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On cross-intersecting families of independent sets in graphs

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Abstract

Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be a collection of families of subsets of an n-element set. We say that this collection is *cross-intersecting* if for any $i, j \in [k]$ with $i \neq j$, $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ implies $A \cap B \neq \emptyset$. We consider a theorem of Hilton which gives a best possible upper bound on the sum of the cardinalities of *uniform* cross-intersecting families. We formulate a graph-theoretic analogue of Hilton's cross-intersection theorem, similar to the one developed by Holroyd, Spencer and Talbot for the Erdős-Ko-Rado theorem. In particular we build on a result of Borg and Leader for signed sets and prove a theorem for uniform cross-intersecting subfamilies of independent vertex subsets of a disjoint union of complete graphs. We proceed to obtain a result for a larger class of graphs, namely chordal graphs, and propose a conjecture for all graphs. We end by proving this conjecture for the cycle on n vertices.

1 Introduction

Let $[n] = \{1, \ldots, n\}$. Denote the family of all subsets of $[n]$ by $2^{[n]}$ and the subfamily of $2^{[n]}$ containing subsets of size r by $\binom{[n]}{r}$. A family $\mathcal{A} \subseteq 2^{[n]}$ is called *intersecting* if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$. Consider a collection of k subfamilies of $2^{[n]}$, say $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Call this collection *cross-intersecting* if for any $i, j \in [k]$ with $i \neq j$, $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$ implies $A \cap B \neq \emptyset$. Note that the individual families themselves do not need to be either non-empty or intersecting, and a subset can lie in more than one family in the collection. We will be interested in *uniform* cross-intersecting families, i.e. cross-intersecting subfamilies of $\binom{[n]}{r}$ for suitable values of r. There are two main kinds of problems concerning uniform cross-intersecting families that have been investigated, the *maximum product* problem and the *maximum sum* problem. One of the main results for the maximum product problem due to Matsumoto and Tokushige [12] states that for $r \leq n/2$ and $k \geq 2$, the product of the cardinalities of k cross-intersecting subfamilies $\{\mathcal{A}_1, \dots, \mathcal{A}_k\}$ of $\binom{[n]}{r}$ is maximum if $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \{A \subseteq \binom{[n]}{r} : x \in A\}$ for some $x \in [n]$.

In this paper however, we will be more interested in the maximum sum problem, particularly the following theorem of Hilton [9], which establishes a best possible upper bound on the sum of cardinalities of cross-intersecting families and also characterizes the extremal structures.

Theorem 1.1 (Hilton). *Let $r \leq n/2$ and $k \geq 2$. Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be cross-intersecting subfamilies of $\binom{[n]}{r}$, with $\mathcal{A}_1 \neq \emptyset$. Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r} & \text{if } k \leq n/r \\ k\binom{n-1}{r-1} & \text{if } k \geq n/r \end{cases}$$

If equality holds, then

1. $\mathcal{A}_1 = \binom{[n]}{r}$ *and* $\mathcal{A}_i = \emptyset$*, for each* $2 \leq i \leq k$*, if* $k < \frac{n}{r}$.
2. $|\mathcal{A}_i| = \binom{n-1}{r-1}$ *for each* $i \in [k]$ *if* $k > \frac{n}{r}$.
3. $\mathcal{A}_1, \dots, \mathcal{A}_k$ *are as in case* 1 *or* 2 *if* $k = \frac{n}{r} > 2$.

It is simple to observe that Theorem 1.1 is a generalization of the fundamental Erdős-Ko-Rado theorem [7] in the following manner: put $k > n/r$, let $\mathcal{A}_1 = \cdots = \mathcal{A}_k$, and we obtain the EKR theorem.

Theorem 1.2 (Erdős-Ko-Rado). *For $r \leq n/2$, let $\mathcal{A} \subseteq \binom{[n]}{r}$ be an intersecting family. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

There have been a few analogues of Hilton's cross-intersection theorem, most recently for permutations by Borg ([2] and [3]) and for uniform cross-intersecting subfamilies of independent sets in the graph M_n, which is the perfect matching on $2n$ vertices, by Borg and Leader [5]. Borg and Leader proved an extension of Hilton's theorem for *signed* sets, which we will state in the language of graphs as we are interested in formulating a graph-theoretic analogue of Theorem 1.1 similar to the one developed in [10] for Theorem 1.2. For a graph G, let $\mathcal{J}^{(r)}(G)$ be the family of all independent sets of size r in G. Also for any vertex $x \in V(G)$, let $\mathcal{J}_x^r(G) = \{A \in \mathcal{J}^r(G) : x \in A\}$.

Theorem 1.3 (Borg-Leader [5]). *Let $r \leq n$ and $k \geq 2$. Let $\mathcal{A}_1, \dots, \mathcal{A}_k \subseteq \mathcal{J}^r(M_n)$ be cross-intersecting. Then*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{r}2^r & \text{if } k \leq 2n/r \\ k\binom{n-1}{r-1}2^{r-1} & \text{if } k \geq 2n/r \end{cases}$$

Suppose equality holds and $\mathcal{A}_1 \neq \emptyset$. Then,

- *If $k \leq 2n/r$, then $\mathcal{A}_1 = \mathcal{J}^r(M_n)$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_k = \emptyset$.*
- *If $k \geq 2n/r$, then for some $x \in V(M_n)$, $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{J}^r_x(M_n)$.*
- *If $k = 2n/r > 2$, then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are as in either of the first two cases.*

In fact, Borg and Leader proved a slightly more general result with the same argument, for a disjoint union of complete graphs, all having the same number of vertices s, for some $s \geq 2$. We consider extensions of this result to any disjoint union of complete graphs. Let G be a disjoint union of complete graphs, with each component containing at least 2 vertices. We first prove a theorem which bounds the sum of the cardinalities of cross-intersecting subfamilies $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of $\mathcal{J}^r(G)$ when k is sufficiently small.

Theorem 1.4. *Let $G_1, \ldots, G_n$ be n complete graphs with $|V(G_i)| \geq 2$ for each $1 \leq i \leq n$. Let G be the disjoint union of these n graphs and let $r \leq n$. For some $2 \leq k \leq \min_{i=1}^{n}\{|V(G_i)|\}$, let $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{J}^r(G)$ be cross-intersecting families. Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq |\mathcal{J}^{(r)}(G)|.$$

This bound is best possible, and can be obtained by letting $\mathcal{A}_1 = \mathcal{J}^r(G)$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_k = \emptyset$.

1.1 Cross-intersecting pairs

We now restrict our attention to *cross-intersecting pairs* in $\mathcal{J}^r(G)$, i.e. we fix $k = 2$. The following corollary of Theorem 1.3 is immediately apparent.

Corollary 1.5. *Let $r \leq n$. Let $(\mathcal{A}, \mathcal{B})$ be a cross-intersecting pair in $\mathcal{J}^r(M_n)$. Then,*

$$|\mathcal{A}| + |\mathcal{B}| \leq 2^r \binom{n}{r}.$$

If $r < n$, then equality holds if and only if $\mathcal{A} = \mathcal{J}^r(M_n)$ and $\mathcal{B} = \emptyset$ (or vice-versa).

We give an alternative proof of Corollary 1.5. The bound in the statement of Corollary 1.5 will follow immediately from Theorem 1.4, while a theorem of Bollobás and Leader [1] is used to characterize the extremal structures.

We now consider this problem for a larger class of graphs, but with a slightly stronger restriction on r. A graph G is *chordal* if it has no induced cycles on more than 3 vertices. For a graph G, let $\mu = \mu(G)$ be the minimum size of a maximal independent set in G. We prove the following theorem for chordal graphs.

Theorem 1.6. *Let G be a chordal graph and let $r \leq \mu(G)/2$. Then for any cross-intersecting pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{J}^r(G)$, $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{J}^r(G)|$.*

We conjecture that the statement of Theorem 1.6 should hold for all graphs.

Conjecture 1.7. *Let G be a graph and let $r \leq \mu(G)/2$. If $(\mathcal{A}, \mathcal{B})$ is a cross-intersecting pair in $\mathcal{J}^r(G)$, then $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{J}^r(G)|$.*

We end by proving Conjecture 1.7 when $G = C_n$, the cycle on $n \geq 2$ vertices[1], which is non-chordal when $n \geq 4$. In fact we prove the following stronger statement.

Theorem 1.8. *For $r \geq 1$, $n \geq 2$, and any cross-intersecting pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{J}^r(C_n)$, $|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{J}^r(G)|$.*

The main tools we use to prove Theorems 1.6 and 1.8 are variations of the well-known shifting technique, suitably modified according to the structural properties of the respective graphs. These tools were first employed by Talbot [13] to prove an Erdős–Ko–Rado theorem for uniform families of independent sets of a cycle, and later extended by Holroyd, Spencer and Talbot [10] to prove similar intersection theorems for other special classes of graphs. Frankl [8] presents a more general survey of the shifting technique, particularly as applied to theorems in extremal set theory.

2 Disjoint union of complete graphs

We start by giving a proof of Theorem 1.4. We will use a strategy of Borg [2] in conjunction with a result of Holroyd, Spencer and Talbot [10]. The strategy is to construct an intersecting family from a collection of cross-intersecting families and obtain the cross-intersection result by invoking the result in [10], the full statement of which we recall below.

Theorem 2.1 (Holroyd-Spencer-Talbot [10])**.** *Let G be a disjoint union of $n \geq r$ complete graphs, each on at least 2 vertices. If $\mathcal{A} \subseteq \mathcal{J}^r(G)$ is intersecting, then $|\mathcal{A}| \leq \max_{x \in V(G)} |\mathcal{J}^r_x(G)|$.*

Proof of Theorem 1.4. Let G be a disjoint union of n complete graphs $G_1, \ldots, G_n$ with $|V(G_i)| \geq 2$ for each $i \in [n]$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be cross-intersecting subfamilies of $\mathcal{J}^r(G)$, with $r \leq n$ and $2 \leq k \leq \min_{i=1}^n\{|V(G_i)|\}$.

We create an auxiliary graph $G' = G \cup G_{n+1}$ where $G_{n+1} = K_k$, the complete graph on k vertices and $V(G_{n+1}) = \{v_1, \ldots, v_k\}$. Let $V(G') = V(G) \cup V(G_{n+1})$ and $E(G') = E(G) \cup E(G_{n+1})$. For each $1 \leq i \leq k$, let $\mathcal{A}'_i = \{A \cup \{v_i\} : A \in \mathcal{A}_i\}$. Let $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}'_i$. Clearly, $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}'_i| = \sum_{i=1}^k |\mathcal{A}_i|$ and $\mathcal{A}' \subseteq \mathcal{J}^{r+1}(G')$. We now prove that $\mathcal{A}'$ is intersecting.

Claim 2.2. *$\mathcal{A}'$ is intersecting.*

Proof. Let $A, B \in \mathcal{A}'$. If $A, B \in \mathcal{A}'_i$ for some $i \in [k]$, then $v_i \in A \cap B$, so assume $A \in \mathcal{A}'_i$ and $B \in \mathcal{A}'_j$ for some $i \neq j$. For $A' = A \setminus \{v_i\}$ and $B' = B \setminus \{v_j\}$, we have $A' \in \mathcal{A}_i$ and $B' \in \mathcal{A}_j$, which implies $A' \cap B' \neq \emptyset$. This gives $A \cap B \neq \emptyset$ as required. ◇

[1] For $n = 2$, we define C_n to be a solitary edge.

Using Theorem 2.1 and Claim 2.2, we get $|\mathcal{A}'| \leq |\mathcal{J}_x^{r+1}(G')|$, where x is any vertex in a component with the smallest number of vertices. In particular we can let $x \in V(G_{n+1})$, since $k \leq \min_{i=1}^{n}\{|V(G_i)|\}$. This gives us $|\mathcal{J}_x^{r+1}(G')| = |\mathcal{J}^r(G)|$, completing the proof of the theorem. □

We can now use Theorem 1.4, and an argument similar to the one used in the proof of Theorem 1.4, to give the following short alternative proof of Corollary 1.5. We will require the following result of Bollobás and Leader [1] to characterize the extremal structures.

Theorem 2.3 (Bollobás-Leader)**.** *Let $r \leq n$ and suppose $\mathcal{A} \subseteq \mathcal{J}^r(M_n)$ is intersecting. Then $|\mathcal{A}| \leq 2^{r-1}\binom{n-1}{r-1}$. If $r < n$, then equality holds if and only if $\mathcal{A} = \mathcal{J}_x^r(M_n)$ for some $x \in V(M_n)$.*

Proof of Corollary 1.5. It is clear that when $k = 2$, the bound in Corollary 1.5 follows immediately from Theorem 1.4. So suppose that $r < n$ and $|\mathcal{A}|+|\mathcal{B}| = 2^r\binom{n}{r}$. Assume $\mathcal{A}'$ is defined as in the proof of Theorem 1.4, so $\mathcal{A}' \subseteq \mathcal{J}^{r+1}(M_{n+1})$ is intersecting. Let v_1v_2 be the edge added to M_n to obtain M_{n+1}. Now $|\mathcal{A}'| = |\mathcal{A}| + |\mathcal{B}| = 2^r\binom{n}{r}$. By using the characterization of equality in Theorem 2.3, we get $\mathcal{A}' = \mathcal{J}_x^{r+1}(M_{n+1})$ for some $x \in V(M_{n+1})$. But by the construction of $\mathcal{A}'$, every set in $\mathcal{A}'$ contains either v_1 or v_2, so $x \in \{v_1, v_2\}$ (since $r < n$). Without loss of generality, let $x = v_1$. This implies that no set in $\mathcal{A}'$ contains v_2. Thus we get $\mathcal{A} = \mathcal{J}^r(M_n)$ and $\mathcal{B} = \emptyset$. ◇

3 Chordal graphs

In this section, we prove Theorem 1.6. The technique we employ is modeled on the one used by Borg and Holroyd [4] to prove Erdős–Ko–Rado theorems for certain special classes of graphs, and our argument will closely follow the proof of an Erdős–Ko–Rado theorem for chordal graphs by Hurlbert and Kamat [11].

We begin by fixing some notation. For a graph G and a vertex $v \in V(G)$, let $G - v$ be the graph obtained from G by removing vertex v. Also let $G \downarrow v$ denote the graph obtained by removing v and its set of neighbors from G. We now recall an important characterization of chordal graphs, due to Dirac [6].

Definition 3.1. *A vertex v is called simplicial in a graph G if its neighborhood is a clique in G.*

Consider a graph G on n vertices, and let $\sigma = [v_1, \ldots, v_n]$ be an ordering of the vertices of G. Let the graph G_i be the subgraph obtained by removing the vertex set $\{v_1, \ldots, v_{i-1}\}$ from G. Then σ is called a *simplicial elimination ordering* if v_i is simplicial in the graph G_i, for each $1 \leq i \leq n$.

Theorem 3.2 (Dirac [6])**.** *A graph G is a chordal graph if and only if it has a simplicial elimination ordering.*

We state and prove two facts regarding the graph parameter μ. We note that the proofs of the second parts of both Lemma 3.3 and Corollary 3.4 appear in [4] and the complete proofs of both of these facts also appear in [11]. For the sake of completeness, we reproduce them here. For a vertex $v \in V(G)$, let $N[v] = \{u \in V(G) : u = v \text{ or } uv \in E(G)\}$.

Lemma 3.3. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following inequalities hold:*

1. $\mu(G - v_2) \geq \mu(G)$;
2. $\mu(G \downarrow v_2) + 1 \geq \mu(G)$.

Proof. We begin by noting that the condition $N[v_1] \subseteq N[v_2]$ implies that $v_1 v_2 \in E(G)$.

1. We will show that if I is a maximal independent set in $G - v_2$, then I is also maximally independent in G. Suppose I is not maximally independent in G. Then $I \cup \{v_2\}$ is an independent set in G. Thus for any $u \in N[v_2]$, $u \notin I$. In particular, for any $u \in N[v_1]$, $u \notin I$. Hence $I \cup \{v_1\}$ is an independent set in $G - v_2$. This is a contradiction. Thus I is a maximal independent set in G.

   Taking I to be the smallest maximal independent set in $G - v_2$, we get $\mu(G - v_2) = |I| \geq \mu(G)$.

2. We will show that if I is a maximal independent set in $G \downarrow v_2$, then $I \cup \{v_2\}$ is a maximal independent set in G. Clearly $I \cup \{v_2\}$ is independent, so suppose it is not maximal. Then for some vertex $u \in G \downarrow v_2$ and $u \notin I \cup \{v_2\}$, $I \cup \{u, v_2\}$ is an independent set. Thus $I \cup \{u\}$ is an independent set in $G \downarrow v_2$, a contradiction.

   Taking I to be the smallest maximal independent set in $G \downarrow v_2$, we get $\mu(G \downarrow v_2) + 1 = |I| + 1 \geq \mu(G)$.

◇

Corollary 3.4. *Let G be a graph, and let $v_1, v_2 \in G$ be vertices such that $N[v_1] \subseteq N[v_2]$. Then the following statements hold:*

1. *If $r \leq \frac{1}{2}\mu(G)$, then $r \leq \frac{1}{2}\mu(G - v_2)$;*
2. *If $r \leq \frac{1}{2}\mu(G)$, then $r - 1 \leq \frac{1}{2}\mu(G \downarrow v_2)$.*

Proof. 1. This follows trivially from the first part of Lemma 3.3.

2. To prove this part, we use the second part of Lemma 3.3 to show

$$r - 1 \leq \frac{1}{2}\mu(G) - 1 = \frac{\mu(G) - 2}{2} \leq \frac{\mu(G \downarrow v_2)}{2} - \frac{1}{2}.$$

◇

We now proceed with the proof of Theorem 1.6. We do induction on r, the base case being $r = 1$. Since $\mu(G) \geq 2$, G has at least two vertices so the bound follows trivially. Let $r \geq 2$ and let G be a chordal graph with $\mu(G) \geq 2r$. We now do induction on $|V(G)|$. If $|V(G)| = \mu(G)$, G is the empty graph on $|V(G)|$ vertices, and we are done by Theorem 1.1. So let $|V(G)| > \mu(G) \geq 2r$. This implies that there is a component of G, say H on at least 2 vertices. It is clear from the definition of chordal graphs that any induced subgraph of a chordal graph is also chordal. So by using Theorem 3.2 for H, we can find a simplicial elimination ordering in H. Let this ordering be $[v_1, \ldots, v_m]$ where $m = |V(H)|$ and let $v_1 v_i \in E(H)$ for some $2 \leq i \leq m$. Let $\mathcal{A}$ and $\mathcal{B}$ be a cross-intersecting pair in $\mathcal{J}^r(G)$.

We define two compression operations $f_{1,i}$ and $g_{1,i}$ for sets in the families $\mathcal{A}$ and $\mathcal{B}$ respectively. Before we give the definitions, we note that $N[v_1] \subseteq N[v_i]$ and that if A is an independent set with $v_i \in A$, then $A \setminus \{v_i\} \cup \{v_1\}$ is also independent.

$$f_{1,i}(A) = \begin{cases} A \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in A, A \setminus \{v_i\} \cup \{v_1\} \notin \mathcal{A} \\ A & \text{otherwise} \end{cases}$$

$$g_{1,i}(B) = \begin{cases} B \setminus \{v_i\} \cup \{v_1\} & \text{if } v_i \in B, v_1 \notin B, B \setminus \{v_i\} \cup \{v_1\} \notin \mathcal{B} \\ B & \text{otherwise} \end{cases}$$

We define $\mathcal{A}' = \{f_{1,i}(A) : A \in \mathcal{A}\}$. We also define $\mathcal{B}'$ in an analogous manner. Note that $|\mathcal{A}'| = |\mathcal{A}|$ and $|\mathcal{B}'| = |\mathcal{B}|$. Next, we define the following families for $\mathcal{A}'$.

$$\mathcal{A}'_i = \{A \in \mathcal{A}' : v_i \in A\},$$

$$\bar{\mathcal{A}}'_i = \mathcal{A}' \setminus \mathcal{A}'_i, \text{ and}$$

$$\mathcal{A}''_i = \{A \setminus \{v_i\} : A \in \mathcal{A}'_i\}.$$

We also define $\mathcal{B}'_i$, $\bar{\mathcal{B}}'_i$ and $\mathcal{B}''_i$ in an identical manner. It is not hard to observe that $|\mathcal{A}| = |\mathcal{A}'| = |\mathcal{A}''_i| + |\bar{\mathcal{A}}'_i|$ and $|\mathcal{B}| = |\mathcal{B}'| = |\mathcal{B}''_i| + |\bar{\mathcal{B}}'_i|$. Consider the pair $(\mathcal{A}''_i, \mathcal{B}''_i)$ and the pair $(\bar{\mathcal{A}}'_i, \bar{\mathcal{B}}'_i)$. We will prove the following lemma about these pairs.

Lemma 3.5. *1. $(\mathcal{A}''_i, \mathcal{B}''_i)$ is a cross-intersecting pair in $\mathcal{J}^{r-1}(G \downarrow v_i)$.*

2. $(\bar{\mathcal{A}}'_i, \bar{\mathcal{B}}'_i)$ is a cross-intersecting pair in $\mathcal{J}^r(G - v_i)$.

Proof. 1. Let $A \in \mathcal{A}''_i$ and $B \in \mathcal{B}''_i$. Then $A_1 = A \cup \{v_i\} \in \mathcal{A}$ and $B_1 = B \cup \{v_i\} \in \mathcal{B}$. Also, $A_2 = A \cup \{v_1\} \in \mathcal{A}$, otherwise A_1 could have been shifted to A_2 by $f_{1,i}$. Since $B_1 \cap A_2 \neq \emptyset$ and $v_1 \notin B_1$ (due to the fact that $v_1 v_i \in E(G)$ and B_1 is independent), we get $A \cap B \neq \emptyset$ as required.

2. Let $A \in \bar{\mathcal{A}}'_i$ and $B \in \bar{\mathcal{B}}'_i$. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we are done, so suppose $A \notin \mathcal{A}$. Then we must have $v_1 \in A$. Assuming $v_1 \notin B$, we get $B \in \mathcal{B}$. Since $(A \setminus \{v_1\} \cup \{v_i\}) \in \mathcal{A}$, we have $(A \setminus \{v_1\} \cup \{v_i\}) \cap B \neq \emptyset$, implying $A \cap B \neq \emptyset$ as required.

$\diamond$

We are now in a position to complete the proof of Theorem 1.6 as follows, using Lemma 3.5. We can use Corollary 3.4 to infer that $G - v_i$ satisfies the induction hypothesis for r and $G \downarrow v_i$ satisfies the induction hypothesis for $r-1$.

$$\begin{aligned} |\mathcal{A}| + |\mathcal{B}| &= (|\bar{\mathcal{A}}_i'| + |\bar{\mathcal{B}}_i'|) + (|\mathcal{A}_i''| + |\mathcal{B}_i''|) \\ &\leq |\mathcal{J}^r(G - v_i)| + |\mathcal{J}^{r-1}(G \downarrow v_i)| \\ &= |\mathcal{J}^r(G)|. \end{aligned} \tag{1}$$

The last equality can be explained by a simple partitioning of the family $\mathcal{J}^r(G)$ based on whether or not a set in the family contains v_i. There are exactly $|\mathcal{J}^{r-1}(G \downarrow v_i)|$ sets which contain v_i and $|\mathcal{J}^r(G - v_i)|$ sets which do not contain v_i. $\square$

4 Cycles

Proof of Theorem 1.8. As mentioned earlier, the main tool we use to prove Theorem 1.8 is a shifting operation first employed by Talbot [13] to prove an EKR theorem for the cycle. Proceeding by induction on r as before with $r = 1$ being the trivial base case, we suppose $r \geq 2$ and do induction on n. The statement is vacuously true when $n \in \{2, 3\}$, so suppose $n \geq 4$. Let $V(C_n) = \{1, \ldots, n\}$ and $E(C_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$. Suppose $(\mathcal{A}, \mathcal{B})$ is a cross-intersecting pair in $\mathcal{J}^r(C_n)$. Consider the graph obtained by contracting the edge $e_1 = \{n-1, n\}$ in C_n. We will identify this contraction by the function $c : [n] \to [n-1]$ defined by $c(n) = n-1$ (and $c(x) = x$ elsewhere), so the resulting graph is C_{n-1}. Similarly identify the graph obtained from C_{n-1} by contracting the edge $e_2 = \{n-2, n-1\}$ as C_{n-2}. We define the following two subfamilies of $\mathcal{A}$. Let $\mathcal{A}_1 = \{A \setminus \{n\} : n-2, n \in A \in \mathcal{A}\}$ and $\mathcal{A}_2 = \{A \setminus \{n-1\} : n-1, 1 \in A \in \mathcal{A}\}$. Define $\mathcal{B}_1$ and $\mathcal{B}_2$ similarly. Now no set in either $\mathcal{A}_1$ or $\mathcal{B}_1$ contains 1. Similarly no set in either $\mathcal{A}_2$ or $\mathcal{B}_2$ contains $n-2$. Moreover, no set in any of the families $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ contains either n or $n-1$. This implies that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{J}^{r-1}(C_{n-2})$. Let $\mathcal{A}_1' = \{A \in \mathcal{A} : n-2, n \in A\}$ and $\mathcal{A}_2' = \{A \in \mathcal{A} : 1, n-1 \in A\}$, with $\mathcal{B}_1'$ and $\mathcal{B}_2'$ defined similarly. We consider the families $\mathcal{A}^* = \mathcal{A} \setminus (\mathcal{A}_1' \cup \mathcal{A}_2')$ and $\mathcal{B}^* = \mathcal{B} \setminus (\mathcal{B}_1' \cup \mathcal{B}_2')$. Note that $(\mathcal{A}^*, \mathcal{B}^*)$ is a cross-intersecting pair in $\mathcal{J}^r(C_n)$. We will now define two shifting operations, one for $\mathcal{A}^*$ and one for $\mathcal{B}^*$ with respect to the vertices n and $n-1$.

$$f(A) = \begin{cases} A \setminus \{n\} \cup \{n-1\} & \text{if } n \in A, A \setminus \{n\} \cup \{n-1\} \notin \mathcal{A}^* \\ A & \text{otherwise} \end{cases}$$

$$g(B) = \begin{cases} B \setminus \{n\} \cup \{n-1\} & \text{if } n \in B, B \setminus \{n\} \cup \{n-1\} \notin \mathcal{B}^* \\ B & \text{otherwise} \end{cases}$$

Let $f(\mathcal{A}^*) = \{f(A) : A \in \mathcal{A}^*\}$ and $f(\mathcal{B}^*) = \{f(B) : B \in \mathcal{B}^*\}$. As before, we partition $f(\mathcal{A}^*)$ (and similarly, $f(\mathcal{B}^*)$) into two parts as follows. Let $\mathcal{A}' = \{A \in f(\mathcal{A}^*) : n \notin A\}$ and let $\mathcal{A}_3 = \{A \setminus \{n\} : A \in f(\mathcal{A}^*) \setminus \mathcal{A}'\}$. We have $\mathcal{A}', \mathcal{B}' \subseteq \mathcal{J}^r(C_{n-1})$. Also $\mathcal{A}_3, \mathcal{B}_3 \subseteq \mathcal{J}^{r-1}(C_{n-2})$ because for any set $S \in \mathcal{A}_3 \cup \mathcal{B}_3$, $S \cap \{1, n-1, n\} = \emptyset$. Let $\tilde{\mathcal{A}} = \bigcup_{i \in [3]} \mathcal{A}_i$ and $\tilde{\mathcal{B}} = \bigcup_{i \in [3]} \mathcal{B}_i$. We consider the pair $(\mathcal{A}', \mathcal{B}')$ in $\mathcal{J}^r(C_{n-1})$ and the pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ in $\mathcal{J}^{r-1}(C_{n-2})$. We first state and prove some claims about these families.

Claim 4.1. *1. Let $A \in \mathcal{A}_3$. Then $A \cup \{n-1\} \in \mathcal{A}^*$.*

2. Let $B \in \mathcal{B}_3$. Then $B \cup \{n-1\} \in \mathcal{B}^*$.

Proof. It suffices to prove the claim for $\mathcal{A}_3$. We know that $A \cup \{n\} \in f(\mathcal{A}^*)$. This means that $A \cup \{n\} \in \mathcal{A}^*$ and $A \cup \{n\}$ was not shifted to $A \cup \{n-1\}$ by f, implying $A \cup \{n-1\} \in \mathcal{A}^*$. ◇

The next claim will show that $\tilde{\mathcal{A}} = \bigcup_{i\in[3]} \mathcal{A}_i$ and $\tilde{\mathcal{B}} = \bigcup_{i\in[3]} \mathcal{B}_i$ are disjoint unions.

Claim 4.2. *1. For any $i, j \in [3]$ with $i \neq j$, $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$.*

2. For any $i, j \in [3]$ with $i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$.

Proof. As before, it suffices to prove the claim for the $\mathcal{A}_i$'s. It is clear from the definitions of $\mathcal{A}_1$ and $\mathcal{A}_2$ that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Since every $(r-1)$-set in $\mathcal{A}_3$ is obtained by removing n from an r-set, no set in $\mathcal{A}_3$ contains 1. So it remains to prove that no set in $\mathcal{A}_3$ contains $n-2$. By the previous claim we know that for any $A \in \mathcal{A}_3$, $A \cup \{n-1\} \in \mathcal{A}^*$. This gives $n-2 \notin A$ as required. ◇

Claim 4.3. *1. $(\mathcal{A}', \mathcal{B}')$ is a cross-intersecting pair in $\mathcal{J}^r(C_{n-1})$.*

2. $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is a cross-intersecting pair in $\mathcal{J}^{r-1}(C_{n-2})$.

Proof.
1. Suppose $A \in \mathcal{A}'$ and $B \in \mathcal{B}'$. If $A \in \mathcal{A}^*$ and $B \in \mathcal{B}^*$, then $A \cap B \neq \emptyset$, so suppose $A \notin \mathcal{A}^*$. This gives $n-1 \in A$. Assume $n-1 \notin B$, so $B \in \mathcal{B}^*$. Since $A_1 = (A \setminus \{n-1\} \cup \{n\}) \in \mathcal{A}^*$, we have $A_1 \cap B \neq \emptyset$, which gives $A \cap B \neq \emptyset$.

2. Let $A \in \tilde{\mathcal{A}}$ and $B \in \tilde{\mathcal{B}}$. So $A \in \mathcal{A}_i$ and $B \in \mathcal{B}_j$ for some $i, j \in [3]$. First consider the case when $i = j$. Each set in $\mathcal{A}_1$ and $\mathcal{B}_1$ has $n-2$, while each set in $\mathcal{A}_2$ and $\mathcal{B}_2$ has 1, so let $A \in \mathcal{A}_3$ and $B \in \mathcal{B}_3$. We have $A \cup \{n\} \in \mathcal{A}^*$. Also, $B \cup \{n-1\} \in \mathcal{B}^*$ by Claim 4.1, so $(A \cup \{n\}) \cap (B \cup \{n-1\}) \neq \emptyset$, giving $A \cap B \neq \emptyset$ as required. Next, let $i \neq j$. We only consider cases when $i < j$, since the other cases follow identically. Suppose $i = 1$ and $j = 2$. In this case we have $(A \cup \{n\}) \in \mathcal{A}$, $(B \cup \{n-1\}) \in \mathcal{B}$, which gives $A \cap B \neq \emptyset$. If $i = 1$ and $j = 3$, we again have $A \cup \{n\} \in \mathcal{A}$ while Claim 4.1 implies $B \cup \{n-1\} \in \mathcal{B}$, giving $A \cap B \neq \emptyset$. Similarly for $i = 2$ and $j = 3$ we have $A \cup \{n-1\} \in \mathcal{A}$ and $B \cup \{n\} \in \mathcal{B}$.

◇

The final claim we prove is regarding the size of $\mathcal{J}^r(C_n)$.

Claim 4.4. $|\mathcal{J}^r(C_n)| = |\mathcal{J}^r(C_{n-1})| + |\mathcal{J}^{r-1}(C_{n-2})|$.

Proof. Consider all sets in $\mathcal{J}^r(C_n)$ which contain neither n nor both $n-1$ and 1. The number of these sets is clearly $|\mathcal{J}^r(C_{n-1})|$. Now consider the subfamily containing the remaining sets, i.e. those which either have n or both 1 and $n-1$. Call it $\mathcal{F}$. We define the following correspondence between $\mathcal{F}$ and $\mathcal{J}^{r-1}(C_{n-2})$. For $A \in \mathcal{F}$, define $f(A) = A - \{n\}$ if $n \in A$ and $f(A) = A - \{n-1\}$ if $1, n-1 \in A$. Clearly $f(A) \in \mathcal{J}^{r-1}(C_{n-2})$ and f is bijective, giving $|\mathcal{F}| = |\mathcal{J}^{r-1}(C_{n-2})|$ as required. ◇

We can now finish the proof of Theorem 1.8 as follows, using Claim 4.2, Claim 4.3 and the inductive hypothesis. The final equality follows from Claim 4.4.

$$\begin{aligned} |\mathcal{A}|+|\mathcal{B}| &= |\mathcal{A}^*|+|\mathcal{B}^*|+\sum_{i=1}^{2}(|\mathcal{A}_i|+|\mathcal{B}_i|) \\ &= (|\mathcal{A}'|+|\mathcal{B}'|)+\sum_{i=1}^{3}(|\mathcal{A}_i|+|\mathcal{B}_i|) \\ &= (|\mathcal{A}'|+|\mathcal{B}'|)+(|\tilde{\mathcal{A}}|+|\tilde{\mathcal{B}}|) \\ &\leq |\mathcal{J}^r(C_{n-1})|+|\mathcal{J}^{r-1}(C_{n-2})| \\ &= |\mathcal{J}^r(C_n)|. \end{aligned} \tag{2}$$

□

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