DEFICIENCIES OF CONNECTED REGULAR TRIANGLE FREE GRAPHS

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ABSTRACT

Let G be a finite simple graph having a maximum matching M. The deficiency def(G) of G is the number of M-unsaturated vertices in G. In an earlier paper we determined an upper bound for def(G) when G is regular and connected. This upper bound is in general not sharp when G is triangle free. In this paper we study the case when G is triangle free and r-regular. We present an upper bound for def(G) and determine the set of all possible values of def(G) when G is r-regular and (r-2)-edge-connected.

1. INTRODUCTION

In this paper the graphs are finite, loopless and have no multiple edges. For the most part our notation and terminology follow Bondy and Murty [2]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges. However we denote the complement of G by $\bar{G}$.

A **matching** M in G is a subset of E(G) in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M′ of G. A vertex v is **saturated** by M if some edge of M is incident with v; otherwise v is said to be **unsaturated**. A matching M is **perfect** if it saturates every vertex of the graph. The **deficiency** def(G) of G is the number of vertices unsaturated by a maximum matching M of G. Observe that def(G) = $\nu(G) - 2|M|$. Consequently, def(G) has the same parity as $\nu(G)$, and def(G) = 0 if and only if G has a perfect matching.

Many problems concerning matchings and deficiency in graphs have been investigated in the literature - see, for example Lovász and Plummer [6]. We have studied the function def(G) for: the case when G is a tree with each vertex having degree 1 or k, $k \geq 2$ [3]; the case when G is a cubic graph [4]; and the more general case when G is r-regular [5].

It is convenient to let $\mathcal{G}(n,r,k)$ denote the class of r-regular, k-edge-connected graphs on n vertices. The set of triangle free members of $\mathcal{G}(n,r,k)$ is denoted by $\mathcal{G}'(n,r,k)$.

In [4] we obtained the set of all possible values of def(G) when $G \in \mathcal{G}(n,r,k)$ for $k \geq 2$. In this paper we focus on the problem of determining the set of all possible values of def(G) when $G \in \mathcal{G}'(n,r,k)$ for $r \geq 4$; the case r=3 was resolved in [4]. Here we resolve this problem when k = r-2 and present an upper bound on def(G) for the general case.

2. UPPER BOUND

An upper bound for def(G) when $G \in \mathcal{G}(n,r,k)$ was determined in [5]. This bound is generally not sharp when G is triangle free. In this section we present an upper bound for def(G) when $G \in \mathcal{G}'(n,r,k)$ which is sharp for k = r-2.

The following lemma is easily established by simple counting and application of Turan's theorem.

Lemma 2.1: Let $G \in \mathcal{G}'(n,r,1)$, $r \geq 3$ and $S \subset V(G)$. If G_0 is an odd component of $G-S$ which is joined to S by at most $r-2$ edges, then $\nu(G_0) \geq 2r+1$. □

Our next lemma was proved in [5].

Lemma 2.2: Let G be an r-regular, connected graph having $\text{def}(G) \neq 1$. Suppose that for any $\phi \neq V_1 \subset V(G)$ every odd component of $G - V_1$ is joined to V_1 by not less than m edges, $1 \leq m \leq r-2$ ($m \equiv r(\text{mod } 2)$). Then there exists a non-empty set $S \subset V(G)$ such that $G-S$ has $\ell \geq \frac{r}{r-m}\,\text{def}(G)$ odd components joined to S by at most $r-2$ edges. □

We now establish an upper bound on def(G).

Theorem 2.1: Let $G \in \mathcal{G}'(n,r,1)$, $r \geq 4$. If for any non-empty set $S \subset V(G)$ every odd component of $G-S$ is joined to S by at least m edges, where $1 \leq m \leq r-2$ and $m \equiv r(\text{mod } 2)$, then

(a) $\text{def}(G) \leq 2\left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor\right\rfloor$, if n is even ;

(b) $\text{def}(G) = 1$, if n is odd and $n < \frac{2r^2 + r+m}{r} \left\lceil \frac{3r}{r-m} \right\rceil$;

(c) $\text{def}(G) \leq 1 + 2\left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor - \frac{1}{2} \right\rfloor$, otherwise.

Proof: The result is trivially true when $\text{def}(G) = 0$ and also when $\text{def}(G) = 1$ as in this case n must be odd. So suppose $\text{def}(G) \geq 2$. Lemma 2.2 implies that there exists a non-empty set $S \subset V(G)$ such that $G - S$ has

$$\ell \geq \frac{r}{r-m}\,\text{def}(G) \tag{2.1}$$

odd components, $G_1, G_2, \ldots, G_\ell$ say, each of which is joined to S by at most r-2 edges.

By simply counting the edges between S and these odd components we can conclude that $r|S| \geq \ell m$ and hence $|S| \geq \frac{\ell m}{r}$. Now

$$n \geq |S| + \sum_{i=1}^{\ell} \nu(G_i) \geq \frac{\ell m}{r} + \ell(2r+1) . \qquad \text{(Lemma 2.1)}$$

Consequently

$$\ell \leq \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor . \tag{2.2}$$

(2.1) and (2.2) together yield

$$\mathrm{def}(G) \leq \frac{r-m}{r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor .$$

Now when n is even, def(G) must be even and thus we can write

$$\mathrm{def}(G) \leq 2\left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor\right\rfloor$$

proving (a). When n is odd, def(G) must be odd and hence

$$3 \leq \mathrm{def}(G) \leq \frac{r-m}{r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor .$$

Therefore

$$\left\lceil \frac{3r}{r-m} \right\rceil \leq \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor ,$$

and hence

$$n \geq \frac{2r^2 + r+m}{r} \left\lceil \frac{3r}{r-m} \right\rceil = n_0 .$$

Thus if $n < n_0$, then def(G) = 1, proving (b). If $n \geq n_0$, then we can write

$$\mathrm{def}(G) \leq 1 + 2\left\lfloor \frac{r-m}{2r} \left\lfloor \frac{rn}{2r^2 + r+m} \right\rfloor - \frac{1}{2} \right\rfloor ,$$

proving (c). This completes the proof of the theorem. □

For the case when $G \in \mathcal{G}'(n,r,k)$ we have the following corollaries of Theorem 2.1.

Corollary 1: Let $G \in \mathcal{G}'(n,r,k)$ with $r \geq 4$ and $1 \leq k \leq r-2$. Then

(a) $\operatorname{def}(G) \leq 2\left\lfloor \frac{r-k'}{2r} \left\lfloor \frac{rn}{2r^2 + r+k'} \right\rfloor\right\rfloor$, if n is even ;

(b) $\operatorname{def}(G) = 1$, if n is odd and $n < \frac{2r^2 + r+k'}{r} \left\lceil \frac{3r}{r-k'} \right\rceil$;

(c) $\operatorname{def}(G) \leq 1 + 2\left\lfloor \frac{r-k'}{2r} \left\lfloor \frac{rn}{2r^2 + r+k'} \right\rfloor - \frac{1}{2} \right\rfloor$, otherwise;

where k′ is the least integer not less than k which has the same parity as r. □

Corollary 2: Let $G \in \mathcal{G}'(n,r,k)$ with $r \geq 4$, $1 \leq k \leq r-2$ and n even. If G has no perfect matching, then

$$n \geq \frac{2r^2 + r+k'}{r} \left\lceil \frac{2r}{r-k'} \right\rceil ,$$

where k′ is the least integer not less than k which has the same parity as r. □

3. THE CLASS $\mathcal{G}'(n,r,r-2)$

In this section we determine the set

$$D(n,r,r-2) = \{\operatorname{def}(G) : G \in \mathcal{G}'(n,r,r-2)\} ,$$

for $r \geq 4$. We begin with some constructions. The graph $A(2n,r)$ is defined as follows. Take the empty graph $\bar{K}_{2n}$ with vertices $u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n$. Form the Hamilton cycle H_i, $1 \leq i \leq \lfloor \frac{1}{2}r \rfloor$, by joining u_j to v_{2i+j-2} and v_{2i+j-1} for each j, $1 \leq j \leq n$; all integers are reduced modulo n when necessary. Define the matching M as

$M = \{u_i v_j : 1 \le i \le n \text{ and } j \equiv i + r-1 \pmod n\}$. Now we define $A(2n,r)$ as

$$A(2n,r) = \begin{cases} \bigcup_{i=1}^{\frac{1}{2}r} H_i & \text{, if } r \text{ is even,} \\ \bigcup_{i=1}^{\frac{1}{2}(r-1)} H_i \cup M & \text{, if } r \text{ is odd.} \end{cases}$$

Observe that $A(2n,r)$ is an r-regular graph with a perfect matching. As to the edge-connectivity of $A(2n,r)$ we have

Lemma 3.1:

(a) $\kappa'(A(2n,r)) = r$, and

(b) $\kappa'(A(2n,r)-x) \ge r-2$ for any vertex x of $A(2n,r)$.

Proof: We prove only (b) as (a) has already been observed by Bollobás and Eldridge [1]. Without loss of generality consider $A(2n,r)-u_1$.

Suppose that $\kappa'(A(2n,r)-u_1) = t < r-2$ and let $(V_1,\bar{V}_1)$ be an edge-cut set of size t with $|V_1| \ge |\bar{V}_1|$. Then $|\bar{V}_1| \ge 2$ and there are two Hamilton paths, P_1 and P_2 say each having exactly one edge of the cut $(V_1,\bar{V}_1)$. Then from the construction of $A(2n,r)$ we have

$$P_1 = v_{2i}\, u_2\, v_{2i+1}\, u_3 \dots v_{2i-1}$$

and $$P_2 = v_{2j}\, u_2\, v_{2j+1}\, u_3 \dots v_{2j-1},$$

for some $i \ne j$. With no loss of generality let $v_{2i} \in V_1$. Then since $|V_1| \ge 2$ we must have $u_2 \in V_1$, as otherwise P_1 would have at least two edges of the cut $(V_1,\bar{V}_1)$. Let $w_z w'_z$, z = 1, 2, be the edge of P_z in the cut $(V_1,\bar{V}_1)$. Then

$$V_1 = \{v_{2i}, u_2, v_{2i+1}, u_3, \ldots, w_1\}$$

and also $$V_1 = \{v_{2j}, u_2, v_{2j+1}, u_3, \ldots, w_2\} .$$

Thus if V_1 contains p of the vertices $v_1, v_2, \ldots, v_n$, then

$$\{v_{2i}, v_{2i+1}, \ldots, v_{2i+p-1}\} = \{v_{2j}, v_{2j+1}, \ldots, v_{2j+p-1}\} .$$

Hence $2j \equiv 2i + t \pmod{n}$ for some positive integer t. Now when $|\bar{V}_1| \geq 2$ we must have $p < n$ and $1 \leq t \leq p - 1$. But then $2j + p - t \equiv 2i + p \pmod{n}$ implying that $v_{2j-t+p} = v_{2i+p} \in V_1$, contradicting the fact that $p < n$. This proves (b). □

The graph $B(2n+1,r)$ on $(2n+1)$ vertices is defined as follows. Take the graph $A(2n,r) - u_1$ and add two new vertices x and y. Join x to y and to each v_i, $1 \leq i \leq \lfloor \frac{1}{2}r \rfloor$, and join y to each v_i, $\lfloor \frac{1}{2}r \rfloor + 1 \leq i \leq r$. Call the resulting graph $B(2n+1,r)$. Note that x and y have degree $\lfloor \frac{1}{2}r \rfloor + 1$ and $\lceil \frac{1}{2}r \rceil + 1$ respectively and every other vertex has degree r. Also $\kappa'(B(2n+1,r)) \geq \frac{1}{2}r$.

The graphs $A(2n,r)$ and $B(2n+1,r)$ are the basic building blocks in our constructions. We next construct a triangle free r-regular graph $G(m,r)$ of odd order $m \geq \frac{5}{2}r$ having deficiency 1. Our construction depends on the parity of $\frac{1}{2}r$.

Consider the graph $A(2n,r)$. Observe that the subgraph of $A(2n,r)$ induced by the vertices $\{u_1, u_2, \ldots, u_{\frac{1}{2}r}, v_{\frac{1}{2}r+1}, v_{\frac{1}{2}r+2}, \ldots, v_r\}$ is the complete bipartite graph $K_{\frac{1}{2}r,\frac{1}{2}r}$ with bipartitioning sets $\{u_1, u_2, \ldots, u_{\frac{1}{2}r}\}$ and $\{v_{\frac{1}{2}r+1}, v_{\frac{1}{2}r+2}, \ldots, v_r\}$. The edges of this subgraph can be partitioned into $\frac{1}{2}r$ disjoint matchings $M_1, M_2, \ldots, M_{\frac{1}{2}r}$. We may take $M_1 = \{u_i v_{\frac{1}{2}r+i} : 1 \le i \le \frac{1}{2}r\}$.

For $\frac{1}{2}r$ odd we form the graph $G(2n + \frac{1}{2}r, r)$ from $A(2n,r) \setminus \{M_1, M_2, \ldots, M_{\frac{1}{2}r}\}$ by adding $\frac{1}{2}r$ new vertices, $w_1, w_2, \ldots, w_{\frac{1}{2}r}$ say, and joining each of these to the r vertices $u_1, u_2, \ldots, u_{\frac{1}{2}r}, v_{\frac{1}{2}r+1}, v_{\frac{1}{2}r+2}, \ldots, v_r$. Observe that $G(2n + \frac{1}{2}r, r)$ is a triangle free graph on $2n + \frac{1}{2}r$ vertices that is r-regular. We will later establish that this graph is r-edge-connected.

Now we consider the case when $\frac{1}{2}r$ is even. Recall that

$$H_{\frac{1}{4}r+1} = u_1 v_{\frac{1}{2}r+2} u_2 v_{\frac{1}{2}r+3} u_3 \ldots v_{\frac{1}{2}r+1} u_1$$

and thus $M_1 \subseteq H_{\frac{1}{4}r+1}$. Form the graph $G(2n+2,r)$ as follows. Take $A(2n,r) \setminus \{u_i v_{\frac{1}{2}r+1} : 1 \le i \le r\}$ and add two vertices, x and y, say. Join x to $u_1, u_2, \ldots, u_r$, and join y to $v_{\frac{1}{2}r+1}, v_{\frac{1}{2}r+2}, \ldots, v_r$. Call the resulting graph $G(2n+2,r)$. Observe that the graph is triangle free, has $2n+2$ vertices and is r-regular. Further, $G(2n+2,r)$ contains as a spanning subgraph the graph G_0 whose edge-set is specified as :

$$E(G_0) = (H_{\frac{1}{4}r+1} \setminus \{u_i v_{\frac{1}{2}r+i} : 1 \le i \le r\}) \bigcup_{i=1}^{r} \{xu_i, yv_{\frac{1}{2}r+i}\} .$$

Observe that G_0 is the union of r edge-disjoint (x,y)-paths and thus is 2-edge-connected. Further, the graph $(G(2n+2,r)-\{x,y\}) \setminus E(G_0)$ consists of the $\frac{1}{2}r-1$ Hamilton cycles $H_2, H_3, \ldots, H_{\frac{1}{2}r}$ and thus is (r-2)-edge-connected. It is easily established that $G(2n+2,r)$ is r-edge-connected.

For $r \geq 4$ we form the graph $G(2n + \frac{1}{2}r+1,r)$ from $G(2n+2,r)\setminus\{M_2,M_3,\ldots,M_{\frac{1}{2}r}\}$ by adding $\frac{1}{2}r-1$ new vertices $w_1,w_2,\ldots,w_{\frac{1}{2}r-1}$ and joining each of these to the r vertices $u_1,u_2,\ldots,u_{\frac{1}{2}r},v_{\frac{1}{2}r+1}, v_{\frac{1}{2}r+2},\ldots,v_r$. The resulting graph is triangle free, has $2n + \frac{1}{2}r + 1$ vertices and is r-regular. Further, this graph is r-edge-connected because of the following result.

Lemma 3.2: Let G be a k-edge-connected graph, $k \geq 1$, and M be a matching in G of size $m \geq \lceil \frac{1}{2}k \rceil$ saturating the vertices $v_1,v_2,\ldots,v_{2m}$. Then the graph G′ obtained by adding a vertex u to $G\setminus M$ and joining it to the vertices $v_1,v_2,\ldots,v_{2m}$ is k-edge-connected.

Proof: Suppose $\kappa'(G') = t < k$ and let E_1 be a t-edge cut of G′ and let E_2 denote those elements of E_1 that are incident to u. Note that $E_2 \neq \phi$ since G is k-edge-connected. Let X denote the set of M-saturated vertices of G that are, in G′, incident to E_2 and M′ denote the set of edges of M incident to exactly one vertex in X. The set

$$E' = (E_1 \setminus E_2) \cup M'$$

is an edge cut in G. But

$$|E'| = |E_1| - |E_2| + |M'| \leq |E_1| \leq k-1 ,$$

contradicting the fact that $\kappa'(G) \geq k$. This proves the lemma. □

Application of Lemma 3.2 to the graphs $G(m,r)$ for odd $m = 2n + \frac{1}{2}r$ or $2n + \frac{1}{2}r+1$ establishes the r-edge-connectedness of these graphs. It thus follows from Lemma 2.2 that def(G(m,r)) = 1.

We make use of the following lemma proved in [7] to establish our main result in this section.

Lemma 3.3: For odd n, $\mathcal{G}'(n,r,1) \neq \phi$ if and only if r is even and $n \geq \frac{5}{2}r$. □

Now we are ready to determine D(n,r,r-2).

Theorem 3.1: For $r \geq 4$,

(a) $D(n,r,r-2) = \phi$, if n and r are odd or $n < 2r$ or $n < \frac{5}{2}r$ is odd;

(b) $D(n,r,r-2) = \{d : 0 \leq d \leq 2 \lfloor \frac{n}{2(r^2 + r-1)} \rfloor$, d is even}, if $n \geq 2r$ is even;

(c) $D(n,r,r-2) = \{1\}$, if n is odd and $\frac{5}{2} \leq n < 3(r^2 + r-1)$;

(d) $D(n,r,r-2) = \{d : 1 \leq d \leq 1 + 2 \lfloor \frac{n}{2(r^2 + r-1)} - \frac{1}{2} \rfloor$, d is odd}, otherwise.

Proof: When $\mathcal{G}'(n,r,r-2) \neq \phi$, then at least one of n or r is even, and by Turan's theorem $n \geq 2r$. Further, by Lemma 3.3, if n is odd, then $n \geq \frac{5}{2}r$. This proves (a). So suppose at least one of n or r is even, $n \geq 2r$ and $n \geq \frac{5}{2}r$ if n is odd. The upper bound of def(G), $G \in \mathcal{G}'(n,r,r-2)$, is determined in Corollary 1.

First we consider the case when n is even. The graph $A(n,r) \in \mathcal{G}'(n,r,r-2)$ and has a perfect matching. This gives the lower bound of def(G) and proves (b) when $n < 2(r^2 + r-1)$. Now let

$n \ge 2(r^2 + r-1)$ and d be an even integer, $2 \le d \le 2 \lfloor \frac{n}{2(r^2 + r-1)} \rfloor$. We construct a graph $G \in \mathcal{G}'(n,r,r-2)$ with def(G) = d for each possible value of d as follows.

Let $\ell = \frac{1}{2}rd$ and $s = \frac{1}{2}rd - d$. Take the empty graph $\bar{K}_s$ with vertices $u_1, u_2, \ldots, u_s$, $\ell-1$ copies $G_1, G_2, \ldots, G_{\ell-1}$ of $B(2r+1,r)$ and a copy G_ℓ of $B(n-d(r^2+r-1)+2r+1,r)$. Note that $n - d(r^2 + r-1) + 2r+1$ is odd and is at least 2r+1 because of the upper bound on d. Further, for $1 \le i \le \ell$, G_i has two vertices, x_i and y_i say, of degree $\lfloor \frac{1}{2}r \rfloor + 1$ and $\lceil \frac{1}{2}r \rceil + 1$ respectively. Then join each x_i and u_j and each y_i to u_z, where $i \le j \le i + \lfloor \frac{1}{2}r \rfloor - 1 < z \le r-2$. The resulting graph G is triangle free, r-regular and has

$$\begin{aligned}\nu(G) &= (\ell-1)(2r+1) + (n-d(r^2 + r-1) + 2r+1) + s \\ &= (\tfrac{1}{2}rd-1)(2r+1) + (n-d(r^2 + r-1) + 2r+1) + (\tfrac{1}{2}rd-d) \\ &= n .\end{aligned}$$

We will now show that G is (r-2)-edge-connected. Suppose that $\kappa'(G) = t < r-2$, then there is t-edge cut, $(V_1,\bar{V}_1)$ say, of G. Lemma 3.1 implies that for each i, $G_i-\{x_i,y_i\}$ is (r-2)-edge-connected, and hence the vertices of $G_i-\{x_i,y_i\}$ are all in V_1 or all in $\bar{V}_1$.

Let $U_1 = V_1 \cap \{u_1,u_2,\ldots,u_s\}$ and $U_2 = \bar{V}_1 \cap \{u_1,u_2,\ldots,u_s\}$. We first prove that $U_i \ne \phi$, i = 1,2. Without loss of generality suppose that $U_1 = \phi$. Then $V(G_i) \nsubseteq V_1$ for any i. Further $V(G_i) \cap V_1 \ne \phi$ for some i. Since $t < r-2$ and the vertices of $G_i-\{x_i,y_i\}$ are all in V_1 or all in $\bar{V}_1$, then V_1 contains exactly one of x_i or y_i. But each of these possibilities results in $t \ge r-2$. Thus $U_i \ne \phi$, i = 1,2.

Since, for each i, G_i is $\frac{1}{2}r$-edge-connected, then there cannot be more than one G_i for which $V(G_i) \cap V_1 \ne \phi$ and $V(G_i) \cap \bar{V}_1 \ne \phi$. We now define a graph G′ as follows. If there exists a G_i with $V(G_i) \cap V_1 \ne \phi$

and $V(G_i) \cap \bar{V}_1 \neq \phi$, then let

$$G' = G - \bigcup_{j=i}^{i+d-1} V(G_j)$$

where the integers are reduced modulo ℓ when necessary. If there is no such G_i, then take

$$G' = G - \bigcup_{j=1}^{d} V(G_j) .$$

Then there exists an edge-cut set $(V_2,\bar{V}_2)$ of G' with less than r-2 edges. If we contract every G_i in G' to a single vertex, then the resulting graph G^* is isomorphic to the graph $A(2s,r-2)$. Further there is an edge-cut set $(V_3,\bar{V}_3)$ of G^* with $|(V_3,\bar{V}_3)| < r-2$. This contradicts the fact that $A(2s,r-2)$ is (r-2)-edge-connected. Hence $G \in \mathcal{G}(n,r,r-2)$.

Now take $S = \{u_1,u_2,\ldots,u_s\}$. Then

$$\begin{aligned} o(G-S) - |S| &= \ell - s \\ &= \frac{1}{2}rd - (\frac{1}{2}rd + d) \\ &= d, \end{aligned}$$

and so $\mathrm{def}(G) \geq d$. Further, every component G_i of G-S has $\mathrm{def}(G_i) = 1$, and hence $\mathrm{def}(G) \leq d$. Thus $\mathrm{def}(G) = d$ and this completes the proof of (b).

Now consider the case when n is odd. The graph $G(n,r) \in \mathcal{G}'(n,r,r-2)$ and has deficiency one. This gives the lower bound of def(G) and proves (c). Let $n \geq \frac{5}{2}r$. For each odd d,

$$3 \leq d \leq 1 + 2 \left\lfloor \frac{n}{2(r^2 + r-1)} - \frac{1}{2} \right\rfloor ,$$

we construct a graph $G \in \mathcal{G}'(n,r,r-2)$ with $\mathrm{def}(G) = d$ following the procedure described for the case when n is even. This completes the proof. □

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