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Inflection points of coherent reliability polynomials

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Abstract

Examples of coherent reliability polynomials with more than one inflection point are given. They are created by examining the structure of a reliability polynomial using a convex basis.

1 Introduction

Let $G = (V, E)$ be an undirected graph. Let $H = (V, E')$ be the random subgraph of G where each $e \in E$ independently belongs to E' with probability p. The probability that H is connected is given by a polynomial in p; this polynomial is called the *all-terminal reliability polynomial* of G.

The reliability polynomial can also be defined in a more general setting. Consider a base set X with n elements. Let $\mathcal{W}$ be a subset of $\mathcal{P}(X)$ such that $\mathcal{W}$ is closed under taking supersets, i.e. if $W \in \mathcal{W}$ and $W \subseteq V$, then $V \in \mathcal{W}$. Let $S \subseteq X$ be a set resulting in selecting each element of X with probability p. Then the reliability polynomial of $\mathcal{W}$ on X is the probability that $S \in \mathcal{W}$. This polynomial will be denoted by $\mathrm{Rel}_{\mathcal{W}}(p; X)$, or $\mathrm{Rel}_{\mathcal{W}}(p)$ if the choice of X is clear. A reliability polynomial formed from this general structure is called a *coherent* reliability polynomial. Throughout this paper, the term reliability polynomial will be used to signify a coherent reliability polynomial.

There is an easy way to compute the reliability polynomial; simply add up the probability of each set in $\mathcal{W}$ occurring. This yields the formula:

$$\mathrm{Rel}_{\mathcal{W}}(p) = \sum_{i=0}^{n} N_i p^i (1-p)^{n-i}$$

where N_i is the number of sets in $\mathcal{W}$ of size i. This representation of the reliability polynomial is called the *N-form*, and the vector $(N_0, N_1, \ldots, N_n)$ is called the *N-vector*.

The second section of this paper gives a new form for reliability polynomials in terms of Bernstein polynomials. A *Bernstein polynomial* is a polynomial of the form

$$b_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-1}.$$

We define the polynomial $B_{k,n}(p)$ to be the following sum of Bernstein polynomials:

$$B_{k,n}(p) = \sum_{i=k}^{n} b_{i,n}(p).$$

In fact, $B_{k,n}$ is a reliability polynomial. If we define $\mathcal{W}_{k,n}$ to be the set of all subsets of size greater than or equal to k of a set of size n, then $B_{k,n}(p)$ is the reliability polynomial of $\mathcal{W}_{k,n}$. The first main result (see Theorem 1) of this paper is that any reliability polynomial can be written as a convex combination of $B_{k,n}$'s. In other words, if f is any reliability polynomial, then

$$f = \lambda_0 B_{0,n} + \lambda_1 B_{1,n} + \cdots \lambda_n B_{n,n}$$

where $0 \le \lambda_i \le 1$ for all i and $\sum_{i=0}^{n} \lambda_i = 1$. We call the vector

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$$

the λ-*vector* of f.

This new method of describing a reliability polynomial gives a different insight into how reliability polynomials behave. Reliability polynomials have been extensively studied in network designs (see [2]). In particular, the shape of reliability polynomials has been studied since Moore and Shannon first popularized this representation of reliability. In their pair of papers [6], they proved that a reliability polynomial can cross the diagonal line $f(p) = p$ at most once. Shortly thereafter, Birnbaum, Esary and Saunders [1] showed exactly what conditions are necessary for a reliability polynomial to cross the diagonal exactly once.

A reliability polynomial is clearly an increasing function; thus, another aspect of a reliability polynomial's shape that can be explored is the rate which a reliability polynomial increases. Of course, this naturally leads to considering derivatives and then to the exploration of inflection points. In his 1993 paper "Some Open Problems on Reliability Polynomials," Colbourn posed the question, "Is it true that an (all-terminal or two-terminal) reliability polynomial has at most one point of inflection in the range $p \in (0,1)$?" [3]. Although this question is still open for all-terminal and two-terminal reliability polynomials, examples of coherent reliability polynomials that do have two inflection points can be constructed.

The third section of this paper gives sufficient conditions for a reliability polynomial to have a unique inflection point (see Theorem 4). If f is a reliability polynomial of the form

$$f(p) = (1 - x_1 - \cdots - x_m) B_{k,n} + x_1 B_{k+1,n} + x_2 B_{k+2,n} + \cdots + x_m B_{k+m,n}$$

with $n \geq m^2+1$ and $k \in \{\binom{m}{2}+1, \ldots, n-\binom{m+1}{2}\}$, then f has a unique inflection point between 0 and 1.

Using these conditions, we have a starting point for creating an example of a reliability polynomial with more than one inflection point. For instance, consider a reliability polynomial formed by using a base set $X = \{1,2,3,4,5,6,7\}$. In order for a reliability polynomial on X to have more than one inflection point, we need either $m^2+1 < 7$ or $k < \binom{m}{2}+1$ or $k > 7-\binom{m+1}{2}$. By choosing $m = 2$ and $k = 5$, we satisfy two of these inequalities and thus have a chance of creating a reliability polynomial with at least two inflection points.

Since $k = 5$, the smallest working set must be of size five. Also, since $m = 2$, there must be some subsets of size five and six that are *not* working sets. There are only a limited number of collections of sets satisfying these properties. One of these collections does in fact yield a reliability polynomial with two inflection points. Consider all subsets of X that contain at least five elements and contain the elements 1 and 2. Then, there are ten working sets of size five, five working sets of size six, and one working set of size seven. Thus, the reliability polynomial is:

$$f(p) = 10p^5(1-p)^2 + 5p^6(1-p) + p^7$$

which simplifies to

$$f(p) = 10p^5 - 15p^6 + 6p^7.$$

The second derivative of this reliability polynomial is

$$f''(p) = 200p^3 - 450p^4 + 252p^5$$

or

$$f''(p) = p^3(21p-20)(6p-5).$$

This reliability polynomial has inflection points at $\frac{20}{21}$ and $\frac{5}{6}$ both of which are between 0 and 1. An infinite class of examples of reliability polynomials with more than one inflection point can be found in the last section of this paper.

2 Convex Basis

This section will describe a convex basis for a reliability polynomials and the corresponding λ-vector. For any n-set, the number of possible collections of working sets is finite; hence, there are only a finite number of reliability polynomials of degree n. The goal of this section is find a relationship between all reliability polynomials of a fixed degree. Consider the case where $n = 3$, or a base set $X = \{1,2,3\}$.

There are exactly ten reliability polynomials of degree less than or equal to three.

They are listed below along with their corresponding minimal working sets:

$f_1(p) = p^3$	$\lfloor W_1 \rfloor = \{\{1,2,3\}\}$
$f_2(p) = p^2$	$\lfloor W_2 \rfloor = \{\{1,2\}\}$
$f_3(p) = 2p^2 - p^3$	$\lfloor W_3 \rfloor = \{\{1,2\},\{1,3\}\}$
$f_4(p) = 3p^2 - 2p^3$	$\lfloor W_4 \rfloor = \{\{1,2\},\{1,3\},\{2,3\}\}$
$f_5(p) = p$	$\lfloor W_5 \rfloor = \{\{1\}\}$
$f_6(p) = p + p^2 - p^3$	$\lfloor W_6 \rfloor = \{\{1\},\{2,3\}\}$
$f_7(p) = 2p - p^2$	$\lfloor W_7 \rfloor = \{\{1\},\{2\}\}$
$f_8(p) = 3p - 3p^2 + p^3$	$\lfloor W_8 \rfloor = \{\{1\},\{2\},\{3\}\}$
$f_9(p) = 0$	$\lfloor W_9 \rfloor = \emptyset$
$f_{10}(p) = 1$	$\lfloor W_{10} \rfloor = \{\emptyset\}$

Of course, these minimal working sets are only unique up to relabeling. For instance, $\lfloor W_2 \rfloor$ could also be $\{\{1,3\}\}$ or $\{\{2,3\}\}$. Also, in this paper we are concerned about inflection points of reliability polynomials. Because the identically 0 and identically 1 reliability polynomials have no inflection points, they will be excluded from this point on.

By writing each remaining polynomial in the N-form, there is an associated point $P_i = (0, N_1, N_2, 1)$ to each polynomial. For instance, $f_2(p)$ can be rewritten as $f_2(p) = p^2(1-p) + p^3$ which would be associated with the point $P_2 = (0,0,1,1)$. Even though all eight of these points are in 4-space, they lie on the same plane. Thus, each polynomial corresponds to a point in 2-space, namely (N_1, N_2).

These points can then be graphed in $\mathbb{R}^2$ as shown in Figure 1.

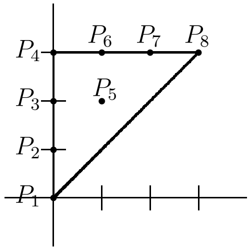

Figure 1: The Graph of 8 Points

This graph shows that the points P_1, P_4, and P_8 are the vertices of the convex hull of these eight points. Thus, all reliability polynomials of degree less than or equal to three can be written as convex combinations of f_1, f_4, and f_8. Specifically:

$$\begin{aligned} f_2 &= \frac{2}{3}f_1 + \frac{1}{3}f_4 \\ f_3 &= \frac{1}{3}f_1 + \frac{2}{3}f_4 \\ f_5 &= \frac{1}{3}f_1 + \frac{1}{3}f_4 + \frac{1}{3}f_8 \\ f_6 &= \frac{2}{3}f_4 + \frac{1}{3}f_8 \\ f_7 &= \frac{1}{3}f_4 + \frac{2}{3}f_8. \end{aligned}$$

A natural question to consider is whether f_1, f_4, and f_8 have some specific property. Notice that f_8 is the reliability polynomial formed by taking all 1-sets of a 3-set to be the minimal working sets; f_4 uses all 2-sets of a 3-set as the minimal working sets; f_1 uses all 3-sets of a 3-set as the minimal working sets.

This result extends to all reliability polynomials of any fixed degree n. In order to state this theorem, first recall that we defined $\mathcal{W}_{k,n}$ to be all subsets of size greater than or equal to k of a set of size n, and the reliability polynomial $B_{k,n}$ as

$$B_{k,n}(p) = \mathrm{Rel}_{\mathcal{W}_{k,n}}(p) = \sum_{i=k}^{n} b_{i,n}$$

where $b_{i,n}$ is a Bernstein polynomial. The convex basis theorem can be stated as follows:

Theorem 1. *Let $X = \{1, 2, \ldots, n\}$ be a base set. If f is a reliability polynomial of degree less than or equal to n, then f can be written as a convex combination of $B_{k,n}$'s. In other words,*

$$f = \lambda_1 B_{1,n} + \lambda_2 B_{2,n} + \cdots + \lambda_n B_{n,n}$$

where $0 \leq \lambda_i \leq 1$ for all i, and $\sum_{i=1}^{n} \lambda_i = 1$.

To prove this result, a technical lemma is first needed. The lemma involves minimizing N_i with respect to N_{i+1} and was first studied by Sperner [7] in 1928 and is a well-known result.

Lemma 1. *Let $f(p) = \mathrm{Rel}_{\mathcal{W}}(p)$ have N-vector $(N_1, N_2, \ldots, N_n)$. Then,*

$$0 \leq \frac{N_{i+1}}{\binom{n}{i+1}} - \frac{N_i}{\binom{n}{i}} \leq 1, \qquad 0 \leq i \leq n-1.$$

This lemma is essential in the proof of Theorem 1 which follows below.

Proof. Let $f = N_1p^1(1-p)^{n-1}+N_2p^2(1-p)^{n-2}+\cdots+N_np^n$ be a reliability polynomial. (Notice that the first few N_i's could in fact be 0.) Now, let

$$\lambda_1 = \frac{N_1}{n}, \quad \lambda_i = \frac{N_i}{\binom{n}{i}} - \frac{N_{i-1}}{\binom{n}{i-1}}, \quad i = 2, \ldots n.$$

Then,

$$\begin{aligned}\lambda_1 B_{1,n}(p) + \cdots + \lambda_n B_{n,n}(p) &= \sum_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \sum_{j=1}^{i} \lambda_j \\ &= \frac{N_1}{n} np(1-p)^{n-1} + \sum_{i=2}^{n} \binom{n}{i} p^i (1-p)^{n-i} \frac{N_i}{\binom{n}{i}} \\ &= N_1 p(1-p)^{n-1} + \sum_{i=2}^{n} N_i p^i (1-p)^{n-i} \\ &= f(p).\end{aligned}$$

The only thing left to show is that $\sum \lambda_i = 1$ and that $0 \leq \lambda_i \leq 1$. By the way λ_i was defined, it is clear that $\sum \lambda_i = \frac{N_n}{\binom{n}{n}} = 1$. Also, $0 \leq N_1 \leq n$, so $0 \leq \lambda_1 \leq 1$, and $0 \leq \lambda_i \leq 1$ by the previous lemma. □

3 Sufficient Conditions

The goal of this section is to use the convex basis to find sufficient conditions for a reliability polynomial to have exactly one inflection point. In order to find inflection points, a closed form for the derivative of a basis element is useful. It is well-known (see [5]) that the derivative of a Bernstein polynomial satisfies

$$b'_{i,n}(x) = n(b_{i-1,n-1}(x) - b_{i,n-1}(x))$$

where $b_{i,n} = 0$ if $i > n$.

Lemma 2. *The derivative of $B_{k,n}$ can be written as:*

$$B'_{k,n}(p) = n\binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}.$$

Proof. Notice that

$$B_{k,n}(p) = \sum_{i=k}^{n} b_{i,n}(p),$$

where $b_{i,n}$ is a Bernstein polynomial. Differentiating yields the equation

$$B'_{k,n}(p) = \sum_{i=k}^{n} n(b_{i-1,n-1}(p) - b_{i,n-1}(p))$$

or more simply

$$B'_{k,n}(p) = n(b_{k-1,n-1}(p)).$$

□

The second derivative of a basis element can also be written concisely.

Lemma 3. *The second derivative of $B_{k,n}$ can be written as:*

$$B''_{k,n}(p) = n\binom{n-1}{k-1}p^{k-2}(1-p)^{n-k-1}(-np+p+k-1)$$

for all $0 < p < 1$.

Proof. The derivative of $B'_{k,n}(p)$ is

$$B''_{k,n}(p) = n\binom{n-1}{k-1}[(k-1)p^{k-2}(1-p)^{n-k} - (n-k)p^{k-1}(1-p)^{n-k-1}].$$

which can be simplified to

$$B''_{k,n}(p) = n\binom{n-1}{k-1}p^{k-2}(1-p)^{n-k-1}(-np+p+k-1).$$

□

Knowing the third derivative of these basis elements will also be useful later.

Lemma 4. *The third derivative of $B_{k,n}$ can be written as*

$$\begin{aligned} B'''_{k,n}(p) &= n\binom{n-1}{k-1}p^{k-3}(1-p)^{n-k-2} \\ &\quad \cdot \left[p^2(n-1)(n-2) - 2p(n-2)(k-1) + (k-1)(k-2)\right] \end{aligned}$$

for $0 < p < 1$.

Since the second derivative of a basis element can be written so concisely, the inflection points of a basis element are clear; $B''_{k,n} = 0$ if and only if $p = 0, 1$ or $\frac{k-1}{n-1}$. This result is summarized in the following theorem:

Theorem 2. *If $1 < k < n$, then $B_{k,n}(p)$ has exactly one inflection point between 0 and 1 occurring at $p = \frac{k-1}{n-1}$ and*

$$B''_{k,n}(p) > 0 \text{ if } p < \frac{k-1}{n-1}$$
$$B''_{k,n}(p) < 0 \text{ if } p > \frac{k-1}{n-1}.$$

Furthermore, $B_{1,n}$ and $B_{n,n}$ have no inflection points between 0 and 1.

The closed form of the second derivative of a basis element simplifies the process of finding sufficient conditions for a reliability to have more than one inflection point between 0 and 1. Consider a reliability polynomial formed by a convex combination of three consecutive basis elements. Let

$$f(p) = (1 - x - y)B_{k,n}(p) + xB_{k+1,n}(p) + yB_{k+2,n}(p)$$

be a reliability polynomial where $n \geq 3$. Then

$$f''(p) = (1 - x - y)B''_{k,n}(p) + xB''_{k+1,n}(p) + yB''_{k+2,n}(p).$$

An inflection point of f occurs if there are values of p between 0 and 1 so that

$$(1 - x - y)B''_{k,n}(p) + xB''_{k+1,n}(p) + yB''_{k+2,n}(p) = 0.$$

Only certain values of p need to be considered. By Theorem 2, if $p < \frac{k-1}{n-1}$, then $B''_{k,n}(p) > 0$, $B''_{k+1,n}(p) > 0$ and $B''_{k+2,n}(p) > 0$. Since x, y and $1 - x - y$ are all positive, the polynomial $f''(p)$ is positive if $p < \frac{k-1}{n-1}$. Similarly, $f''(p)$ is negative if $p > \frac{k+1}{n-1}$. Thus, the only possibility for inflection points occur when

$$\frac{k-1}{n-1} \leq p \leq \frac{k+1}{n-1}.$$

Notice that the equation

$$(1 - x - y)B''_{k,n}(p) + xB''_{k+1,n}(p) + yB''_{k+2,n}(p) = 0$$

really has 3 variables: p, x, and y. Previously, it was assumed that x and y were fixed with p a variable; now, the case where p is fixed while both x and y are variables will be considered. It is clear that when p is fixed, this equation is simply a linear equation in x and y. For each p there is a corresponding line; thus, for $f''(p)$ to have two roots between 0 and 1, two of these lines must cross. If two lines cross, then there is an inflection point at each distinct value of p corresponding to those two lines. The x and y values of the intersection would represent a reliability polynomial with two inflection points.

Consider the change in the x- and y-intercepts of these lines as p increases. Let

$$g_x(p) = \frac{B''_{k,n}(p)}{B''_{k,n}(p) - B''_{k+1,n}(p)} \quad \text{and}$$
$$g_y(p) = \frac{B''_{k,n}(p)}{B''_{k,n}(p) - B''_{k+2,n}(p)}$$

be the functions representing the x and y intercepts respectively. If these functions are both increasing as functions of p when $\frac{k-1}{n-1} \leq p \leq \frac{k+1}{n-1}$, then no two of these lines can cross. The following two lemmas explain when these functions are increasing.

Lemma 5. *The function $g_x(p)$ is always increasing when $0 < p < 1$.*

Proof. Notice that $g_x(p)$ is increasing if and only if $g'_x(p) > 0$ which occurs when

$$B''_{k,n}B'''_{k+1,n} - B'''_{k,n}B''_{k+1,n} > 0.$$

By using the closed forms for the derivatives of basis elements from Lemma 3 and its corollary, the left side of this inequality can be simplified to

$$n^2\binom{n-1}{k-1}\binom{n-1}{k}p^{2k-4}(1-p)^{2n-2k-4} \cdot \left[p^2(n-1)(n-2) - 2p(n-1)(k-1) + k(k-1)\right]$$

Thus, it is clear that the expression is greater than zero if and only if

$$p^2(n-1)(n-2) - 2p(n-1)(k-1) + k(k-1) > 0.$$

The roots of this polynomial occur at

$$\frac{k-1}{n-2} \pm \frac{\sqrt{(n-1)(k-2)(k-n+1)}}{(n-1)(n-2)}.$$

Because these roots are complex, the polynomial is always positive. □

Lemma 6. *Let $n \geq 5$ and $2 \leq k \leq n-3$. Then, the function $g_y(p)$ is increasing for all $0 < p < 1$.*

Proof. The function $g_y(p)$ is increasing if and only if $g'_y(p) \geq 0$ which occurs when

$$B''_{k,n}B'''_{k+2,n} - B'''_{k,n}B''_{k+2,n} \geq 0.$$

Now, by using the closed forms for the derivatives of basis elements, $g'_y(p) \geq 0$ when

$$p^2(n-2)(n-1) + p(n-1)(1-2k) + (k-1)(k+1) \geq 0.$$

Call this polynomial $h(p)$. The roots of $h(p)$ occur at

$$r_1 = \frac{2k-1}{2(n-2)} - \frac{\sqrt{(n-1)(5n-4nk+4k+4k^2-9)}}{2(n-2)(n-1)}$$

and

$$r_2 = \frac{2k-1}{2(n-2)} - \frac{\sqrt{(n-1)(5n-4nk+4k+4k^2-9)}}{2(n-2)(n-1)}.$$

If r_1 and r_2 are real, $h(p) \geq 0$ if and only if

$$p \leq r_1 \text{ or } p \geq r_2.$$

But, it can be shown by direct algebraic computation that $r_1 < 0 < 1 < r_2$. Since $p \in (0,1)$, p is never less than r_1 nor ever less than r_2. Thus, the case where r_1 and r_2 are real can be discarded.

On the other hand, if r_1 and r_2 are complex, $h(p)$ is always positive. In order for the roots to be complex, the following inequality must hold:

$$4k^2 + 4k(1-n) + 5n - 9 < 0.$$

This can be considered as a polynomial in k denoted by $s(k)$ with roots at

$$k_1 = \frac{n-1}{2} - \frac{\sqrt{(n-5)(n-2)}}{2}, \qquad k_2 = \frac{n-1}{2} + \frac{\sqrt{(n-5)(n-2)}}{2}.$$

For $s(k)$ to be less than zero, it is needed that

$$k_1 < k < k_2.$$

The roots k_1 and k_2 are real if $n \geq 5$. Also, if $n \geq 5$,

$$1 < k_1 \leq 2$$

and

$$n-3 \leq k_2 < n-2.$$

Since k is an integer, it is clear that

$$k > r_1 \iff k \geq 2$$

and

$$k < r_2 \iff k \leq n-3.$$

Thus, $s(k) < 0$ if and only if $n \geq 5$ and $2 \leq k \leq n-3$, and our original function is increasing under these conditions. ☐

If both the x-intercepts and y-intercepts are increasing, then no lines will ever cross and the reliability will have no more than one inflection point. This leads to the following theorem.

Theorem 3. *Let $f(p) = (1-x-y)B_{k,n} + xB_{k+1,n} + yB_{k+2,n}$ be a reliability polynomial with $n \geq 5$ and $k \in \{2, 3, \ldots, n-3\}$. Then f has exactly one inflection point between 0 and 1.*

Proof. It was already shown that f has no more than one inflection point between 0 and 1 when n and k meet the stated conditions. The reliability polynomial f has at least one inflection point between 0 and 1, because $f''(p) > 0$ when $p < \frac{k-1}{n-1}$ and $f''(p) < 0$ when $p > \frac{k+1}{n-1}$. Since $f''(p)$ is a continuous function, it must have a root between these points. ☐

These results can be extended when more basis elements are used. Let

$$f(p) = (1 - x_1 - x_2 - \cdots - x_m)B_{k,n} + x_1 B_{k+1,n} + x_2 B_{k+2,n} + \cdots + x_m B_{k+m,n}.$$

Again, start by considering a fixed p. Then, $(1-x_1-x_2-\cdots-x_m)B_{k,n}+x_1B_{k+1,n}+x_2B_{k+2,n}+\cdots+x_mB_{k+m,n}=0$ is simply a linear equation in m variables. This hyperplane has an x_j-intercept at

$$x_j = \frac{B''_{k,n}(p)}{B''_{k,n}(p)-B''_{k+j,n}(p)}.$$

There is a different hyperplane for each value of p, and the reliability polynomial has two inflection points only if two of these hyperplanes intersect. Because $f''(p)$ is positive if $p<\frac{k-1}{n-1}$ and $f''(p)$ is negative if $p>\frac{k+m-1}{n-1}$, it is only necessary to consider values of p where

$$\frac{k-1}{n-1}\le p\le\frac{k+m-1}{n-1}$$

To see when these intercepts are increasing with respect to p, define m functions representing these intercepts:

$$g_{x_j}(p) = \frac{B''_{k,n}(p)}{B''_{k,n}(p)-B''_{k+j,n}(p)}.$$

If all of these functions are increasing when $\frac{k-1}{n-1}\le p\le\frac{k+m-1}{n-1}$, then no two hyperplanes can cross. Now, $g_{x_j}(p)$ is increasing if and only if

$$B''_{k,n}B'''_{k+j,n}-B'''_{k,n}B''_{k+j,n}>0.$$

By using the closed forms for these derivatives, this occurs when

$$p^2(n-1)(n-2)-p(n-1)(2k+j-3)+(k-1)(k+j-1)>0.$$

Call this polynomial $h_j(p)$. Using the quadratic formula, it is easy to find the roots of $h_j(p)$ occurring at

$$\frac{2k+j-3}{2(n-2)}\pm\frac{\sqrt{(n-1)(k^2-4k(n-j+1)+nj^2-2nj+5n-j^2-2j-1)}}{2(n-1)(n-2)}.$$

Notice that if $h_j(p)$ has no real roots, then it is always greater than 0. The polynomial $h_j(p)$ has no real roots if and only if

$$k^2-4k(n-j+1)+nj^2-2nj+5n-j^2-2j-1<0.$$

For notation purposes, call the polynomial on the left $s_j(k)$. Now, $s_j(k)<0$ if and only if $s_j(k)$ has two real roots and k is some value between those roots. Again, by using the quadratic formula, it can be seen that $s_j(k)$ has two real roots as long as $n\ge j^2+1$ and these roots occur at

$$k_1(j)=\frac{n-j+1}{2}-\frac{\sqrt{(n-2)(n-j^2-1)}}{2}$$

and

$$k_2(j) = \frac{n-j+1}{2} + \frac{\sqrt{(n-2)(n-j^2-1)}}{2}.$$

Thus, for $s_j(k)$ to be less than 0, it is needed that

$$k_1(j) < k < k_2(j).$$

In order for *every* hyperplane to have an increasing intercept, it is needed that this last expression is true for all j. By considering the derivative of the function $k_1(j)$, it is clear that $k_1(j)$ is increasing if $j \geq 1$. Similarly, $k_2(j)$ is decreasing for all j. So, if there are values of k so that

$$k_1(m) < k < k_2(m),$$

then $s_j(k) < 0$ for all j. In other words, only the mth hyperplane's intercept matters.

To get more workable bounds, it can be shown that

$$\binom{m}{2} + 1 \geq k_1(m)$$

by the following inequalities:

$$\begin{aligned} & \binom{m}{2} + 1 \geq k_1(m) \\ \iff \quad & \frac{m^2-m+2}{2} \geq \frac{n-m+1}{2} - \frac{\sqrt{(n-2)(n-m^2-1)}}{2} \\ \iff \quad & \sqrt{(n-2)(n-m^2-1)} \geq -m^2-1+n \\ \iff \quad & (n-m^2-1)(n-2) \geq (n-m^2-1)^2. \end{aligned}$$

Similarly, it can be shown that

$$k_2(m) \geq n - \binom{m+1}{2}.$$

So, if

$$\binom{m}{2} + 1 \leq k \leq n - \binom{m+1}{2}$$

then $s_j(k) < 0$ for all j; thus, all of our intercepts are increasing as p increases.

The following theorem summarizes the main results:

Theorem 4. *Let*

$$f(p) = (1 - x_1 - \cdots - x_m)B_{k,n} + x_1 B_{k+1,n} + x_2 B_{k+2,n} + \cdots + x_m B_{k+m,n}$$

be a reliability polynomial with $n \geq m^2 + 1$ *and* $k \in \{\binom{m}{2} + 1, \ldots, n - \binom{m+1}{2}\}$. *Then* f *has exactly one inflection point between 0 and 1.*

Proof. It was already shown that f has no more than one inflection point between 0 and 1 when n and k meet the stated conditions because no two hyperplanes can cross. But f has at least one inflection point between 0 and 1 since $f''(p)$ is positive when $p < \frac{k-1}{n-1}$ and $f''(p)$ is negative when $p > \frac{k+m-1}{n-1}$. □

4 Examples

Although the hypotheses of Theorem 4 are fairly strong, it is not a vacuously true statement. For instance, let n, k, and m satisfy the hypotheses of the theorem and consider the reliability polynomial with N-vector:

$$N_j = \begin{cases} 0 & \text{if } j < k \\ \binom{n}{j} - \binom{k+m-1}{j} & \text{if } k \le j < k+m \\ \binom{n}{j} & \text{if } j \ge k+m \end{cases}$$

This N-vector can be described as a collection of sets $\mathcal{W}$ on a base set $X = \{1, 2, \ldots, n\}$ where $\mathcal{W}$ is any subset of size at least k that is *not* a subset of $\{1, 2, \ldots, k+m-1\}$. It is clear that $\mathcal{W}$ is closed under taking supersets so it does describe a reliability polynomial.

The reliability polynomial, $g(p)$, that is described above, can be written in the form

$$g(p) = (1 - x_1 - \cdots - x_m)B_{k,n} + x_1 B_{k+1,n} + x_2 B_{k+2,n} + \cdots + x_m B_{k+m,n}$$

where

$$x_j = \frac{\binom{n-k-j}{n-k-m}}{\binom{n}{k+m-1}}.$$

Thus, there are examples of reliability polynomials that satisfy the conditions of the theorem, and all such reliability polynomials have only one inflection point.

Examining polynomials without the stated properties in the theorem gives a good place to look for reliability polynomials with more than one inflection point. An infinite class of polynomials with more than one inflection point can be found in the following manner.

Consider the polynomial

$$f_n(p) = \frac{(n-3)(n-2)}{n(n-1)} B_{n-2,n} + \frac{2(n-2)}{n(n-1)} B_{n-1,n} + \frac{2}{n} B_{n,n}$$

which has as its working sets any set of size at least $n-2$ that contains the subset $\{1, 2\}$. Using the notation from Theorem 4, we have $k = n-2$ and $m = 2$. Notice that $k \notin \{2, 3, \ldots, n-3\}$, so this reliability has the possibility of having more than one inflection point. This polynomial has second derivative

$$\begin{aligned} f_n''(p) = {} & \frac{(n-3)^2(n-2)^2}{2} p^{n-4} - (n-2)^2(n-4)(n-1)p^{n-3} \\ & + \frac{n(n-1)(n-3)(n-4)}{2} p^{n-2} \end{aligned}$$

which has roots at

$$p = \frac{n-2}{n(n-3)} \left[(n-2) \pm \sqrt{\frac{n^2 - 9n + 16}{(n-4)(n-1)}} \right]$$

both of which are between 0 and 1.

Other examples of reliability polynomials with more than one inflection point can be found using series extensions of k-out-of-n systems. Consider polynomials of the form

$$g_{s,k}(p) = p^{s-1}B_{k,k+s}$$

with $s \geq 3$. Then $g_s(p)$ is a series extension of a k-out-of-$(k+s)$ system. One can think of this example as a system with $k+2s-1$ components. In order for the system to run, the first $s-1$ components must operate and at least k of the remaining $k+s$ components must operate.

Using the notation from Theorem 4, we have $n = k+2s-1$ and $m = s$. Although $n \geq m^2+1$ for sufficiently large k, we also have that $k > n - \binom{m+1}{2}$ for $s \geq 5$. Since the hypotheses of the theorem are not completely satisfied, this reliability polynomial has the possibility of having more than one inflection point.

Computing the second derivative of $g_{s,k}(p)$ yields

$$g''_{s,k}(p) = (s-1)(s-2)p^{s-3}B_{k,k+s} + 2(s-1)p^{s-2}B'_{k,k+s} + p^{s-1}B''_{k,k+s}.$$

The last two summands can be simplified to

$$(k+s)\binom{k+s-1}{k-1}p^{k+s-3}(1-p)^{s-1}\left[p(-3s+3-k)+2s+k-3\right]$$

using the closed forms computed previously.

In order to show that this polynomial does indeed have multiple inflection points for specific values of k, we will show that the second derivative is positive at $p_0 = \frac{k+s-1}{k+2s-1}$, negative at $p_1 = \frac{k+2s-1}{k+3s-3}$, and positive at $p_2 = 1$. First, notice that $g''_{s,k}(1) = (s-1)(s-2)$ which is clearly greater than zero. Also,

$$\begin{aligned} g''_{s,k}(p_0) &= (s-1)(s-2)(p_0)^{s-3}B_{k,k+s}(p_0) \\ &\quad + (k+s)\binom{k+s-1}{k-1}(p_0)^{k+s-3}(1-p_0)^{s-1}\frac{s(s-2)}{k+2s-1} \end{aligned}$$

where it is clear that each summand is positive and thus $g''_{s,k}(p_0) > 0$.

To prove that $g''_{s,k}(p_1) < 0$ requires that k be chosen carefully. First notice that in order for $g''_{s,k}(p_1) < 0$, we need

$$B_{k,k+s}(p_1) < \frac{2\binom{k+s-1}{s}(k+s)(k+2s-1)^k(s-2)^{s-2}}{(s-1)(k+3s-3)^{k+s-1}}.$$

Because $B_{k,k+s}(p_1) < 1$, showing that the right-hand side of the inequality is greater than 1 would suffice. The key to proving this statement is noticing that

$$\left(\frac{k+2s-1}{k+3s-3}\right)^k \geq \frac{1}{e^{s-2}}$$

and

$$\binom{k+s-1}{s}(k+s) = \binom{k+s}{s}k \geq \frac{k(k+s)^s}{s^s}.$$

So, we have

$$\frac{2\binom{k+s-1}{s}(k+s)(k+2s-1)^k(s-2)^{s-2}}{(s-1)(k+3s-3)^{k+s-1}} \geq \left(\frac{2k(k+s)^s}{(k+3s-3)^{s-1}}\right)\left(\frac{2(s-2)^{s-2}}{(s-1)s^s e^{s-2}}\right)$$

which is greater than 1 for sufficiently large k.

Thus, for any s there exists a series extension of a k-out-of-$(k+s)$ system with more than one inflection point.

5 Concluding Remarks

Theorem 4 does not find sufficient conditions for a reliability polynomial to have more than one inflection point; rather, it states sufficient conditions for a reliability polynomial to have exactly one inflection point. These conditions are fairly strong, and it would be nice to find a set of weaker hypotheses. Also, the reliability polynomials with two inflection points listed above do not represent the all-terminal reliability polynomials of a graph. Proving that all-terminal reliability polynomials have exactly one inflection point is still an open question.

Another logical step to take in this research would be to find necessary conditions for a reliability polynomial to have more than one inflection point. Some partial results can be found in [4], but this approach seems quite difficult. Other questions arise such as, "Can a reliability polynomial have more than two inflection points?"

Finally, in the above examples, the inflection points seem to be rather close together. Another direction of research to consider would be how far apart these inflection points can be. Is it possible to find a reliability polynomial with one inflection point close to 0 and another close to 1?

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