A Sufficient Condition for Hamiltonian Cycles in Bipartite Tournaments

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Abstract

We prove a new sufficient condition on degrees for a bipartite tournament to be Hamiltonian, that is, if an $n \times n$ bipartite tournament T satisfies the condition $d_T^-(u) + d_T^+(v) \geq n - 1$ whenever uv is an arc of T, then T is Hamiltonian, except for two exceptional graphs. This result is shown to be best possible in a sense.

$T(X, Y, E)$ denotes a bipartite tournament with bipartition (X, Y) and vertex-set $V(T) = X \cup Y$ and arc-set $E(T)$. If $|X| = m$ and $|Y| = n$, such a bipartite tournament is called an *$m \times n$ bipartite tournament*. For a vertex v of T and a subdigraph S of T, we define $N_s^-(v)$ and $N_s^+(v)$ to be the set of vertices of S which, respectively, dominate and are dominated by, the vertex v. Put

$$N_T^-(S) = \bigcup_{v \in s} N_T^-(v); \quad N_T^+(S) = \bigcup_{v \in s} N_T^+(v);$$

$$d_T^-(v) = |\ N_T^-(v)\ |; \quad d_T^+(v) = |\ N_T^+(v)\ |.$$

Let P be a subset of X and Q a subset of Y; $P \to Q$ (resp. $Q \to P$) denotes $pq \in E(T)$ (resp. $qp \in E(T)$) for all $p \in P$ and all $q \in Q$. If $P = \{x\}$ this becomes $x \to Q$. To simplify notation, we denote also $B_1 \to B_2, B_2 \to B_3, \cdots$, by $B_1 \to B_2 \to B_3 \to \cdots$. Moreover, a *factor* of T is a spanning subdigraph H of T such that $d_H^-(v) = d_H^+(v) = 1$ for all $v \in V(T)$. T is said to be *strong* if for any two vertices u and v, there is a path from u to v and a path from v to u. A *component* of T is a maximal strong subdigraph.

$T(b_1, b_2, b_3, b_4)$ denotes the bipartite tournament, whose vertex-set may be partitioned into four independent sets B_i, $i = 1,2,3,4$, such that $|B_i| = b_i \geq 0$ and $B_1 \to B_2 \to B_3 \to B_4 \to B_1$. Other terms and symbols not defined in this paper can be found in [**1**].

Up to now, there are very few conditions that imply the existence of Hamiltonian cycles for bipartite tournaments. An obvious necessary condition for an $m \times n$ bipartite tournament to be Hamiltonian is $m = n$. Therefore, we are only interested in researching Hamiltonian properties in $n \times n$ bipartite tournaments. We recall now the well-known conditions for an $n \times n$ bipartite tournament to have Hamiltonian cycles.

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Theorem 1 *(Jackson [2]). If an $n \times n$ strong bipartite tournament T satisfies*

$$vu \notin E(T) \Rightarrow d_T^-(u) + d_T^+(v) \geq n,$$

then T is Hamiltonian.

Theorem 2 *(Wang [3]). If an $n \times n$ bipartite tournament T satisfies*

$$vu \notin E(T) \Rightarrow d_T^-(u) + d_T^+(v) \geq n - 1,$$

then T is Hamiltonian, unless n is odd and T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$.

Obviously, Theorem 2 improves Theorem 1. In this paper we prove another condition that is weaker than the conditions of the two theorems above, ensuring an $n \times n$ bipartite tournament to be Hamiltonian, except for two described cases. In showing the main result we will use the following theorem:

Theorem 3 *(Haggkvist and Manoussakis [4]). A bipartite tournament T is Hamiltonian if and only if T is strong and contains a factor.*

Theorem 4 *If an $n \times n$ bipartite tournament T satisfies*

$$uv \in E(T) \Rightarrow d_T^-(u) + d_T^+(v) \geq n - 1,$$

then T is Hamiltonian, unless T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ when n is odd or $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$ when n is even.

Proof. Suppose that T is an $n \times n$ bipartite tournament satisfying the hypotheses of the theorem. We first establish two claims.

Claim 1. If $n \geq 3$, then T is strong.

Assume that T is not strong and has components $B_1, B_2, \cdots, B_m$ with $m \geq 2$ such that $X(B_i) \to Y(B_j)$ and $Y(B_i) \to X(B_j)$ whenever $i \leq j$. Then B_1 contains a vertex u such that $d_T^-(u) \leq \frac{|V(B_1)|}{4}$. Such a vertex exists because

$$\sum_{v \in V(B_1)} d_T^-(v) = |E(B_1)| \leq \frac{|V(B_1)|^2}{4}.$$

Without loss of generality we may assume that $u \in X$.

Case 1. $Y(B_m) \neq \emptyset$. If there is a vertex v in $Y(B_m)$ such that $d_T^+(v) \leq \frac{|V(B_m)|}{4}$, then we have

$$d_T^-(u) + d_T^+(v) \leq \frac{|V(B_1)| + |V(B_m)|}{4} \leq \frac{n}{2}.$$

In particular, $uv \in E(T)$ implies

$$d_T^-(u) + d_T^+(v) \geq n - 1$$

and so $n-1 \leq n/2$ or $n \leq 2$, a contradiction. Now assume that $d_T^+(v) > \frac{|V(B_m)|}{4}$ for all $v \in Y(B_m)$, which implies $|X(B_m)| \neq \emptyset$. Since B_m is strong, we must have $|V(B_m)| \geq 4$ and so $d_T^+(v) \geq 2$ for all $v \in Y(B_m)$. In this case we can easily deduce that $|X(B_m)| \geq 3$. Furthermore, we can conclude that there is a vertex w in $X(B_m)$ such that $d_T^+(w) < \frac{|V(B_m)|}{4}$. Otherwise put $|X(B_m)| = a$ and $|Y(B_m)| = b$. Then $|V(B_m)| = a+b$ and hence

$$\begin{aligned} ab &= \sum_{w\in X(B_m)} d_{B_m}^+(w) + \sum_{w\in X(B_m)} d_{B_m}^-(w) \\ &= \sum_{w\in X(B_m)} d_{B_m}^+(w) + \sum_{v\in Y(B_m)} d_{B_m}^+(v) \\ &= \sum_{w\in X(B_m)} d_T^+(w) + \sum_{v\in Y(B_m)} d_T^+(v) \\ &> \frac{a(a+b)}{4} + \frac{b(a+b)}{4} = \frac{(a+b)^2}{4}, \end{aligned}$$

which implies $(a-b)^2 < 0$. This is impossible. Thus we have

$$d_T^-(u) + d_T^+(w) < \frac{|V(B_1)| + |V(B_m)|}{4} \leq \frac{n}{2}. \tag{1}$$

On the other hand, since B_m is strong, there is a vertex v in $Y(B_m)$ such that $uv, vw \in E(T)$. It follows that

$$d_T^-(u) + d_T^+(v) \geq n-1 \quad \text{and}$$
$$d_T^-(v) + d_T^+(w) \geq n-1,$$

and so

$$d_T^-(u) + d_T^+(w) \geq n-2. \tag{2}$$

It follows from (1) and (2) that $n-2 < n/2$, which implies $n < 4$ contradicting $n \geq |X(B_1)| + |X(B_m)| \geq 4$.

Case 2. $Y(B_m) = \emptyset$. This implies that B_m is a vertex w of X with $d_T^+(w) = 0$. Since the case $Y(B_1) \neq \emptyset$ is transformed into Case 1 by considering the converse digraph of T, it is sufficient to consider the case $Y(B_1) = \emptyset$. Noting that B_1 is strong, we have $V(B_1) = X(B_1) = \{u\}$ and hence $d_T^-(u) = 0$. Therefore we obtain

$$d_T^-(u) + d_T^+(w) = 0. \tag{3}$$

Moreover, it is easy to see that there is a vertex v in Y such that $uv, vw \in E(T)$. Hence

$$d_T^-(u) + d_T^+(v) \geq n-1,$$
$$d_T^-(v) + d_T^+(w) \geq n-1,$$

and so

$$d_T^-(u) + d_T^+(w) \geq n - 2. \tag{4}$$

It follows from (3) and (4) that $n - 2 \leq 0$ or $n \leq 2$ contradicting $n \geq 3$. This proves Claim 1.

Claim 2. Either T contains a factor, or else T is isomorphic to $T(k, k-1, n-k, n-k-1)$, $\frac{n}{2} \leq k \leq \frac{n+1}{2}$.

Suppose that T contains no factor. It follows from a well-known theorem of Hall-Konig on matchings (see [1], p.72) that there exists a subset P either of X or of Y such that $|P| > |N_T^+(P)|$. Without loss of generality, assume that $P \subseteq X$. Put $N_T^+(P) = Q$, $R = X \setminus P$, and $S = Y \setminus Q$. Then $S \neq \emptyset$ and $S \to P$. Consider the vertices p in P and s in S. We now see that $N_T^-(s) \subseteq R$ and $N_T^+(p) \subseteq Q$ and hence

$$d_T^-(s) + d_T^+(p) \leq |R| + |Q| < |R| + |P| = n.$$

Combining this with the fact that $sp \in E(T)$, implying $d_T^-(s) + d_T^+(p) \leq n - 1$, we get

$$d_T^-(s) + d_T^+(p) = |R| + |Q| = n - 1.$$

It follows from this, and the arbitrariness of s and p, that $P \to Q$ and $R \to S$. Furthermore, we can conclude that $Q \to R$ for otherwise there are vertices q in Q and r in R such that $rq \in E(T)$ and therefore $d_T^-(r) + d_T^+(q) \leq |Q| - 1 + |R| - 1 = n - 3$, contradicting the hypothesis of the theorem. Set $|P| = k$. Then it follows from $|Q| + |R| = n - 1$ and $|P| + |R| = n$ that $|Q| = k - 1$, and so $|R| = n - k$ and $|S| = n - k - 1$. Thus we conclude that T is isomorphic to $T(k, k-1, n-k, n-k-1)$. We now prove that $\frac{n}{2} \leq k \leq \frac{n}{2} + 1$ by considering the arcs pq and rs, respectively. By the assumption of the theorem and the fact obtained above, we have

$$n - 1 \leq d_T^-(p) + d_T^+(q) = |R| + |S| = 2n - 2k - 1 \quad \text{and}$$

$$n - 1 \leq d_T^-(r) + d_T^+(s) = |P| + |Q| = 2k - 1.$$

It follows that $\frac{n}{2} \leq k \leq \frac{n}{2} + 1$. In particular, we deduce easily that $k = \frac{n+1}{2}$ when n is odd and $k = \frac{n}{2}$ or $\frac{n}{2} + 1$ when n is even. Hence T is isomorphic to either $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ or $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$ or $T(\frac{n+2}{2}, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2})$. However, it is easy to see that $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}) \cong T(\frac{n+2}{2}, \frac{n}{2}, \frac{n-2}{2}, \frac{n}{2})$. This proves Claim 2.

If $n \geq 3$, the theorem follows by Theorem 3 and Claims 1 and 2. Only the cases $n = 1$ and $n = 2$ remain. In the first case T is only $T(1,1,0,0) = T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$. In the second case we can easily verify that $T \cong T(1,1,1,1)$ or $T \cong T(1,2,1,0)$. Clearly, the former is Hamiltonian and the latter is $T(\frac{n}{2}, \frac{n+2}{2}, \frac{n}{2}, \frac{n-2}{2})$. The proof of the theorem is complete. □

Remark 1. Theorem 4 is the best possible in the sense that it becomes false if the condition on the degrees is relaxed by one. To see this we construct the bipartite tournament $T = B_1 \cup \{u\} \to B_2 \cup \{v\} \to B_3 \to B_4 \to B_1 \cup \{u\}/\{uv\} \cup \{vu\}$ with $|B_1| = |B_2| = \frac{n+1}{2}$ and $|B_3| = |B_4| = \frac{n-3}{2}$. It is easy to check that T satisfies

$d^-_T(x) + d^+_T(y) \geq n-2$ for all $xy \in E(T)$ and is not one of $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ and $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$. However, T is obviously nonhamiltonian.

Remark 2. Theorem 4 is stronger than Theorem 2. This can be demonstrated by the bipartite tournament $B_1 \cup \{u\} \to B_2 \cup \{v\} \to B_3 \cup \{w\} \to B_4 \to B_1 \cup \{u\} / \{uv, vw\} \cup \{vu, wv\}$ with $|B_2| = \frac{n}{2}$ and $|B_i| = \frac{n-2}{2}$ for i =1,3,4. We verify easily that it satisfies the condition of Theorem 4 and so is Hamiltonian, however, not that of Theorem 2, and hence Theorem 2 does not apply.

Remark 3. In fact, an $n \times n$ bipartite tournament satisfying the condition of Theorem 4 is not only Hamiltonian but satisfies an even stronger result, unless T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ or $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n-2}{2})$. To see this, we first recall a result by Beineke and Little [5] that every Hamiltonian bipartite tournament either contains cycles of all possible even lengths, or else is isomorphic to $T(k,k,k,k)$ for some $k > 1$. Zhang [6] and Haggkvist and Manoussakis [4] generalized this result by showing that every vertex of a Hamiltonian bipartite tournament is contained in cycles of all possible even lengths, unless T is isomorphic to $T(k,k,k,k)$ for some $k > 1$. Combining this with Theorem 4 we can establish the following result.

Theorem 5 *Let T be an $n \times n$ bipartite tournament satisfying*

$$uv \in E(T) \Rightarrow d^-_T(u) + d^+_T(v) \geq n-1,$$

and w be any vertex of T. There are cycles of all possible even lengths through w, unless T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$ when n is odd or $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$ when n is even.

Moreover, Theorem 4 has the following two immediate consequences.

Corollary 4. 1 *If an $n \times n$ bipartite tournament T satisfies*

$$vu \notin E(T) \Rightarrow d^-_T(u) + d^+_T(v) \geq n-1,$$

then T is Hamiltonian, unless n is odd and T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$.

Proof. Since $T(\frac{n}{2}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2})$ obviously does not satisfy the condition of the corollary, the conclusion follows from Theorem 4. □

Corollary 4. 2 *An $n \times n$ bipartite tournament T of minimum indegree and outdegree at least $(n-1)/2$ is Hamiltonian, unless n is odd and T is isomorphic to $T(\frac{n+1}{2}, \frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2})$.*

Proof. This follows immediately from Corollary 4.1. □

Finally, a possible version of Theorem 4 for an oriented graph D is that D is Hamiltonian if $d^-_D(u) + d^+_D(v) \geq n-2$ whenever uv is an arc of D. We believe this to be true.

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