# The spectrum for 3-perfect 9-cycle systems

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ABSTRACT: A decomposition of $K_n$ into 3-perfect 9-cycles is shown to exist if and only if $n \equiv 1$ or 9 (modulo 18), $n \neq 9$.

## 1 Introduction

A great deal of work in recent years has been done on decompositions of the complete graph into edge-disjoint cycles; see the survey [5] for example. Specifically, we call a decomposition of the complete graph $K_v$ into disjoint cycles of length $m$ an *$m$-cycle system of $K_v$*. In other words, an $m$-cycle system of $K_v$ is an ordered pair $(V, C)$ where $V$ is the vertex set of $K_v$ and $C$ is a set of edge disjoint $m$-cycles which partition the edge set of $K_v$.

It is possible to require additional structure of the decomposition. For example, suppose we have an $m$-cycle system of $K_v$ so that when, for each cycle, we take the graph formed by joining all vertices distance $i$ apart, we again have a decomposition of $K_v$. Then this is called an *$i$-perfect $m$-cycle decomposition of $K_v$*, or an *$i$-perfect $m$-cycle system.* Previous papers ([4,3,1]) have considered 2-perfect $m$-cycle systems. Here we completely determine the spectrum (that is, the set of all values of $v$) for 3-perfect 9-cycle systems of $K_v$. In particular, we prove:

**THEOREM 1.1** *The necessary and sufficient conditions for a 3-perfect 9-cycle decomposition of $K_v$ are $v \equiv 1$ or 9 (mod 18) and $v \neq 9$.*

So that the reader does not get the erroneous idea that 3-perfect 9-cycle systems are contrived, we point out that the distance 3 graph inside each 9-cycle consists of three

*Research supported by the Australian Research Council
†Research supported by NSF grant DMS–8913576 and NSA grant MDA–904–89–H–2016.

disjoint 3-cycles (or triples) and that the collection of these triples is a Steiner triple system. In other words, a 9-cycle system is 3-perfect if and only if the collection of three disjoint triples inside each 9-cycle is a Steiner triple system. While it is virtually trivial [5] to construct 9-cycle systems for every $v \equiv 1$ or $9 \pmod{18}$, a bit of reflection will convince the reader that the additional requirement of being 3-perfect is far less trivial a matter. We point out that the analogous problem for 6-cycle systems or *hexagon* systems is the problem of constructing 2-perfect hexagon systems. This problem has been completely settled in [3].

First we verify the necessary conditions for a 3-perfect 9-cycle system to exist. Certainly the number of edges of $K_v$, namely $v(v-1)/2$, must be divisible by 9. Moreover, the degree of each vertex, $v-1$, must be even, and so $v$ must be odd. These requirements mean that $v$ must be 1 or 9 modulo 18.

The graph $K_9$ cannot be decomposed into 3-perfect 9-cycles. The underlying triple system would be the (unique) affine plane of order three, and it is not hard to verify that there is *no* possible way that the triples can fit inside four 9-cycles to form a 3-perfect 9-cycle system.

Subsequently we shall show that for all other $v \equiv 1$ or $9$ (modulo 18), there exists a 3-perfect 9-cycle decomposition of $K_v$.

## 2 The case $v \equiv 1$ modulo 18

We start with a couple of necessary examples. In what follows we shall denote the $m$-cycle consisting of the edges $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \dots, \{x_{m-1}, x_m\}, \{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, \dots, x_m)$ or $(x_1, x_m, x_{m-1}, \dots, x_2)$.

**EXAMPLE 2.1** *There are 3-perfect 9-cycle decompositions of $K_v$ for $v = 19$ and 37.*

For $v = 19$, define

$$C_{19} = \{(0,12,17,1,2,10,6,4,13) + i \mid 0 \le i \le 18\},$$

where $(v_0, v_1, \dots, v_8) + i = (v_0 + i, v_1 + i, \dots, v_8 + i)$, with entries reduced modulo 19. Then $(\mathbb{Z}_{19}, C_{19})$ is a 3-perfect 9-cycle system of $K_{19}$.

For $v = 37$, define

$$C_{37} = \{(0,16,5,6,8,1,13,7,15) + i, (0,18,23,3,36,2,15,1,28) + i \mid 0 \le i \le 36\}.$$

Then $(\mathbb{Z}_{37}, C_{37})$ is a 3-perfect 9-cycle system of $K_{37}$. □

The next example is crucial to the constructions used in both cases, 1 modulo 18 and 9 modulo 18.

**EXAMPLE 2.2** *There is a* 3*-perfect* 9*-cycle decomposition of* $K_{9,9,9}$.

Let the vertices of $K_{9,9,9}$ be $V_9 = \{(i,j) \mid 0 \le i \le 8, 1 \le j \le 3\}$. We fix the second components and work modulo 9 with the first component. Also for brevity, we shall write $ij$ for $(i,j)$. Then $(V_9, C_9)$ is a 3-perfect 9-cycle decomposition of $K_{9,9,9}$ where

$$\begin{aligned} C_9 = \{ & (01,22,61,02,33,82,03,11,13) + i0, \\ & (01,23,41,32,31,02,63,72,53) + i0, \\ & (01,42,63,62,23,81,33,61,72) + i0 \mid 0 \le i \le 8\} \end{aligned}$$

(first component reduced modulo 9). □

The following result is well-known; we include it for completeness.

**LEMMA 2.3** *There is a group divisible design on* $2n \ge 6$ *elements with block size* 3 *and group size* 2 *whenever* $2n \equiv 0$ *or* 2 *(mod* 6*); there is a group divisible design on* $2n \ge 10$ *elements with block size* 3*, one group of size* 4 *and the rest of size* 2*, when* $2n \equiv 4$ *(mod* 6*).*

**Proof:** The cases $2n \equiv 0$ or 2 (mod 6) first appeared in Hanani [2], Lemma 6.3; such group divisible designs also arise from any Steiner triple system by deleting one point. For the case $2n \equiv 4$ (mod 6), see for example [6, page 276]. This gives a pairwise balanced design with number of elements congruent to 5 (mod 6), and with one block of size five and the rest of size three. Deletion of a point from the block of size five yields a suitable group divisible design, with one group of size four and the rest of size two. □

We are now ready to give a general construction.

**THEOREM 2.4** *There exists a* 3*-perfect* 9*-cycle decomposition of* $K_v$ *for all* $v \equiv 1$ *modulo* 18.

**Proof:** Let $v = 18n + 1$. The cases $n = 1$ and $n = 2$ are done in Example 2.2. So let $n \ge 3$, and let the vertex set $V$ of $K_v$ be $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 2n, \ 1 \le j \le 9\}$. By Lemma 2.3, there exists a group divisible design (GDD) on $\{(i,j) \mid 1 \le i \le 2n\}$ with all groups of size 2, except possibly one of size 4.

Then take 9-cycles as follows:

(i) For each block $\{(x,j),(y,j),(z,j)\}$ in the GDD, place a copy of the decomposition $(V_9, C_9)$ of $K_{9,9,9}$ from Example 2.2 on the vertices

$$\{(x,j) \mid 1 \le j \le 9\} \cup \{(y,j) \mid 1 \le j \le 9\} \cup \{(z,j) \mid 1 \le j \le 9\}.$$

(ii) For each group $\{(a_1,j),(a_2,j)\}$ of size 2 of the GDD, place a copy of $(\mathbb{Z}_{19}, C_{19})$ on the vertices $\{\infty\} \cup \{(a_1,j),(a_2,j) \mid 1 \le j \le 9\}$. (See Example 2.1.)

(iii) When $2n \equiv 4$ (mod 6), one group is of size 4; say it is $\{(a_i,j) \mid 1 \le i \le 4\}$. Then place a copy of $(\mathbb{Z}_{37}, C_{37})$ on the vertices $\{\infty\} \cup \{(a_i,j) \mid 1 \le i \le 4, 1 \le j \le 9\}$. (See Example 2.1.)

It is clear that this is now a 3-perfect 9-cycle decomposition of $K_v$. □

## 3 The case $v \equiv 9$ modulo 18

Here we let $v = 18n + 9$. We have already pointed out the impossibility of the case $n = 0$. The cases $n = 1$ and 2 follow.

**EXAMPLE 3.1** *There is a 3-perfect 9-cycle decomposition of $K_{27}$.*

Let the 27 elements be $\{\infty\} \cup \{(i,j) \,|\, 0 \le i \le 12,\ 1 \le j \le 2\}$. Then the following 9-cycles give a suitable decomposition:

$$((0,2),\infty,(0,1),(1,2),(12,2),(1,1),(4,2),(12,1),(4,1)) + (i,0);$$
$$((0,1),(1,1),(11,1),(2,1),(6,2),(0,2),(3,2),(8,1),(6,1)) + (i,0);$$
$$((5,2),(8,1),(2,2),(10,2),(1,2),(0,2),(1,1),(7,2),(5,1)) + (i,0);$$

here $0 \le i \le 12$. (So the second component is fixed, $\infty$ is of course fixed, and the first component is cycled modulo 13.) □

**EXAMPLE 3.2** *There is a 3-perfect 9-cycle decomposition of $K_{45}$.*

This time the 45 elements are $\{\infty\} \cup \{(i,j) \,|\, 0 \le i \le 10,\ 1 \le j \le 4\}$. There are ten "starter" 9-cycles, modulo 11 on the first component of each element. The element $\infty$ is fixed, and the second component of each element is also fixed. The starters are:

$$(\infty,(1,2),(8,2),(0,1),(7,3),(10,1),(0,2),(5,4),(8,4)),$$
$$(\infty,(8,1),(3,2),(0,3),(10,2),(6,4),(0,4),(4,1),(5,3)),$$
$$((0,1),(1,1),(1,2),(2,1),(2,4),(0,4),(5,1),(8,3),(5,3)),$$
$$((0,1),(4,1),(2,4),(1,1),(3,2),(5,2),(5,4),(1,4),(10,3)),$$
$$((0,2),(7,1),(10,1),(6,2),(1,3),(8,3),(7,4),(10,3),(10,4)),$$
$$((0,3),(10,3),(10,1),(1,3),(1,2),(6,2),(4,3),(9,1),(1,4)),$$
$$((0,4),(4,3),(0,1),(2,4),(10,3),(8,2),(6,4),(2,1),(10,4)),$$
$$((0,4),(7,3),(4,2),(1,4),(8,2),(10,1),(5,1),(7,1),(5,3)),$$
$$((0,2),(6,4),(8,3),(1,2),(3,4),(9,1),(3,2),(7,3),(1,4)),$$
$$((0,2),(8,1),(7,4),(4,2),(1,2),(2,2),(7,3),(1,3),(10,3)).$$

□

For the main construction in the case $v \equiv 9 \pmod{18}$ we also use a 3-perfect 9-cycle decomposition of $K_{27} \backslash K_9$; this is a decomposition into 3-perfect 9-cycles of the graph on 27 vertices in which all pairs of vertices are adjacent except for pairs occurring in a distinguished set of nine vertices. We shall refer to the 9 distinguished vertices as the "hole".

**LEMMA 3.3** *There is a 3-perfect 9-cycle decomposition of $K_{27} \backslash K_9$.*

**Proof:** Our elements are

$$\{(i,j) \,|\, i = 0,1,2; 1 \le j \le 6\} \cup \{\mathrm{A,B,C,D,E,F,G,H,I}\}.$$

A simple count shows that thirty-five 9-cycles are required. We take two fixed 9-cycles, and a further 11 starters, where we fix the second component of each element, and cycle modulo 3 on the first component of the elements; the elements in the hole are fixed and not cycled at all. The two fixed cycles are:

$$((0,1),(0,3),(0,5),(1,1),(1,3),(1,5),(2,1),(2,3),(2,5)),$$
$$((0,2),(1,4),(0,6),(1,2),(2,4),(1,6),(2,2),(0,4),(2,6)).$$

Then the remaining 11 cycles which are starter cycles modulo (3,–) (and with hole elements fixed) are:

$$((0,1),(1,3),(2,3),(0,2),(1,5),(0,6),(0,3),(1,4),(0,4)),$$
$$((0,1),(1,1),(1,5),(0,5),(2,4),(2,6),(0,6),(2,2),(1,2)),$$
$$((0,1),(1,6),(1,5),(2,2),\mathrm{D},(2,4),\mathrm{A},(1,3),\mathrm{F}),$$
$$((0,5),(2,3),(1,4),(2,6),\mathrm{F},(2,2),\mathrm{A},(0,1),\mathrm{D}),$$
$$((0,3),(0,2),(1,1),(2,4),\mathrm{F},(2,5),\mathrm{A},(0,6),\mathrm{D}),$$
$$((2,1),(1,3),(0,2),(2,4),\mathrm{G},(2,5),\mathrm{C},(0,6),\mathrm{B}),$$
$$((0,1),(2,6),(2,2),(2,5),\mathrm{B},(1,3),\mathrm{H},(0,4),\mathrm{E}),$$
$$((1,1),(0,4),(0,3),(2,6),\mathrm{I},(1,5),\mathrm{E},(2,2),\mathrm{C}),$$
$$((2,6),(1,3),(0,5),(1,4),\mathrm{B},(2,2),\mathrm{H},(0,1),\mathrm{G}),$$
$$((2,1),(2,6),(1,5),(1,4),\mathrm{C},(2,3),\mathrm{G},(0,2),\mathrm{I}),$$
$$((0,4),(0,2),(0,1),(1,5),\mathrm{H},(2,6),\mathrm{E},(1,3),\mathrm{I}).$$

□

We are now ready to give the main construction when $v \equiv 9 \pmod{18}$.

**THEOREM 3.4** *There exists a 3-perfect 9-cycle decomposition of $K_v$ for all $v \equiv 9$ modulo 18, $v > 9$.*

**Proof:** Let $v = 18n + 9$. The case $n = 1$ is dealt with in Example 3.1 and the case $n = 2$ in Example 3.2. So now assume that $n \geq 3$. Let the vertex set $V$ of $K_v$ be

$$\{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E},\mathrm{F},\mathrm{G},\mathrm{H},\mathrm{I}\} \cup \{(i,j) \mid 1 \leq i \leq 2n, \ \ 1 \leq j \leq 9\}.$$

By Lemma 2.3, there exists a group divisible design on $\{(i,j) \mid 1 \leq i \leq 2n\}$ with all groups of size 2, except one of size 4 when $2n \equiv 4 \pmod 6$.

Take 9-cycles as follows:

(i) For each block $\{(x,j),(y,j),(z,j)\}$ in the group divisible design, place a copy of the decomposition $(V_9, C_9)$ of $K_{9,9,9}$ from Example 2.2 on the vertices

$$\{(x,j) \mid 1 \leq j \leq 9\} \cup \{(y,j) \mid 1 \leq j \leq 9\} \cup \{(z,j) \mid 1 \leq j \leq 9\}.$$

(ii) If $2n \equiv 0$ or $2 \pmod 6$, for each group of the GDD, $\{(a_1,j),(a_2,j)\}$ of size two, *except one*, place a copy of the decomposition of $K_{27}\backslash K_9$ (given in Lemma 3.3) on the vertices

$$\{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E},\mathrm{F},\mathrm{G},\mathrm{H},\mathrm{I}\} \cup \{(a_1,j),(a_2,j) \mid 1 \leq j \leq 9\}.$$

For the remaining group of size two, on its 18 vertices (as $j$ varies from 1 to 9) together with the vertices {A, B, ..., I}, place a decomposition of $K_{27}$ (see Example 3.1).

(iii) If $2n \equiv 4 \pmod 6$, then one group of the GDD has size four; suppose that group is $\{(b_1, j), (b_2, j), (b_3, j), (b_4, j)\}$. Then on

$$\{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{I}\} \cup \{(b_i, j) \mid 1 \le i \le 4,\ \ 1 \le j \le 9\}$$

place a decomposition of $K_{45}$, given in Example 3.2. Finally, for the remaining groups $\{(a_1, j), (a_2, j)\}$ of the GDD of size two, on the vertex set

$$\{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{I}\} \cup \{(a_1, j), (a_2, j) \mid 1 \le j \le 9\}$$

place a decomposition of $K_{27} \backslash K_9$, given in Lemma 3.3.

The result is a 3-perfect 9-cycle decomposition of $K_v$. □

Theorems 2.4 and 3.4 complete the proof of the main Theorem 1.1.

# 4 Concluding remarks.

There is a never-ending list of problems involving the existence of $m$-cycle systems with additional properties, such as being $i$-perfect. In view of the results in [**3**] and this paper, one avenue of research is certainly the determination of the spectrum of $k$-perfect $3k$-cycle systems for $k \ge 4$. A good place to start is with the spectrum of 4-perfect 12-cycle systems. Good luck!

# References


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