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Optimal orientation of the tensor product of a small diameter graph and a complete graph

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Abstract

For a graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. The *orientation number* of G, denoted by $\vec{d}(G)$, is defined as $\min\{d(D) \mid D \in \mathcal{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, a sufficient condition is given for $\vec{d}(G \times K_n) = d(G \times K_n)$, where G is a graph with $d(G)$ two or three and $\times$ is the tensor product of graphs.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity,* denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\{d_G(v,x) \mid x \in V(G)\}$, where $d_G(v,x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e_G(v) \mid v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, arcs with same tail and same head). For $v \in V(D)$, the notions $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $x, y \in V(D)$, we write $x \to y$ or $y \leftarrow x$ if $(x,y) \in A(D)$. For sets $X, Y \subseteq V(D)$, $X \to Y$ denotes $\{(x,y) \in A(D) : x \in X \text{ and } y \in Y\}$. For distinct vertices $v_1, v_2, \ldots, v_k$, $v_1 \to v_2 \to \cdots \to v_k$ represents the directed path in D with arcs $v_1 \to v_2$, $v_2 \to v_3$, $\ldots$, $v_{k-1} \to v_k$. For subsets $V_1, V_2, \ldots, V_k$ of V, we write $V_1 \to V_2 \to \cdots \to V_k$ for the set of all directed paths of length $k-1$ whose ith vertex is in V_i, $1 \le i \le k$. For $x \in V(D)$ and $V' \subseteq V(D)$, by $d_D(x, V') \le k$, we mean $d_D(x, v') \le k$, for all $v' \in V'$.

For graphs G and H, the *tensor product*, $G \times H$, of G and H, is the graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u,v)(x,y) : ux \in E(G) \text{ and } vy \in$

$E(H)\}$. For $x \in V(G)$, the *G-layer* G_x is the subset $\{(x, y) : y \in V(H)\}$ of vertices of $G \times H$, and similarly, for $y \in V(H)$, the *H-layer* H_y of $G \times H$ is $\{(x, y) : x \in V(G)\}$.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edges. By abuse of notation, by D we mean an orientation of G and also the digraph arising out of the orientation of G.

A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' celebrated one-way street theorem [7] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. Throughout this paper, whenever an orientation of a graph G is considered, we assume that G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathcal{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G, denoted by $\vec{d}(G)$, is defined as $\min\{d(D) \mid D \in \mathcal{D}(G)\}$. In [3], $\vec{d}(G) - d(G)$ is defined as $\rho(G)$. Any orientation D in $\mathcal{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. The problem of evaluating the orientation number of an arbitrary connected graph is very difficult, as Chvátal and Thomassen [2] have shown that the problem of deciding whether a graph admits an orientation of diameter 2 is NP-hard. For results on orientations of graphs, see a survey by Koh and Tay [3].

Let P_n, C_n and K_n denote the path, cycle and complete graph of order n, respectively. Notation and terminology not defined here can be seen in [1].

Let n be a positive integer and let L be a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. A *circulant* $X(n; L)$ is a simple graph with vertex set $V(X(n; L)) = \mathbb{Z}_n$ and edge set $E(X(n; L)) = \{\{i, i + \ell\} : i \in \mathbb{Z}_n, \ell \in L\}$, where $\mathbb{Z}_n$ is the set of integers modulo n.

Except for few pairs (r, s), we have evaluated $\rho(G \times H)$ for combinations of graphs including $K_r \times K_s$, $P_r \times K_s$ and $C_r \times K_s$; see [4], [5] and [6]. Some results were obtained in [5] for tensor products of circulant graphs with some special graphs. In the sequel, we assume that G is a graph with $d(G) = 2$ or 3. In this paper, we find a sufficient condition for $\rho(G \times K_n) = 0$.

2 Optimal orientations of $G \times K_n$

A subset $\hat{L}$ of $\mathbb{Z}_n$ is called a $\mathbb{Z}_n^2$-*set*, if when $j \in \hat{L}$, then $(n - j) \bmod n \notin \hat{L}$ and every element $i \in \mathbb{Z}_n \backslash \{0\}$ can be written as $(a_1 + a_2) \bmod n$ for some $a_1, a_2 \in \hat{L}$. For a $\mathbb{Z}_n^2$-set $\hat{L}$, we associate a set L as follows: $L = \{i \in \hat{L} : 0 < i < \frac{n}{2}\} \cup \{n - i : i \in \hat{L} \text{ and } \frac{n}{2} < i < n\}$.

An unordered pair $\{A, B\}$ of distinct subsets of $\mathbb{Z}_n$ is called a $\mathbb{Z}_n^2$-*pair*, if both A and B are $\mathbb{Z}_n^2$-sets and every element $i \in \mathbb{Z}_n$ can be written as $(a + b) \pmod n$ for some $a \in A$ and $b \in B$. The set $\mathcal{F} = \{A_1, A_2, \ldots, A_r\}$ of distinct subsets of $\mathbb{Z}_n$ is called a $\mathbb{Z}_n^2$-*family*, if $\{A_i, A_j\}$ is a $\mathbb{Z}_n^2$-pair, for every $i, j \in \{1, 2, \ldots, r\}$ with $i \neq j$.

For $x \in \mathbb{Z}_n$ and $A \subseteq \mathbb{Z}_n$, we define $x+A = \{(x+a) \bmod n : a \in A\}$. For $a, b \in \mathbb{Z}_n$, $[a, b] = \{a, (a+1) \bmod n, (a+2) \bmod n, \ldots, b\}$.

Lemma 2.1. *For each $i \in \{1, 2, \ldots, \lfloor \frac{n-5}{2} \rfloor\}$, let $F_i = [i+1, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil\} \cup [n-i, n-1]$. For $n \geq 15$ or $n \in \{11, 13\}$, the family $\mathcal{F} = \{F_i : i \in \{1, 2, \ldots, \lfloor \frac{n-5}{2} \rfloor\}\}$ is a $\mathbb{Z}_n^2$-family.*

Proof. Claim 1. F_i is a $\mathbb{Z}_n^2$-set.

Let

$$\begin{aligned}
A &= [i+1, \lfloor \tfrac{n-3}{2} \rfloor] + [n-i, n-1] = [1, \lfloor \tfrac{n-5}{2} \rfloor], \\
B &= [i+1, \lfloor \tfrac{n-3}{2} \rfloor] + \lceil \tfrac{n+1}{2} \rceil = [i + \lceil \tfrac{n+3}{2} \rceil, n-1], \\
C &= [i+1, \lfloor \tfrac{n-3}{2} \rfloor] + [i+1, \lfloor \tfrac{n-3}{2} \rfloor], \\
D &= \lceil \tfrac{n+1}{2} \rceil + [n-i, n-1] = [\lceil \tfrac{n+1}{2} \rceil - i, \lceil \tfrac{n-1}{2} \rceil], \\
E &= [n-i, n-1] + [n-i, n-1] = [n-2i, n-2].
\end{aligned}$$

Note that $C = [2i+2, n-4]$ if n is even and $C = [2i+2, n-3]$ if n is odd.

If $i \leq \lfloor \frac{n-7}{4} \rfloor$, then $A \cup C \cup B = [1, n-1]$ and if $i \geq \lceil \frac{n+3}{4} \rceil$, then $A \cup E = [1, n-2]$ and $\lfloor \frac{n-3}{2} \rfloor + \lceil \frac{n+1}{2} \rceil = n-1$.

If $i = \lfloor \frac{n-3}{4} \rfloor$, then $A \cup D \cup C \cup B = [1, n-1]$ and if $i = \lceil \frac{n-1}{4} \rceil$, then $A \cup D \cup E \cup B = [1, n-1]$. Finally, if $n \equiv 2 \pmod 4$ and $i = \frac{n-2}{4}$, then $A \cup D \cup C \cup B = [1, n-1]$.

Thus we have Claim 1.

Claim 2. $\{F_i, F_j\}$ is a $\mathbb{Z}_n^2$-pair.

Assume that $i < j$. From the notation, $F_i = [i+1, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil\} \cup [n-i, n-1]$ and $F_j = [j+1, \lfloor \frac{n-3}{2} \rfloor] \cup \{\lceil \frac{n+1}{2} \rceil\} \cup [n-j, n-1]$. Let $A_{ij} = [i+1, \lfloor \frac{n-3}{2} \rfloor] + [j+1, \lfloor \frac{n-3}{2} \rfloor]$ (A_{ij} is $[i+j+2, n-4]$ if n is even and is $[i+j+2, n-3]$ if n is odd), $B_{ij} = [i+1, \lfloor \frac{n-3}{2} \rfloor] + \{\lceil \frac{n+1}{2} \rceil\} = [i + \lceil \frac{n+1}{2} \rceil + 1, n-1]$, $C_{ij} = [i+1, \lfloor \frac{n-3}{2} \rfloor] + [n-j, n-1] = [n+i-j+1, \lfloor \frac{n-5}{2} \rfloor]$, $D_{ij} = \{\lceil \frac{n+1}{2} \rceil\} + [n-j, n-1] = [\lceil \frac{n+1}{2} \rceil - j, \lceil \frac{n-1}{2} \rceil]$ and $E_{ij} = [n-i, n-1] + [n-j, n-1] = [n-i-j, n-2]$.

If $i+j \leq \lfloor \frac{n-7}{2} \rfloor$, then $C_{ij} \cup A_{ij} \cup B_{ij} = [0, n-1]$ and if $i+j \geq \lceil \frac{n+3}{2} \rceil$, then $C_{ij} \cup E_{ij} \supseteq [0, n-2]$ and $n-1 \in B_{ij}$.

$C_{ij} \cup D_{ij} \cup A_{ij} \cup B_{ij} = [0, n-1]$ if $i+j \in \{\lfloor \frac{n-5}{2} \rfloor, \lfloor \frac{n-3}{2} \rfloor\}$ or n is even and $i+j = \frac{n-2}{2}$. Also $C_{ij} \cup D_{ij} \cup E_{ij} \supseteq [0, n-2]$ and $n-1 \in B_{ij}$, if $i+j \in \{\lceil \frac{n-1}{2} \rceil, \lceil \frac{n+1}{2} \rceil\}$.

This completes the proof of Claim 2. ■

For $n = 9$, $\{\{1,2,4,6\}, \{1,5,6,7\}, \{4,6,7,8\}\}$ is a $\mathbb{Z}_9^2$-family; for $n = 12$, $\{\{1,2,3,5,8\}, \{1,2,4,5,9\}, \{1,3,4,5,10\}\}$ is a $\mathbb{Z}_{12}^2$-family and for $n = 14$, $\{\{1,2,3,4,6,9\}, \{1,2,3,5,6,10\}, \{1,2,4,5,6,11\}, \{1,3,4,5,6,12\}\}$ is a $\mathbb{Z}_{14}^2$-family. Thus we have:

Lemma 2.2. *For each $n \geq 11$, there exists a $\mathbb{Z}_n^2$-family of cardinality $\lfloor \frac{n-5}{2} \rfloor$, and for $n = 9$, there exists a $\mathbb{Z}_9^2$-family of cardinality 3.*

Theorem 2.1. *Let G be a graph and let $\mathcal{F}$ be a $\mathbb{Z}_n^2$-family containing $\chi'(G)$ sets, where $\chi'(G)$ is the edge-chromatic number of G. If $d(G) \in \{2,3\}$, then $\vec{d}(G \times K_n) = d(G \times K_n)$, and hence $\rho(G \times K_n) = 0$.*

Proof. Let $\chi'(G) = r$, $\mathcal{F} = \{A_1, A_2, \ldots, A_r\}$ be a $\mathbb{Z}_n^2$-family and let $(E_1, E_2, \ldots, E_r)$ be an r-edge colouring of G. Orient $G \times K_n$ so that for every edge uv of G, $(u,i) \to \{(v, i+\ell) : \ell \in A_j\}$ whenever $uv \in E_j$. Let D be the resulting digraph. Observe that, for $n \geq 3$:

(i) if $d(G) = 3$, then $d(G \times K_n) = 3$;

(ii) if $d(G) = 2$ and there is an edge of G which is not in a triangle of G, then $d(G \times K_n) = 3$;

(iii) if $d(G) = 2$ and every edge of G is in a triangle of G, then $d(G \times K_n) = 2$.

We shall show that $d(D) = d(G \times K_n)$. This implies that $\rho(G \times K_n) = 0$. To show that $d(D) = d(G \times K_n)$, it is enough to determine the eccentricity $e_D((x,i))$ for each (x,i) in $V(D)$. By the nature of the orientation, we need only consider $(x,0)$.

First we claim that, for $i \in \mathbb{Z}_n \backslash \{0\}$, $d_D((x,0),(x,i)) \leq 2$.

Let $xy \in E(G)$. Now $xy \in E_j$ for some $j \in \{1,2,\ldots,r\}$. Since A_j is a $\mathbb{Z}_n^2$-set, $i = (a+b) \pmod{n}$ for some $a, b \in A_j$. The existence of the path $(x,0) \to (y,a) \to (x,i)$ in D proves the claim.

Next, we consider $d_D((x,0),(z,i))$, for $z \in V(G)$, $z \neq x$, and $i \in \mathbb{Z}_n$.

Since $d(G) \in \{2,3\}$, there exists an (x,z)-path $x = v_0, v_1, \ldots, v_{k-1}, v_k = z$ of length k (≤ 3) in G.

Case 1. $k = 2$.

Since xv_1 and v_1z are adjacent edges, $xv_1 \in E_s$ and $v_1z \in E_t$ for some $s, t \in \{1,2,\ldots,r\}$ with $s \neq t$. Since $\{A_s, A_t\}$ is a $\mathbb{Z}_n^2$-pair, any element of $\mathbb{Z}_n$ can be expressed as $c+d$ for some $c \in A_s$ and $d \in A_t$. The existence of the path $(x,0) \to (v_1,c) \to (z, c+d)$ in D shows that $d_D((x,0),G_z) \leq 2$.

Case 2. $k \neq 2$ and $d(G) = 3$ or $d(G) = 2$ and there is an edge of G which is not in a triangle of G.

If $k = 1$, then by the above claim, $d_D((x,0),G_x) \leq 2$ and hence $d_D((x,0),G_z) \leq 3$. If $k = 3$, then by Case 1, $d_D((x,0),G_{v_2}) \leq 2$ and hence $d_D((x,0),G_z) \leq 3$.

Case 3. $k \neq 2$, $d(G) = 2$ and every edge of G is in a triangle of G.

In this case $k = 1$, and so xz is an edge of G. Then, by hypothesis, there is a vertex y such that $xyzx$ is a triangle of G. By Case 1, $d_D((x,0),G_z) \leq 2$.

Thus $d(D) = d(G \times K_n)$. ■

Corollary 2.1. *Let G be a graph and let $d(G) \in \{2,3\}$. If $n \geq \max\{2\chi'(G)+5, 15\}$, then $\rho(G \times K_n) = 0$.*

Corollary 2.2. *Let G be a graph and let $d(G) \in \{2,3\}$. If $\chi'(G) = 3$, then $\rho(G \times K_n) = 0$ for $n \geq 11$ or $n = 9$. If $\chi'(G) = 4$, then $\rho(G \times K_n) = 0$ for $n \geq 13$.*

Corollary 2.3. *Let $n \geq 11$ or $n = 9$. Then $\rho(Q_3 \times K_n) = 0$, where Q_3 denotes the* 3-*cube.*

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