CYCLES AND PATHS IN MULTIGRAPHS

by

R B Eggleton
Department of Mathematics, Universiti Brunei Darusalam
Bandar Seri Begawan, Brunei

D A Holton
Department of Mathematics and Statistics, University of Otago
Dunedin, New Zealand

and

C H C Little
Department of Mathematics and Statistics, Massey University
Palmerston North, New Zealand

Abstract

We consider cycles and paths in multigraphic realizations of a degree sequence $\underline{d}$. In particular we show that there exists a realization of $\underline{d}$ in which no cycle has order greater than three and no path has length greater than four.

In addition we show which orders of cycles and which lengths of paths exist in some realization of $\underline{d}$.

1. Introduction

Throughout we consider $\underline{d} = (d_1, d_2, \ldots, d_n)$ to be a sequence of n non-negative integers. The sequence $\underline{d}$ will be called a **degree sequence** if there is a multigraph G (without loops) with vertex set $\{v_i : i = 1, 2, \ldots, n\}$ such that deg $v_i = d_i$ for $i = 1, 2, \ldots, n$. We say that $\underline{d}$ is an **ordered** sequence if $d_i \geq d_j$ whenever $i \leq j$. It is well known (see [2]) that any ordered sequence $\underline{d}$ is a degree sequence if and only if

(i) $\sum_{i=1}^{n} d_i \equiv 0 \pmod 2$, and

(ii) $d_1 \leq \sum_{i=2}^{n} d_i$.

Furthermore $\underline{d}$ is a **positive** degree sequence if $d_i > 0$ for all i.

For convenience we will use $\Sigma\underline{d}$ for $\sum_{i=1}^{n} d_i$ and we abbreviate such sequences as (2, 2, ..., 2) (a twos) and (2, 2, ..., 2, 1, ..., 1) (a twos and b ones) as (2^a) and $(2^a, 1^b)$, respectively. By max $\underline{d}$ we mean the largest term in the sequence $\underline{d}$. Thus if $\underline{d}$ is ordered, max $\underline{d} = d_1$.

The **skeleton**, skel (G), of a multigraph G is the simple graph with vertex set VG such that two vertices are adjacent if and only if they are adjacent in G. We say that the multigraph G is **acyclic** if and only if skel(G) is a forest, even if G contains a cycle isomorphic to C_2.

In [1] Erdős and Gallai showed that if $\Sigma\underline{d} \geq k(n-1)$, then there is some simple realization of $\underline{d}$ which contains a cycle of order at least k. Here we consider the same problem for multigraphs and also determine what paths can be found in multigraphic realizations of $\underline{d}$.

2. Cycles and Paths which are Forced.

In this section we show that given any multigraphic sequence $\underline{d}$, there exists a realization which contains at most one triangle and no longer cycle and there exists a P_5 but no longer path.

Theorem 2.1:

The sequence $\underline{d}$ has an acyclic realization if and only if it has a bipartite realization.

Proof:

Clearly an acyclic realization is bipartite.

If G is a bipartite realization of $\underline{d}$, then it is already acyclic, or it contains a cycle C_{2m} for some $m \geq 2$. Let a be the smallest multiplicity of any edge on this cycle. Let $v_0, v_1, \ldots, v_{2m-1}$ be the vertices of C_{2m}, where v_i is joined to v_j if and only if $i \equiv j \pm 1$ (mod 2m). Suppose the multiplicity of v_0v_1 is a. If we decrease the multiplicities of $v_0v_1, v_2v_3, \ldots, v_{2m-2}v_{2m-1}$ by a and increase the multiplicities of the remaining edges by the same amount then the cycle C_{2m} is destroyed while the degree sequence of the original multigraph is preserved. Every even cycle can be destroyed in this way to give an acyclic realization of $\underline{d}$.

Having destroyed the even cycles, we may then operate on odd cycles in multigraphs by similar methods and reduce them to triangles. These techniques can therefore be used to show that there is some multigraphic realization of $\underline{d}$ which contains no cycle of order greater than three. However the following theorem proves a stronger result.

Theorem 2.2:

Let $\underline{d}$ be a multigraphic sequence. Then there is some realization of $\underline{d}$ which does not contain P_m for any $m \geq 6$ and which contains no cycle of order greater than three.

Proof:

Without loss of generality we assume that $\underline{d}$ is positive. Clearly if $n \leq 2$, then the theorem is true, and so we may assume that $n \geq 3$.

Let $m_i = \sum_{k=i}^{n} (-1)^{k-i} d_k$ for all $i \in \{1, 2, ..., n\}$. Let j_0 be the smallest value of j for which $2 \sum_{i=3}^{j} m_i \geq m_1$. Such a value of j exists since

$$2\sum_{i=3}^{n} m_i - m_1 = 2 \sum_{k=0}^{[(n-3)/2]} d_{2k+3} - \sum_{k=1}^{n} (-1)^{k-1} d_k$$

$$= -d_1 + \sum_{k=2}^{n} d_k,$$

and this expression is non-negative since $\underline{d}$ is multigraphic. Define $\delta = 2 \sum_{i=3}^{j_0} m_i - m_1$. Note that δ is even since m_1 is even.

Case 1: $j_0 = 3$. Then we may realize $\underline{d}$ as follows:

join	v_1 to v_2	with	$d_2 - \frac{1}{2}\delta$	edges;
join	v_1 to v_3	with	$m_3 - \frac{1}{2}\delta$	edges;
join	v_2 to v_3	with	$\frac{1}{2}\delta$	edges;
join	v_3 to v_{2k+2}	with	$d_{2k+2} - d_{2k+3}$	edges for $k = 1, 2, ..., \left[\frac{n-3}{2}\right]$;
join	v_3 to v_n	with	d_n	edges if n is even;
join	v_{2k+2} to v_{2k+3}	with	d_{2k+3}	edges for $k = 1, 2, ..., \left[\frac{n-3}{2}\right]$.

The skeleton of this realization is a subgraph of the graph shown in Figure 2.1, when n is even. Clearly this realization contains no path longer than P_5.

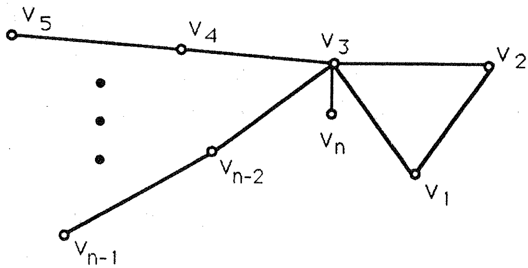

Figure 2.1

Case 2: $j_0 > 3$. Then we may realize $\underline{d}$ as follows:

join	v_1 to v_i	with	d_i	edges for $i = 3, ..., j_0 - 2$;
join	v_1 to v_{j_0-1}	with	$d_{j_0-1} - \frac{1}{2}\delta$	edges;
join	v_1 to v_2	with	$d_2 - d_{j_0+1}$	edges;
join	v_1 to v_{j_0+2k}	with	$d_{j_0+2k} - d_{j_0+2k+1}$	edges for $k = 1, 2, ..., \left[\frac{1}{2}(n-j_0)\right]$;
join	v_1 to v_n	with	d_n	edges if $n \equiv j_0 \pmod 2$;
join	v_1 to v_{j_0}	with	$d_{j_0} - \frac{1}{2}\delta$	edges;
join	v_2 to v_{j_0+1}	with	d_{j_0+1}	edges if $j_0 < n$;
join	v_{j_0+2k} to v_{j_0+2k+1}	with	d_{j_0+2k+1}	edges for $k = 1, 2, ..., \left[\frac{1}{2}(n-j_0-1)\right]$;
join	v_{j_0-1} to v_{j_0}	with	$\frac{1}{2}\delta$	edges.

It is clear that this realization of $\underline{d}$ contains no path longer than P_5.

On the other hand we now show that every path from P_2 to P_5 must occur in every realization of some degree sequence.

Clearly P_2 must occur in every realization of every degree sequence where $\Sigma\underline{d} > 0$.

The sequence $\underline{d} = (2^3)$ is uniquely realizable and contains a P_3 in this realization.

The sequence $\underline{d}$ = (18, 15, 13, 12) is such that every realization is connected. This follows since no two terms in the sequence are equal. Inspection of the residue classes modulo 3 of the terms in the sequence reveals that no two distinct subsequences of $\underline{d}$ have equal sum and so $\underline{d}$ does not have a bipartite realization. Hence every realization of $\underline{d}$ contains a triangle. The remaining vertex in any realization of $\underline{d}$ must be adjacent to one of the vertices of the triangle since the realization is connected. Thus every realization of $\underline{d}$ contains a P_4.

Consider the sequence $\underline{d}$ = (24, 18, 15, 13, 12). By previous arguments we see that every realization of $\underline{d}$ is connected but none is bipartite. If every realization of $\underline{d}$ contains C_5, then they all contain P_5. Hence we may suppose that some realization contains a triangle. The only way such a realization does not contain P_5 is for the two vertices v, w, remaining to be adjacent to the same vertex u of the triangle. But then deg u > deg v + deg w. However the smallest possible value of deg v + deg w is 25 which exceeds any degree of the sequence. Hence every realization of $\underline{d}$ contains a P_5.

But the longest cycle that can be found in every realization of $\underline{d}$ is a triangle and if $\underline{d}$ contains a bipartite realization then it will have an acyclic realization as we have already noted in Theorem 2.1.

3. Possible Orders of Cycles and Paths

In this section we find the range of possible cycle orders and path lengths which occur in realizations of the degree sequence $\underline{d}$.

Theorem 3.1

Let $\underline{d}$ be a positive degree sequence. If $\underline{d} \neq (2^s)$ for any $s \geq 3$, then $\underline{d}$ has a realization with a cycle isomorphic to C_ℓ if and only if $2 \leq \ell \leq \min\left\{\frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4), m\right\}$, where m is the number of terms of $\underline{d}$ which are greater than one. If $\underline{d} = (2^s)$ for some $s \geq 3$, then $\underline{d}$ contains a realization with a cycle isomorphic to $C\ell$ whenever $2 \leq \ell \leq s-2$ or $\ell = s$.

Proof

We may assume that $\underline{d}$ is ordered. Suppose $\underline{d}$ contains a realization G with a cycle isomorphic to C_ℓ. Then $2 \le \ell \le m$ and the degree sequence $\underline{d}' = \underline{d} - (2^\ell)$, of the graph obtained from G by removing C_ℓ, is graphic.

If $\max \underline{d}' = d_1$, then $d_1 \le \sum_{i=2}^{n} d_i - 2\ell$ implies $\ell \le \frac{1}{2}(\Sigma\underline{d} - 2d_1) < \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4)$.

If $\max \underline{d}' = d_1 - 2$, then $d_1 - 2 \le \sum_{i=2}^{n} d_i - 2(\ell-1)$ implies $\ell \le \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4)$.

Finally, if $\max \underline{d}' = d_j = d_1 - 1$, then $d_j \le \sum_{i \ne j} d_i - 2\ell$ implies $\ell \le \frac{1}{2}(\Sigma\underline{d} - 2d_j)$ $= \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 2) < \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4)$. Thus we have the stated restriction on ℓ.

Suppose now that $2 \le \ell \le \min\left[\frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4), m\right]$. Let $\underline{d}' = (d_1', \ldots, d_n')$, where

$$d_i' = \begin{cases} d_i - 2 & 1 \le i \le \ell \\ d_i & i > \ell \end{cases}.$$

Since $\Sigma\underline{d} \equiv 0 \pmod 2$, clearly $\Sigma\underline{d}' \equiv 0 \pmod 2$.

If $\underline{d}'$ is multigraphic, then $\underline{d}$ has a realization with a cycle isomorphic to C_ℓ. Hence we may assume that $\underline{d}'$ is not multigraphic.

Now $\max \underline{d}'$ is either $d_1 - 2$ or $d_{\ell+1}$. If $\max \underline{d}' = d_1 - 2$ we must have

$$d_1 - 2 > \sum_{i=2}^{n} d_i - 2(\ell - 1)$$

i.e. $$d_1 > \sum_{i=2}^{n} d_i - 2\ell + 4$$

i.e. $$\ell > \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4).$$

This contradicts our choice of ℓ.

If max $\underline{d}' = d_{\ell+1}$ we must have

$$d_\ell \geq d_{\ell+1} > \sum_{i=1}^{\ell} d_i + \sum_{i=\ell+2}^{n} d_i - 2\ell$$

$$\geq \sum_{i=1}^{\ell} d_i - 2\ell.$$

Hence $$\sum_{i=1}^{\ell-1} d_i < 2\ell.$$

Now $d_i \geq 2$ for $1 \leq i \leq \ell$, since $\ell \leq m$. Therefore either $d_i = 2$ for $1 \leq i \leq \ell-1$ (and $d_\ell = 2$ since $\ell \leq m$) or $d_1 = 3$ and $d_i = 2$ for $2 \leq i \leq \ell$. So either $\underline{d} = (2^n)$ for $n \geq \ell$, $\underline{d} = (2^\alpha, 1^\beta)$ for $\alpha \geq \ell$, $\beta > 0$ and β even, or $\underline{d} = (3, 2^\alpha, 1^\beta)$ for $\alpha \geq \ell - 1$ and β odd. Since $\underline{d}'$ is not multigraphic, $\underline{d} = (2^{\ell+1})$, the one degree sequence not covered by the theorem.

Corollary 3.2

The positive degree sequence $\underline{d}$ has no realization with a cycle of order greater than two if and only if $d_1 = \sum_{i=2}^{n} d_i$ or $\underline{d}$ has at most two terms greater than one.

Proof

If $d_1 = \sum_{i=2}^{n} d_i$, then $\underline{d}$ has a unique realization whose skeleton is a star. Thus no cycle has order greater than two. If $\underline{d}$ has at most two terms greater than one, then no realization can contain a cycle larger than C_2.

Suppose $\underline{d}$ has no realization with a cycle longer than C_2. Then $\underline{d} \neq (2^s)$ for any $s \geq 3$ and so by the theorem we must have $m \leq 2$ or $\frac{1}{2}(\Sigma\underline{d} - 2d_1 + 4) \leq 2$. The latter condition gives $d_1 \geq \sum_{i=2}^{n} d_i$, but since $\underline{d}$ is multigraphic we must have $d_1 = \sum_{i=2}^{n} d_i$. The corollary then follows.

Corollary 3.3

The positive degree sequence $\underline{d}$ contains a realization with a Hamiltonian cycle if and only if $2d_1 \le \Sigma\underline{d} - 2n + 4$ and $m = n$.

Theorem 3.4

The positive degree sequence $\underline{d}$ has a realization with a path isomorphic to P_ℓ if and only if $1 \le \ell \le \min\left\{\frac{1}{2}(\Sigma\underline{d} - 2d_1 + 6),\ n,\ m+2\right\}$.

Proof

The argument is analogous to that of Theorem 3.1 and we only sketch it here.

If $\underline{d}$ contains a path isomorphic to P_ℓ, then clearly $1 \le \ell \le \min(n, m+2)$. Further $\underline{d} - (2^{\ell-2}, 1^2)$ is graphic and similar inequalities to those of the proof of Theorem 3.1 showthat $\ell \le \frac{1}{2}(\Sigma\underline{d} - 2d_1 + 6)$. We note however that there are four cases to consider since we may subtract 1 from d_1 in forming $\underline{d} - (2^{d-2}, 1^2)$ from $\underline{d}$.

Suppose then that $1 \le \ell \le \min\left\{\frac{1}{2}(\Sigma\underline{d} - 2d_1 + 6),\ n,\ m+2\right\}$. Here we need $\underline{d}' = (d_1', \ldots, d_n')$ defined by $d_i' = \begin{cases} d_i - 2 & 1 \le i \le \ell-2 \\ d_i & \ell-1 \le i \le n-2 \\ d_i - 1 & n-1 \le i \le n \end{cases}$.

If $\underline{d}'$ is multigraphic the theorem follows. Suppose therefore that $\underline{d}'$ is not multigraphic. Arguing as in the proof of Theorem 3.1 we obtain a contradiction unless $\max \underline{d}' = d_{\ell-1}$ where $\ell < n$, or $\max \underline{d}' = d_{n-1} - 1$.

Note that any realization of $\underline{d}$ contains a P_ℓ if $\ell \le 2$, since $\underline{d}$ is positive. Suppose $\max \underline{d}' = d_{\ell-1}$ and $2 < \ell < n$. Since $\underline{d}'$ is not multigraphic, we then have

$$d_{\ell-2} \ge d_{\ell-1} > \sum_{i=1}^{\ell-2} d_i + \sum_{i=\ell}^{n} d_i - 2\ell + 2. \qquad (1)$$

Thus

$$d_{\ell-2} > \sum_{i=1}^{\ell-2} d_i - 2\ell + 4, \qquad (2)$$

since

$$\sum_{i=\ell}^{n} d_i \geq d_{n-1} + d_n \geq 2.$$

On the other hand, if max $\underline{d}' = d_{n-1} - 1$ then ℓ=n and (1) is replaced by

$$d_{\ell-2} > d_{\ell-1} - 1 > \sum_{i=1}^{\ell-2} d_i + d_n - 2\ell + 3$$

and (2) again follows. Thus in either case

$$\sum_{i=1}^{\ell-3} d_i < 2\ell - 4.$$

Now, arguing as in the proof of Theorem 3.1, we find that either $\underline{d} = (2^\alpha, 1^\beta)$ for some $\alpha \geq \ell-2$ and some even $\beta \geq 0$ or $\underline{d} = (3, 2^\alpha, 1^\beta)$ for some $\alpha \geq \ell-3$ and some odd β. As $\underline{d}'$ is not multigraphic, these sequences force $d'_{\ell-1} = 2$ and $\underline{d}_i' = 0$ for all $i \neq \ell-1$. Hence $\underline{d} = (2^{\ell-1}, 1^2)$ and $\underline{d}$ has a realization with a path isomorphic to $P_{\ell+1}$.

Corollary 3.5

The positive degree sequence $\underline{d}$ contains a realization with a Hamiltonian path if and only if $2d_1 \leq \Sigma\underline{d} - 2n + 6$ and $m \geq n-2$.

REFERENCES

[1] P Erdős and T Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hung. 10 (1959) 337-356.

[2] J K Senior, Partitions and their respective graphs, Amer. J. Math. 73 (1951) 663–689.