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Stabilizers for GF(5)-representable matroids

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Abstract

We give a constructive proof for the existence of small size uniquely representable stabilizers for GF(5)-representable matroids.

1 Introduction

The matroid terminology follows Oxley [4]. Two $r \times n$ matrices A and A' representing the same matroid M over a field GF(q) are called *projectively equivalent representations* of M if one can be obtained from the other by elementary row operations and column scaling. Otherwise they are called *projectively inequivalent*. The matrices A and A' are called *equivalent* representations of M if, in addition to elementary row operations and column scaling, A' may be obtained from A by replacing each entry of A by its image under a non-trivial automorphism of GF(q). Otherwise they are called *inequivalent representations*. For prime fields, as in this case, projective equivalence and equivalence coincide as prime fields do not have non-trivial automorphisms.

A minor N of a matroid M stabilizes M over GF(q) if no GF(q)-representation of N can be extended to two projectively inequivalent GF(q)-representations of M. Observe that if N has k projectively inequivalent representations, then M has at most k projectively inequivalent representations. We say that N is a *stabilizer* for GF(q) if N stabilizes each 3-connected GF(q)-representable matroid that contains N as a minor. A line (flat of rank-2) in a simple matroid is called *non-trivial* if it has at least three points. Otherwise it is called *trivial*. The uniform matroid $U_{2,5}$ denotes the five point line and $M(W_3)$ denotes the matroid representing the 3-wheel.

The following theorem is the main result in this note.

Theorem 1.1. *Suppose M is a rank 3 simple GF(5)-representable matroid that can be obtained from an $M(W_3)$-minor by a sequence of extensions where each extension point lies on at least two non-trivial lines. Further suppose M has a $U_{2,5}$-minor. Then M is a uniquely representable stabilizer for GF(5)-representable matroids.*

The significance of this result is that it generalizes the following result by Geelen and Whittle in the case of GF(5): for each prime power q, $PG(2,q)$ is a stabilizer for GF(q) [2]. It may be possible that rank-3 matroids considerably smaller than $PG(2,q)$ that satisfy certain geometric conditions can serve as stabilizers for GF(q). We were able to find such matroids for GF(5).

2 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on two lemmas. The next lemma may be derived from Duke [1, Prop. 3.5]. A short direct proof may be found in [3, Prop 3.1]. We denote the single-element extension of a matroid M by an element e as $M+e$.

Lemma 2.1. *Let M be a connected rank-3 simple GF(q)-representable matroid that extends to a simple single-element extension $M+e$ over GF(q), where e lies on at least two non-trivial lines. Then M stabilizes $M+e$.*

In [6] Whittle proved that in order to check whether a 3-connected matroid N is a stabilizer for GF(q) it is sufficient to check matroids M with $r(M) \leq r(N)+1$ and $r^*(M) \leq r^*(N)+1$. A short proof of the following proposition that uses Whittle's result appears in [3, Prop. 4.1]. The original, rather lengthy, proof appears in [5].

Lemma 2.2. *The matroid $U_{2,5}$ is a stabilizer for GF(5)-representable matroids.*

Proof of Theorem 1.1. First observe that $M(W_3)$ is uniquely representable. Lemma 2.1 implies that if M can be obtained from $M(W_3)$-minor by a sequence of extensions by points that lie on at least two non-trivial lines, then each matroid in the sequence, and consequently M itself, is uniquely representable. Since $U_{2,5}$ is a stabilizer for GF(5)-representable matroids (Lemma 2.2) and M has a $U_{2,5}$-minor, M is a uniquely representable stabilizer for GF(5)-representable matroids. □

It follows directly from the main theorem that $PG(2,5)$ is a uniquely representable stabilizer for 3-connected GF(5)-representable matroids.

3 Small size stabilizers for GF(5)-representable matroids

Using Theorem 1.1 we find that the smallest size uniquely representable stabilizers for GF(5)-representable matroids occur at 9 elements (see Figure 1).

We begin with $M(W_3)$ and look for single-element extensions where the extension point lies on at least two non-trivial lines. The non-Fano matroid, F_7^-, is the only simple single-element extension of $M(W_3)$ that meets this requirement. Figure 2 shows the single-element extensions of F_7^- with the extension point circled. Observe that the second extension is the only one that meets the requirement.

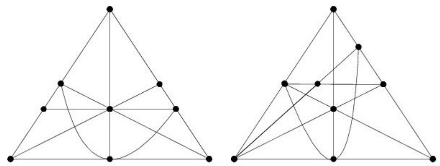

Figure 1: Smallest size uniquely representable stabilizers for GF(5)

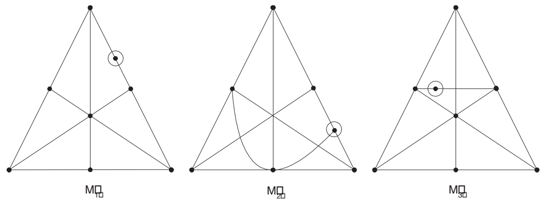

Figure 2: The simple single-element extensions of F_7^- over GF(5)

Continuing this way, the second extension of F_7^- has six simple single-element extensions over GF(5). They are shown in Figure 3.

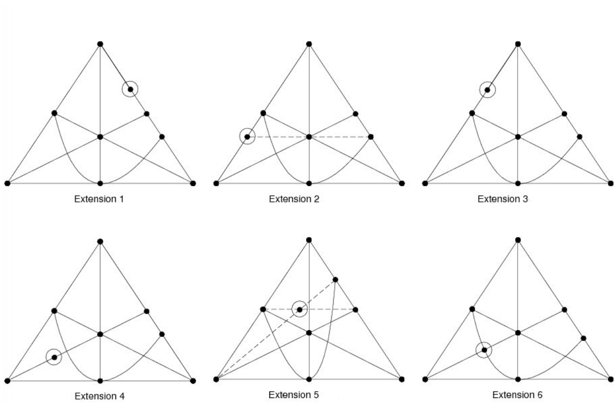

Figure 3: The simple single-element extensions of M_2 over GF(5)

The second and fifth extensions are uniquely representable stabilizers since each has a $U_{2,5}$-minor. The last extension has no $U_{2,5}$-minor even though the extension point lies on at least two non-trivial lines. It is interesting to note that even though the extension point in the third extension lies on only one non-trivial line, we can check that it is stabilized by M_2. Since it has a $U_{2,5}$-minor it is a uniquely representable stabilizer for GF(5)-representable matroids.

The relative ease with which one can find small size stabilizers for GF(5)-representable matroids suggests that Geelen and Whittle's result that $PG(2, q)$ is a uniquely representable stabilizer can be strengthened. However, although one might expect $U_{2,7}$ to be a candidate for a uniquely representable stabilizer for GF(7)-representable matroids, it is not a stabilizer because $U_{2,7}$ stabilizes only 18 of its 20 single-element extensions (see Figure 4). The extensions labeled U_8 and V_8 are not stabilized by $U_{2,7}$. Further it is known that for $q \geq 7$ there are two infinite families of matroids without a bounded number of inequivalent representations [5]. Thus the result obtained for GF(5)-representable matroids is best possible.

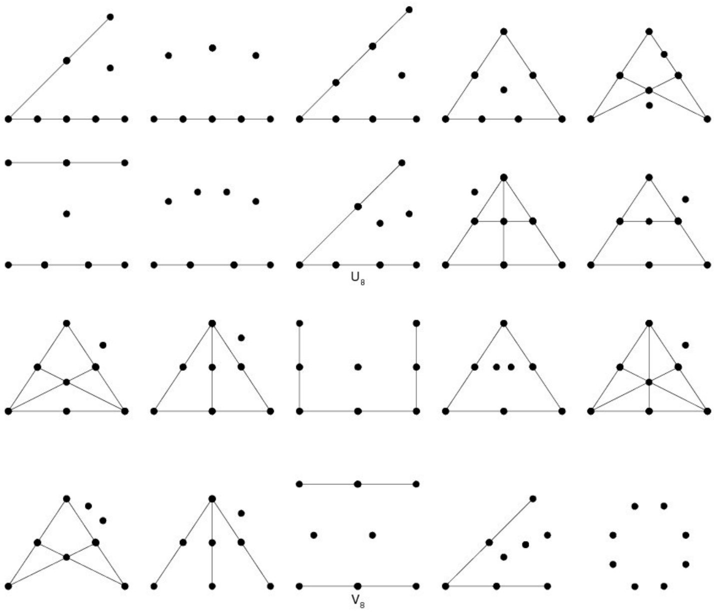

Figure 4: The 3-connected single-element coextensions of $U_{2,7}$ over GF(7)

Acknowledgements

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References

[1] R. Duke, Freedom in matroids, *Ars Combin.* **26B** (1988), 191–216.

[2] J. Geelen and G. Whittle, The projective plane is a stabilizer, *J. Combin. Theory Ser. B* **100** no. 2 (2010), 128–131.

[3] S. R. Kingan, A computational approach to inequivalence in matroids, Proc. Thirty-Seventh Southeastern Int. Conf. Combin., Graph Theory and Computing, *Congr. Numer.* **198** (2009), 63–74.

[4] J. G. Oxley, *Matroid Theory*, (1992), Oxford University Press, New York.

[5] J. G. Oxley, D. Vertigan and G. Whittle, On inequivalent representations of matroids over finite fields, *J. Combin. Theory, Ser. B* **67** (1996), 325–343.

[6] G. Whittle, Stabilizers of classes of representable matroids, *J. Combin. Theory Ser. B* **77** no. 1 (1999), 39–72.

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