CONSTRUCTIONS OF CYCLIC MENDELSOHN DESIGNS

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ABSTRACT. A Mendelsohn design M(k,v) is a pair (V,B), where $|V|=v$ and B is a set of cyclically ordered k-tuples of distinct elements of V, called blocks, such that every ordered pair of distinct elements of V belongs to exactly one block of B. A M(k,v) is called cyclic if it has an automorphism consisting of a single cycle of length v. The spectrum of existence of cyclic M(3,v)'s and M(4,v)'s is known. In this paper we prove that in every cyclic M(k,v) with $k\not\equiv 2 \pmod 4$ v is odd, and we give some constructions which allow us to determine the spectrum of cyclic M(k,v)'s for every k such that $5\le k\le 8$.

1. INTRODUCTION

Given a finite set V, a *Mendelsohn k-tuple* on V, $k\ge 3$, is a set

$$\{(x_1,x_2),\dots,(x_{k-1},x_k),(x_k,x_1)\} ,$$

where $x_1,x_2,\dots,x_k$ are distinct elements of V. A Mendelsohn k-tuple will be denoted by $[x_1,x_2,\dots,x_k]$. Clearly:

$$[x_1,x_2,\dots,x_k]=[x_2,\dots,x_k,x_1]=\dots=[x_k,x_1,\dots,x_{k-1}] .$$

A 2-(v,k,λ) *Mendelsohn design* is a pair (V,B), where $|V|=v$ and B is a collection of Mendelsohn k-tuples on V, called *blocks*, such that every ordered pair of distinct elements of V belongs to exactly λ blocks of B.

A 2-(v,k,1) Mendelsohn design will be denoted by M(k,v). If (V,B) is a M(k,v) then $|B|=\dfrac{v(v-1)}{k}$; it follows that *a necessary condition for the existence of* M(k,v)*'s is* $v(v-1)\equiv 0 \pmod k$, $v\ge k$.

The problem of existence of M(k,v)'s is open; however the spectrum of $M(k,v)$'s is known for every k such that $3\le k\le 16$, $k\ne 15$ ([7],[1],[2],[3]).

A $M(k,v)$ is called *cyclic* if it has an automorphism consisting of a single cycle of length v. In what follows a cyclic $M(k,v)$ will be denoted by $CM(k,v)$.

In [6] it is proved that a $CM(3,v)$ exists if and only if $v\equiv 1$ or $3 \pmod 6$, $v\ne 9$. Further, in [8] it is showed that a $CM(4,v)$ exists if and only if $v\equiv 1 \pmod 4$.

In this paper we study the spectrum of $CM(k,v)$'s. We prove that if a $CM(k,v)$ exists and $k\not\equiv 2 \pmod 4$ then v is odd. Further we give some constructions from which it follows that for $5\le k\le 8$ a $CM(k,v)$ exists if and only if $k=5$ and $v\equiv 1$ or $5 \pmod{10}$, $k=6$ and $v\equiv 0$ or $1 \pmod 3$, $v\ne 6,9$, $k=7$ and $v\equiv 1$ or $7 \pmod{14}$, $k=8$ and $v\equiv 1 \pmod 8$.

2. EXISTENCE OF CM(k,v)'s

Suppose that (V,B) is a $CM(k,v)$. Then we may assume that $V=\mathbb{Z}_v$ and that if $b=[x_1,x_2,\dots,x_k]\in B$ then also every block

$$b+y=[x_1+y,x_2+y,\dots,x_k+y] \quad ,$$

$y\in\mathbb{Z}_v$, belongs to B.

With each block $b=[x_1,x_2,\dots,x_k]\in B$ we can associate a cyclically ordered k-tuple:

$$d(b)=(x_2-x_1,\dots,x_k-x_{k-1},x_1-x_k) \quad ,$$

which will be called *difference block* (briefly d-*block*) of b.

We will say that the set $\overline{B}=\{d(b):\ b\in B\}$ is the *difference family* of $(\mathbb{Z}_v,B)$.

The following result is well known.

THEOREM 1. *A* CM(k,v) *exists if and only if there exists a set* D *of cyclically ordered* k-*tuples of elements of* $\mathbb{Z}_v-\{0\}$ *such that*:

(1) *every* $z\in\mathbb{Z}_v-\{0\}$ *is contained in exactly one* k-*tuple of* D;

(2) *for every* $(z_1,z_2,\dots,z_k)\in D$:

$$\sum_{i=1}^{k} z_i=0 \quad \textit{and} \quad \sum_{i=1}^{m} z_i\neq 0 \quad \textit{for every } m=1,2,\dots,k-1 .$$

Let $(\mathbb{Z}_v,B)$ be a CM(k,v) and let $\bar{B}$ be its difference family. In the following lemmas we determine some properties of $\bar{B}$.

LEMMA 1. *For every* $(z_1,z_2,\dots,z_k)\in\bar{B}$, *if* $z_1=z_i$ *then* $z_2=z_{i+1}$.

Proof. Let $(z_1,z_2,\dots,z_k)\in\bar{B}$ and suppose that $z_1=z_i$. Let

$$w_r=\sum_{j=1}^{r} z_j \text{ and } \bar{w}_r=\sum_{j=r}^{k} z_j,\quad r=1,2,\dots,k.$$

Consider $b=[w_1,w_2,\dots,w_k]\in B$. Since $(\mathbb{Z}_v,B)$ is cyclic, $b+\bar{w}_i\in B$; further, $w_{i-1}+\bar{w}_i=w_k$ and $w_i+\bar{w}_i=w_1$ because $z_1=z_i$. It follows that $b+\bar{w}_i=b$, hence, in particular, $w_2=w_{i+1}+\bar{w}_i$, for that $z_2=z_{i+1}$. ∎

From Lemma 1 it follows that for every $d\in\bar{B}$ there exists a divisor δ of k such that $d=(z_1,z_2,\dots,z_\delta,z_1,z_2,\dots,z_\delta,\dots\dots,z_1,z_2,\dots,z_\delta)$, where $z_1,z_2,\dots,z_\delta$ are distinct elements of $\mathbb{Z}_v-\{0\}$. Let $\pi=z_1+z_2+\dots+z_\delta$, where + is the usual addition between integers. From Theorem 1 $\pi\equiv 0 \pmod v$ if and only if $\delta=k$. We set $T(d)=(\delta,\pi)$.

LEMMA 2. *For every* $d\in\bar{B}$, *if* $T(d)=(\delta,\pi)$ *then there exists* $t\in\{\delta,2\delta,\dots,(k-1)\delta\}$ *such that* $GCD(t,k)=\delta$ *and* $\pi=\dfrac{tv}{k}$.

Proof. Let $\delta' = \frac{k}{\delta}$. Then there exists $h \in \{1,2,\ldots,k-1\}$ such that $\pi\delta' = hv$. Let $t = h\delta$. Then $\pi = \frac{tv}{k}$, with $t \in \{\delta, 2\delta, \ldots, (k-1)\delta\}$.

We prove that $GCD(t,k) = \delta$. Clearly δ is a divisor of t and k. Now, suppose that $\bar{\delta}$ is a divisor of t and k and let $t = \bar{\delta}\bar{h}$ and $k = \bar{\delta}\bar{\delta}'$. Then $\pi\bar{\delta}' = \bar{h}v$. But then from Theorem 1 it follows that $\bar{\delta}' \geq \delta'$ and hence $\bar{\delta} \leq \delta$. ∎

In the following theorem we give a necessary condition for the existence of CM(k,v)'s.

THEOREM 2. *If a* CM(k,v) *exists and* $k \not\equiv 2 \pmod 4$ *then* v *is odd.*

Proof. Let $(\mathbb{Z}_v, B)$ be a CM(k,v) and let $\bar{B}$ be its difference family, with $\bar{B} = \{d_1, d_2, \ldots, d_s\}$ and $T(d_i) = (\delta_i, \pi_i)$, $i = 1,2,\ldots,s$.

From Lemma 2, $\pi_i = \frac{t_i v}{k}$, and from (1) of Theorem 1, $\sum_{i=1}^{s} \pi_i = \frac{v(v-1)}{2}$. It follows that

$$(3) \qquad \sum_{i=1}^{s} t_i = \frac{k(v-1)}{2}.$$

Therefore, if $k \equiv 1$ or $3 \pmod 4$ then v is odd.

Further, from Lemma 2, if k is even then, for every $i = 1,2,\ldots,s$, t_i is even if and only if δ_i is even. Therefore $\sum_{i=1}^{s} t_i$ is even if and only if $\sum_{i=1}^{s} \delta_i$ is even. Suppose now that $k \equiv 0 \pmod 4$; then, from (3), $\sum_{i=1}^{s} t_i$ is even, therefore $\sum_{i=1}^{s} \delta_i = v-1$ is even, so v is odd. ∎

3. CONSTRUCTIONS OF CM(k,v)'s

In this section we give some constructions which allow us to determine the spectrum of CM(k,v)'s for $5 \le k \le 8$.

THEOREM 3. *If* k *is prime*, $k \ge 3$, *then a* CM(k,k) *exists.*

Proof. From Theorem 1, the family of cyclically ordered k-tuples $D=\{(i,i,\dots,i): i=1,2,\dots,k-1\}$ determines a CM(k,k). ■

THEOREM 4. *If* $v \equiv 1$ *or* 5 (mod 10), $v>5$, *then a* CM(5,v) *exists.*

Proof. First consider the case $v \equiv 1$ (mod 10).

Let $v=10h+1$, $h \ge 1$. For every $i=1,2,\dots,h$, let $d_i=(i, h+2i-1, 3h+2i-1, 10h-4i+3, 6h-i+1)$, $d'_i=(h+2i, 3h+2i, 10h-4i+2, 10h-4i+4, 6h+4i-3)$ and let $D=\{d_i: i=1,2,\dots,h\} \cup \{d'_i: i=1,2,\dots,h\}$.

It is a routine matter to verify that (1) and (2) of Theorem 1 hold for D, so a CM(5,v) exists for every $v \equiv 1$ (mod 10).

Now, consider the case $v \equiv 5$ (mod 10).

Let $v=10h+5$, $h \ge 1$. For every $i=1,2,\dots,h$ we consider $d_i=(i, 2h-i+1, 6h+i+3, 8h+i+4, 4h-2i+2)$ and $d'_i=(4h+i+2, 7h+i+3, 6h-i+3, 9h+i+4, 4h-2i+3)$. Further, for every $i=1,2,3,4$, let $d''_i=(i(2h+1), i(2h+1), i(2h+1), i(2h+1), i(2h+1))$ and let $D=\{d_i: i=1,2,\dots,h\} \cup \{d'_i: i=1,2,\dots,h\} \cup \{d''_i: i=1,2,3,4\}$.

In a similar way to the case $v \equiv 1$ (mod 10) we can verify that, in view of Theorem 1, the family D determines a CM(5,v) for every $v \equiv 5$ (mod 10), $v>5$. This complete the proof. ■

In the following theorems we apply Theorem 1 to give constructions of CM(k,v)'s for k=6,7,8. The proofs are similar to that of Theorem 4 and we give only the constructions of CM(k,v)'s, because the verifications are tedious but straightforward.

In [1] it is proved that a M(6,v) exists if and only if $v\equiv 0$ or 1 (mod 3), $v>6$. By Theorem 1 it is easy to verify that there does not exist a CM(6,9).

THEOREM 5. *If* $v\equiv 0$ or 1 (mod 3), $v>6$ *and* $v\neq 9$, *then a* CM(6,v) *exists.*

Proof. First, consider the case $v\equiv 0$ (mod 6).

Let $v=6h$, $h\geq 2$. For every $i=1,2,\dots,h-2$ let $d_i=(i,h+i,2h+i,6h-i,5h-i,4h-i)$. Let $D_1=\{d_i: i=1,2,\dots,h-2\}$, $D_2=\{(h,h,h,h,h,h),(5h,5h,5h,5h,5h,5h),(2h,3h-1,4h+1,2h,3h-1,4h+1),(h-1,2h-1,3h,3h+1,5h+1,4h)\}$ and $D=D_1\cup D_2$. Observe that for $h=2$, $D_1=\emptyset$.

From the family D we can construct a CM(6,6h) for every $h\geq 2$.

Consider now the case $v\equiv 1$ (mod 6).

Let $v=6h+1$, $h\geq 1$. For every $i=1,2,\dots,h$ let $d_i=(i,h+i,2h+i,6h-i+1,4h-i+1,5h-i+1)$ and let $D=\{d_i: i=1,2,\dots,h\}$. The family D determines a CM(6,6h+1) for every $h\geq 1$.

Now we suppose that $v\equiv 3$ (mod 6), $v\neq 9$, and we distinguish two cases: $v\equiv 3$ (mod 12) and $v\equiv 9$ (mod 12).

Let $v=12h+3$, $h\geq 1$. For every $i=1,2,\dots,h-1$ let $d_{i1}=(4h+2i,4h-2i+2,4h+2i,4h-2i+2,4h+2i,4h-2i+2)$, $d_{i2}=(2i-1,8h-2i+3,2i-1,8h-2i+3,2i-1,8h-2i+3)$, $d_{i3}=(4h+2i+1,12h-2i+3,4h+2i+1,$

$12h-2i+3, 4h+2i+1, 12h-2i+3)$, $d_{i4}=(8h-2i+2, 8h+2i+2, 8h-2i+2, 8h+2i+2, 8h-2i+2, 8h+2i+2)$, $d_{i5}=(8h+2i+1, 12h-2i+4, 8h+2i+1, 12h-2i+4, 8h+2i+1, 12h-2i+4)$, and for every $i=1,2,\dots,h-2$ $d_{i6}=(4h-2i+1, 2i, 4h-2i+1, 2i, 4h-2i+1, 2i)$.

For $r=1,2,\dots,5$ let $D_r=\{d_{ir}: i=1,2,\dots,h-1\}$; let $D_6=\{d_{i6}: i=1,2,\dots,h-2\}$, $D_7=\{(2h-2, 2h+3, 2h-2, 2h+3, 2h-2, 2h+3)\}$, $D_8= =\{(2h, 2h+1, 2h, 2h+1, 2h, 2h+1)$, $(2h-1, 2h+2, 2h-1, 2h+2, 2h-1, 2h+2)$, $(10h+2, 10h+3, 10h+2, 10h+3, 10h+2, 10h+3)$, $(10h+1, 10h+4, 10h+1, 10h+4, 10h+1, 10h+4)$, $(4h+1, 6h, 6h+1, 8h+2, 6h+2, 6h+3)\}$.

Observe that for $h=1$, $\bigcup_{r=1}^{7} D_r=\emptyset$ and for $h=2$, $D_6=\emptyset$.

For every $h\geq 1$ the family $D=\bigcup_{r=1}^{8} D_r$ determines a $CM(6,12h+3)$.

Let $v=12h+9$, $h\geq 1$. For every $i=1,2,\dots,h$ let $d_{i1}=(4h+2i+2, 4h-2i+4, 4h+2i+2, 4h-2i+4, 4h+2i+2, 4h-2i+4)$, $d_{i2}=(2i-1, 8h-2i+7, 2i-1, 8h-2i+7, 2i-1, 8h-2i+7)$, $d_{i3}=(8h+2i+5, 12h-2i+10, 8h+2i+5, 12h-2i+10, 8h+2i+5, 12h-2i+10)$, and for every $i=1,2,\dots,h-1$ let $d_{i4}=(4h-2i+3, 2i, 4h-2i+3, 2i, 4h-2i+3, 2i)$, $d_{i5}=(4h+2i+3, 12h-2i+9, 4h+2i+3, 12h-2i+9, 4h+2i+3, 12h-2i+9)$, $d_{i6}=(8h-2i+6, 8h+2i+6, 8h-2i+6, 8h+2i+6, 8h-2i+6, 8h+2i+6)$.

For $r=1,2,3$ let $D_r=\{d_{ir}: i=1,2,\dots h\}$ and for $r=4,5,6$ $D_r=\{d_{ir}: i=1,2,\dots,h-1\}$; let $D_7=\{(2h+1, 2h+2, 2h+1, 2h+2, 2h+1, 2h+2)$, $(2h, 2h+3, 2h, 2h+3, 2h, 2h+3)$, $(10h+6, 10h+9, 10h+6, 10h+9, 10h+6, 10h+9)$, $(10h+7, 10h+8, 10h+7, 10h+8, 10h+7, 10h+8)$, $(4h+3, 6h+3, 6h+4, 8h+6, 6h+5, 6h+6)\}$. Observe that if $h=1$ then $D_4=D_5=D_6=\emptyset$.

For every $h\geq 1$ the family $D=\bigcup_{r=1}^{7} D_r$ determines a $CM(6,12h+9)$.

Finally, consider the case $v \equiv 4 \pmod 6$.

Let $v=6h+4$, $h \geq 1$. For every $i=1,2,\ldots,h$ let $d_i=(i+1,h+i+1,2h+i+1,6h-i+4,4h-i+2,5h-i+3)$. For every $h \geq 1$ the family $D=\{d_i: 1,2,\ldots,h\} \cup \{(1,4h+2,5h+3,1,4h+2,5h+3)\}$ determines a $CM(6,6h+4)$.

This complete the proof. ■

THEOREM 6. *If* $v \equiv 1$ or $7 \pmod{14}$, $v>7$, *then a* $CM(7,v)$ *exists.*

Proof. First, consider the case $v \equiv 1 \pmod{14}$.

Let $v=14h+1$, $h \geq 1$. For every $i=1,2,\ldots,h$ let $d_{i1}=(2h-2i+1,2h-2i+2,8h+6i-4,14h-6i+6,4h+4i-3,8h-4i+3,4h+4i-2)$, $d_{i2}==(2h+2i,8h-4i+4,8h+6i-1,14h-6i+4,8h+6i-3,14h-6i+1,2h+2i-1)$.

From the family $D=\{d_{i1}: i=1,2,\ldots,h\} \cup \{d_{i2}: i=1,2,\ldots,h\}$ we can obtain a $CM(7,14h+1)$, for every $h \geq 1$.

Now, we consider the case $v \equiv 7 \pmod{14}$.

Let $v=14h+7$, $h \geq 1$. For every $i=1,2,\ldots,h$ let $d_{i1}=(i,3h-i+2,h+i,5h-i+3,6h+2i+2,4h-i+2,9h-i+5)$, $d_{i2}=(5h+i+2,14h-i+7,6h+2i+3,12h-i+6,13h-i+7,9h+i+4,11h-i+6)$, and for every $i=1,2,\ldots,6$ let $d_{i3}=(i(2h+1),i(2h+1),i(2h+1),i(2h+1),i(2h+1),i(2h+1),i(2h+1))$.

The family $D=\{d_{i1}: i=1,2,\ldots,h\} \cup \{d_{i2}: i=1,2,\ldots,h\} \cup \{d_{i3}: i=1, 2,\ldots,6\}$ determines a $CM(7,14h+7)$ for every $h \geq 1$. ■

THEOREM 7. *If* $v \equiv 1 \pmod 8$, $v>8$, *then a* $CM(8,v)$ *exists.*

Proof. Let $v=8h+1$, $h \geq 1$. The family $D=\{(i,7h-i+1,2h+i,5h-i+1,6h-i+1,h+i,8h-i+1,3h+i): i=1,2,\ldots,h\}$ determines a $CM(8,8h+1)$ for every $h \geq 1$. ■

Collecting together Theorems 2-7 gives the following theorem:

THEOREM 8. *For* $5 \le k \le 8$ *a* CM(k,v) *exists if and only if* k=5 *and* $v \equiv 1$ *or* 5 (mod 10), *with* $v \ge 5$; k=6 *and* $v \equiv 0$ *or* 1 (mod 3), *with* v>6 *and* $v \ne 9$; k=7 *and* $v \equiv 1$ *or* 7 (mod 14), *with* $v \ge 7$; k=8 *and* $v \equiv 1$ (mod 8), *with* v>8.

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