*

Proof. The proof is similar to the proof of Theorem 5. □

Theorem 7. *Let D be a strongly connected digraph with arc connectivity λ and minimum degree δ. If $\lambda = \delta$, then there exist two disjoint sets $X, Y \subseteq V(D)$ with $X \cup Y = V(D)$ and $|[X, Y]| = \lambda$ such that $|X| = 1$ or $|X| \geqslant \delta$ and $|Y| = 1$ or $|Y| \geqslant \delta$.*

Proof. By the definition of the arc connectivity λ, there exist two disjoint sets $X, Y \subseteq V(D)$ with $X \cup Y = V(D)$ and $|[X, Y]| = \lambda$. For any $x \in X$, $\delta \leqslant d^+(x)$. Then

$$|X|\delta \leqslant \sum_{x \in X} d^+(x) \leqslant |X|(|X| - 1) + \lambda = |X|(|X| - 1) + \delta,$$

and hence $(|X| - 1)\delta \leqslant |X|(|X| - 1)$. We obtain $|X| = 1$ or $|X| \geqslant \delta$. Similarly, $|Y| = 1$ or $|Y| \geqslant \delta$. The proof of Theorem 7 is complete. □

References

[1] Béla Bollobás, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* 28 (3) (1979), 321–323.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, New York, 2007.

[3] Gary Chartrand, A graph-theoretic approach to a communications problem, *SIAM J. Appl. Math.* 14 (4) (1966), 778–781.

[4] Peter Dankelmann and Lutz Volkmann, Degree sequence conditions for maximally edge-connected graphs and digraphs, *J. Graph Theory* 26 (1) (1997), 27–34.

[5] Peter Dankelmann and Lutz Volkmann, Degree sequence conditions for maximally edge-connected graphs depending on the clique number, *Discrete Math.* 211 (1–3) (2000), 217–223.

[6] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *J. Graph Theory* 13 (6) (1989), 657–668.

[7] J. Fàbrega and M.A. Fiol, Bipartite graphs and digraphs with maximum connectivity, *Discrete Appl. Math.* 69 (3) (1996), 271–279.

[8] Donald L. Goldsmith and Arthur T. White, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* 23 (1) (1978), 31–36.

[9] Donald L. Goldsmith and Roger C. Entringer, A sufficient condition for equality of edge-connectivity and minimum degree of a graph, *J. Graph Theory* 3 (3) (1979), 251–255.

[10] Angelika Hellwig and Lutz Volkmann, Maximally edge-connected digraphs, *Australas. J. Combin.* 27 (2003), 23–32.

[11] Angelika Hellwig and Lutz Volkmann, Neighborhood conditions for graphs and digraphs to be maximally edge-connected, *Australas. J. Combin.* 33 (2005), 265–277.

[12] Linda Lesniak, Results on the edge-connectivity of graphs, *Discrete Math.* 8 (4) (1974), 351–354.

[13] Terunao Soneoka, Hiroshi Nakada, Makoto Imase and Claudine Peyrat, Sufficiet conditions for maximally connected dense graphs, *Discrete Math.* 63 (1) (1987), 53–66.

[14] Jerzy Topp and Lutz Volkmann, Sufficient conditions for equality of connectivity and minimum degree of a graph, *J. Graph Theory* 17 (6) (1993), 695–700.

[15] Lutz Volkmann, Edge-connectivity in p-partite graphs, *J. Graph Theory* 13 (1) (1989), 1–6.

[16] Lutz Volkmann, Degree sequence conditions for equal edge-connectivity and minimum degree, depending on the clique number, *J. Graph Theory* 42 (3) (2003), 234–245.

[17] Jun-Ming Xu, A sufficient condition for equality of arc-connectivity and minimum degree of a digraph, *Discrete Math.* 133 (1–3) (1994), 315–318.

(Received 11 Jan 2011; revised 21 Apr 2011)