A NEW UPPER BOUND FOR THE RAMSEY NUMBER $R(5,5)$

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Abstract.

We show that, in any colouring of the edges of K_{53} with two colours, there exists a monochromatic K_5, and hence $R(5,5) \leq 53$. This is accomplished in three stages: a full enumeration of (4,4)-good graphs, a derivation of some upper bounds for the maximum number of edges in (4,5)-good graphs, and a proof of the nonexistence of (5,5)-good graphs on 53 vertices. Only the first stage required extensive help from the computer.

1. Introduction.

The two-colour Ramsey number $R(k,l)$ is the smallest integer n such that, for any graph F on n vertices, either F contains K_k or $\bar{F}$ contains K_l, where $\bar{F}$ denotes the complement of F. A graph F is called (k,l)-*good* if F does not contain a K_k and $\bar{F}$ does not contain a K_l. The best upper bound known previously, $R(5,5) \leq 55$, is due to Walker (1971 [7]). The best lower bound, $R(5,5) \geq 43$, was obtained by Exoo (1989 [1]), who constructed a (5,5)-good graph on 42 vertices.

Throughout this paper we will also use the following notation:

$N_F(x)$	— the neighbourhood of vertex x in graph F
$\deg_F(x)$	— the degree of vertex x in graph F
$n(F)$, $e(F)$	— the number of vertices and edges in graph F
$t(F)$	— the number of triangles in F
$\bar{t}(F)$	— the number of independent 3-sets in graph F; i.e. $t(\bar{F})$
$V(F)$	— the vertex set of graph F
(k,l,n)-good graph	— a (k,l)-good graph on n vertices
$e(k,l,n)$	— the minimum number of edges in any (k,l,n)-good graph
$E(k,l,n)$	— the maximum number of edges in any (k,l,n)-good graph
$t(k,l,n)$	— the minimum number of triangles in any (k,l,n)-good graph

* Supported in part by a grant from the National Science Foundation CCR-8920692

Let $n = |V(F)|$ and let n_i be the number of vertices of degree i in F. The well-known theorem of Goodman [2] says that

$$t(F) + \bar{t}(F) = \binom{n}{3} - \frac{1}{2}\sum_{i=0}^{n-1} i(n-i-1)n_i. \quad (1)$$

In his study of the Ramsey numbers $R(k,l)$, Walker [6] observed that if F is a (k,l,n)-good graph then

$$t(F) + \bar{t}(F) \leq \frac{1}{3}\sum_{i=0}^{n-1}\Big(E(k-1,l,i) - e(k,l-1,n-i-1) + \binom{n-i-1}{2}\Big)n_i.$$

Let $x \in V$ be a fixed vertex in a (k,l)-good graph F and consider the two induced subgraphs of F, G_x and H_x, where $V(G_x) = N_F(x)$ and $V(H_x) = V - (\{x\} \cup V(G_x))$. Note that G_x and H_x are $(k-1,l)$-good and $(k,l-1)$-good graphs, respectively. We define the *edge-deficiency* $\delta(x)$ of vertex x to be

$$\delta(x) = E(k-1,l,n(G_x)) - e(G_x) + e(H_x) - e(k,l-1,n(H_x)).$$

The edge deficiency $\delta(x)$ measures how close to extremal graphs the subgraphs G_x and H_x are. Clearly, $\delta(x) \geq 0$. One can also easily see that

$$\delta(x) = E(k-1,l,n(G_x)) - e(G_x) + E(l-1,k,n(H_x)) - e(\bar{H}_x). \quad (2)$$

It is convenient to define the *edge deficiency* $\Delta(F)$ of a (k,l)-good graph F by

$$\Delta(F) = \sum_{x\in V(F)} \delta(x). \quad (3)$$

The first lemma below, similar to (1) in [6], gives a strong condition which permits us to restrict the search space for (k,l)-good graphs.

Lemma 1. *If n_i is the number of vertices of degree i in a (k,l,n)-good graph F then*

$$0 \leq 2\Delta(F) = \sum_{i=0}^{n-1}\big(2E(k-1,l,i) + 2E(l-1,k,n-i-1) + 3i(n-i-1) - (n-1)(n-2)\big)n_i. \quad (4)$$

Proof. Observe that for all $x \in V(F)$ the number of triangles containing x is equal to $e(G_x)$ and the number of independent 3-sets containing x is equal to $e(\bar{H}_x)$. Hence by (2),

$$\begin{aligned} 3(t(F) + \bar{t}(F)) &= \sum_{x\in V(F)} \big(e(G_x) + e(\bar{H}_x)\big) \\ &= \sum_{x\in V(F)} \big(E(k-1,l,n(G_x)) + E(l-1,k,n(H_x)) - \delta(x)\big), \end{aligned}$$

and so by (3) we have

$$0 \leq \Delta(F) = \sum_{i=0}^{n-1}\big(E(k-1,l,i) + E(l-1,k,n-i-1)\big)n_i - 3\big(t(F) + \bar{t}(F)\big).$$

Now using (1) and $\sum_{i=0}^{n-1} n_i = n$, we obtain (4). ∎

2. Generation of all $(4,4)$-good graphs.

This section describes how we generated the set of all (4,4)-good graphs. Let us denote by $R(4,4,n)$ the set of all (4,4,n)-good graphs and let $R'(4,4,n)$ be the subset of those $F \in R(4,4,n)$ with maximum degree D at most $(n-1)/2$. The result of applying the permutation α to the labels of any labelled object X will be denoted by X^α, and also $\mathrm{Aut}(F)$ is the automorphism group of the graph F, as a group of permutations of $V(F)$.

Suppose that θ is a function defined on $\bigcup_{n\geq 2} R'(4,4,n)$ which satisfies these properties:

(i) $\theta(F)$ is an orbit of $\mathrm{Aut}(F)$,

(ii) the vertices in $\theta(F)$ have maximum degree in F, and

(iii) for any F, and any permutation α of $V(F)$, $\theta(F^\alpha) = \theta(F)^\alpha$.

It is easy to implement a function satisfying the requirements for θ by using the program *nauty* [3]. Given θ, and $F \in R'(4,4,n)$ for some $n \geq 2$, the *parent* of F is the graph $\mathrm{par}(F)$ formed from F by removing the first vertex in $\theta(F)$ and its incident edges. The properties of θ imply that isomorphic graphs have isomorphic parents. It is also easily seen that $\mathrm{par}(F) \in R'(4,4,n-1)$. Since $R'(4,4,1) = \{K_1\}$, we find that the relationship "par" defines a rooted directed tree T whose vertices are the isomorphism classes of $\bigcup_{n\geq 1} R'(4,4,n)$, with the graph K_1 at the root. If ν is a node of T, then the *children* of ν are those nodes ν' of T such that for some $F \in \nu'$ we have $\mathrm{par}(F) \in \nu$. The set of children of ν can be found by the following algorithm, whose correctness follows easily from the definitions:

(a) Let F be any representative of the isomorphism class ν.
Suppose that F has n vertices and maximum degree D.

(b) Let $L = L(F)$ be a list of all subsets X of $V(F)$ such that

(b.1) either $|X| > D$, or $|X| = D$ and X does not include any vertex of degree D,

(b.2) X intersects every independent set of size 3 in F,

(b.3) X does not include any triangle of F, and

(b.4) if $F(X)$ is the graph of order $n+1$ formed by joining a new vertex x to X, then $x \in \theta(F(X))$.

(c) Remove isomorphs from amongst the set $\{F(X) \mid X \in L\}$.
The remaining graphs form a set of distinct representatives for the children of ν.

The primary advantage of this method is that isomorph rejection need only be performed within very restricted sets of graphs. For example, even though $|R'(4,4,12)| = 909767$, no isomorphism class of $R'(4,4,11)$ has more than 58 children.

The full set $\bigcup_{n\geq 1} R'(4,4,n)$ was found by this method. Altogether, 5623547 sets X passed conditions (b.1)-(b.3), and 2165034 passed condition (b.4) as well. The total size of $R'(4,4,n)$ for all n is 2065740, which is only slightly less because most (4,4)-good graphs have no nontrivial automorphisms. There are altogether 3432184 nonisomorphic (4,4)-good graphs. The total execution time on a 12-mip computer was 9.4 hours, or 6 milliseconds per invocation of the program *nauty*. In particular, we obtained the information gathered in Table I.

n	4	5	6	7	8	9	10
$\|R(4,4,n)\|$	9	24	84	362	2079	14701	103706
$E(4,4,n)$	5	8	12	16	21	27	31
$t(4,4,n)$	0	0	0	0	0	1	4
n	11	12	13	14	15	16	17
$\|R(4,4,n)\|$	546356	1449166	1184231	130816	640	2	1
$E(4,4,n)$	36	40	45	50	55	60	68
$t(4,4,n)$	7	10	17	25	38	56	68

Table I. Some data on (4,4)-good graphs

3. Upper bounds for $E(4,5,n)$.

Walker [7] established the best upper bound so far of 28 for $R(4,5)$, so we know that any (4,5)-good graph has at most 27 vertices. No $(4,5,n)$-good graph is known for $n \geq 25$. The goal of this section is to derive some upper bounds for $E(4,5,n)$ for $24 \leq n \leq 27$, provided such graphs exist.

Let F be a $(4,5,n)$-good graph and let a_i denote the number of edges in F contained in i triangles. Note that $a_i = 0$ for $i \geq 5$ since F is (4,5)-good. For each $x \in V(F)$ consider induced subgraphs G_x and H_x as in Section 1, which in this case are (3,5)-good and (4,4)-good graphs, respectively.

Lemma 2.

$$\sum_{x \in V(F)} t(H_x) = 4a_4 - 2a_2 - 2a_1 + \sum_{x \in V(F)} (n/3 + 3 - \deg_F(x))e(G_x). \tag{5}$$

Proof. For an arbitrary triangle $T = ABC$ in F let $b_i(T)$ denote the number of vertices in $V(F) - T$ adjacent to exactly i vertices in T, and let $\deg_F(T) = \deg_F(A) + \deg_F(B) + \deg_F(C)$. Note that $b_i(T) = 0$ for $i \geq 3$, since F has no K_4. By counting the 4-sets of vertices formed by any triangle T and any vertex x not adjacent to T in two different ways we have

$$\sum_{x \in V(F)} t(H_x) = \sum_{T-\text{triangle}} b_0(T), \tag{6}$$

and one also easily notes that for each triangle T

$$b_0(T) = n - 3 - b_1(T) - b_2(T) \tag{7}$$

and

$$b_1(T) + 2b_2(T) + 6 = \deg_F(T). \tag{8}$$

Now (7) and (8) give

$$b_0(T) = n + 3 + b_2(T) - \deg_F(T). \tag{9}$$

Using (9) in (6) we obtain

$$\sum_{x \in V(F)} t(H_x) = (n+3)t(F) + \sum_{T-\text{triangle}} \big(b_2(T) - \deg_F(T)\big). \tag{10}$$

Counting edges adjacent to points in triangles by two methods gives

$$\sum_{T-\text{triangle}} \deg_F(T) = \sum_{x \in V(F)} \deg_F(x) e(G_x), \tag{11}$$

and one can also easily see that

$$3t(F) = \sum_{x \in V(F)} e(G_x) = \sum_{i=1}^{4} i a_i. \tag{12}$$

By recalling the definitions of $b_2(T)$ and a_i we conclude that

$$\sum_{T-\text{triangle}} b_2(T) = \sum_{i=2}^{4} i(i-1)a_i = 4a_4 - 2a_2 - 2a_1 + 2\sum_{i=1}^{4} i a_i. \tag{13}$$

Now applying (11), (12) and (13) in (10) we obtain

$$\sum_{x \in V(F)} t(H_x) = \frac{1}{3}(n+3) \sum_{x \in V(F)} e(G_x) + 4a_4 - 2a_2 - 2a_1 + 2 \sum_{x \in V(F)} e(G_x) - \sum_{x \in V(F)} \deg_F(x) e(G_x),$$

which can be easily converted to (5). ∎

We know that for each vertex x the number of triangles in H_x is at least $t(4, 4, n(H_x))$, where $n(H_x) = n - 1 - \deg_F(x)$. Define the *triangle deficiencies* $\gamma(x)$ of a vertex x and $\Gamma(F)$ of a graph F as

$$\gamma(x) = t(H_x) - t(4, 4, n(H_x)), \quad \Gamma(F) = \sum_{x \in V(F)} \gamma(x). \tag{14}$$

For any vertex x we obviously have $\gamma(x) \geq 0$.

Lemma 3. *If F is any $(4,5,n)$-good graph on at least 24 vertices and F has n_i vertices of degree i for each i, then*

$$0 \leq 3\Gamma(F) \leq \sum_{i=6}^{13}\big((n+9-3i)E(3,5,i)+6i-3t(4,4,n-i-1)\big)n_i. \tag{15}$$

Proof. Since $R(3,5)=14$ and $R(4,4)=18$, by (5) we have

$$3\sum_{x\in V(F)} t(H_x) = 12a_4 - 6a_2 - 6a_1 + \sum_{i=6}^{13}\sum_{\deg_F(x)=i}(n+9-3i)e(G_x).$$

Note that for $n \geq 24$ the coefficient $n+9-3i$ is negative only for $i=13$ or for $i=12$ and $n=24,25,26$, hence we can use $E(3,5,i)$ in place of $e(G_x)$ in the following inequality except in those cases.

$$3\sum_{x\in V(F)} t(H_x) \leq 12a_4 + \sum_{i=6}^{13}(n+9-3i)E(3,5,i)n_i$$
$$+ \sum_{\deg_F(x)\geq 12}(E(3,5,\deg_F(x)) - e(G_x))(3\deg_F(x)-n-9). \tag{16}$$

All (3,5)-good graphs are known ([5] and independently [4]). In particular, there exists a unique (3,5,13)-good graph, which implies that the terms in the last summation for $\deg_F(x) \geq 13$ are equal to zero. It is also known that $E(3,5,12)=24$ is achieved only by 4-regular graphs, and furthermore any (3,5,12)-good graph has only vertices of degree 3 and/or 4. Thus if for some vertex x of degree 12 in F the graph G_x is not maximal, i.e. $e(G_x) < 24$, then for each vertex y of degree 3 in G_x the edge $\{x,y\}$ contributes to a_3, and each edge appearing in three triangles can be accounted at most twice this way. Thus the second summation in the right hand side of (16) is at most $3a_3$ for $n \geq 24$. Hence by $e(F) \geq a_4 + a_3$ and (16) we find

$$3\sum_{x\in V(F)} t(H_x) \leq 12e(F) + \sum_{i=6}^{13}(n+9-3i)E(3,5,i)n_i. \tag{17}$$

Finally, we can easily obtain (15) by using (14), (17) and $12e(F) = \sum_{i=6}^{13} 6in_i$. ∎

Theorem 1. *If we interpret $e(k,l,n)$ as ∞ and $E(k,l,n)$ as 0 for $n \geq R(k,l)$ then $153 \leq e(4,5,27)$ and $E(4,5,27) \leq 160$, $130 \leq e(4,5,26)$ and $E(4,5,26) \leq 154$, $116 \leq e(4,5,25)$ and $E(4,5,25) \leq 148$, $101 \leq e(4,5,24)$ and $E(4,5,24) \leq 139$.*

Proof. Let F be any $(4,5,n)$-good graph for some $24 \leq n \leq 27$ with e edges and n_i vertices of degree i. Consider the set of constraints formed by $\sum_{i=6}^{13} n_i = n$ and the conditions for $\Delta(F)$ and $\Gamma(F)$ given by Lemmas 1 and 3, respectively. This gives a simple instance (for a computer) of a non-negative integer linear programming optimization problem with variables n_i and objective function $2e = \sum_{i=6}^{13} in_i$. For $n = 27$ we have to minimize or maximize

$$9n_9 + 10n_{10} + 11n_{11} + 12n_{12} + 13n_{13}$$

subject to

$$\begin{aligned} 27 &= n_9 + n_{10} + n_{11} + n_{12} + n_{13}, \\ 0 &\leq -21n_9 - 10n_{10} - n_{11} + 2n_{12} - n_{13}, \end{aligned} \tag{18}$$

and

$$0 \leq n_9 + 4n_{10} + 6n_{11} - n_{12} - 17n_{13}, \tag{19}$$

where constraint (18) is obtained from (4) and constraint (19) is obtained from (15), using the numerical data from Table I for $t(4,4,j)$, $E(4,4,i)$, and some of the results listed in [5], namely $E(3,5,i) = 2i$ for $10 \leq i \leq 13$ and $E(3,5,9) = 17$. Also in [5] we find the values $E(3,5,8) = 16$, $E(3,5,7) = 12$ and $E(3,5,6) = 9$, which are needed for the calculations in the cases of $24 \leq n \leq 26$. For $n = 27$ the maximal number of edges e is 160 with the unique possible degree sequence $n_{12} = 23$ and $n_{11} = 4$. The other bounds are obtained similarly. We used a simple computer program to perform these calculations, and another to check them. ∎

The numbers of edges in the known (4,5,24)-good graphs range from 118 to 132 (personal communication from G. Exoo). The lower bounds for $e(4,5,n)$ are not needed for the proof of $R(5,5) \leq 53$; they are included in Theorem 1 for completeness.

4. An upper bound for $R(5,5)$.

We are now in a position to prove our major result.

Theorem 2. $R(5,5) \leq 53$.

Proof. Assume that F is a (5,5)-good graph on 53 vertices and let n_i be the number of vertices of degree i in F. Since $R(4,5) \leq 28$ we have in this case $n_{25} + n_{26} + n_{27} = 53$. The calculation of bounds for $2\Delta(F)$ from Lemma 1, using Theorem 1, gives

$$\begin{aligned} 0 &\leq (2{\cdot}308 + 3{\cdot}25{\cdot}27 - 52{\cdot}51)(n_{25} + n_{27}) + (2{\cdot}308 + 3{\cdot}26{\cdot}26 - 52{\cdot}51)n_{26} \\ &= -11(n_{25} + n_{27}) - 8n_{26}, \end{aligned}$$

which is a contradiction. ∎

The same method does not disprove the existence of a (5,5,52)-good graph, but such a result would be possible if we could sufficiently improve the bounds of Theorem 1.

References.

[1] G. Exoo, A Lower Bound for $R(5,5)$, *Journal of Graph Theory*, **13** (1989) 97–98.

[2] A. W. Goodman, On sets of acquaintances and strangers at any party, *Amer. Math. Monthly*, **66** (1959) 778–783.

[3] B. D. McKay, Nauty User's Guide (version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University (1990).

[4] B. D. McKay and Zhang Ke Min, The value of the Ramsey number $R(3,8)$, submitted.

[5] S. P. Radziszowski and D. L. Kreher, On $(3,k)$ Ramsey Graphs: Theoretical and Computational Results, *J. Combinatorial Math. and Combinatorial Comput.*, **4** (1988) 37–52.

[6] K. Walker, Dichromatic Graphs and Ramsey Numbers, *J. Combinatorial Theory*, **5** (1968) 238–243.

[7] K. Walker, An Upper Bound for the Ramsey Number $M(5,4)$, *J. of Combinatorial Theory*, **11** (1971) 1–10.