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On the work performed by a transformation semigroup

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Abstract

A (partial) transformation α on the finite set $\{1, \ldots, n\}$ moves an element i of its domain a distance of $|i - i\alpha|$ units. The work $w(\alpha)$ performed by α is the sum of all of these distances. We derive formulae for the total work $w(S) = \sum_{\alpha \in S} w(\alpha)$ performed by various semigroups S of (partial) transformations. One of our main results is the proof of a conjecture of Tim Lavers which states that the total work performed by the semigroup of all order-preserving functions on an n-element chain is equal to $(n-1)2^{2n-3}$.

1 Introduction

Fix a positive integer n and write $\mathbf{n} = \{1, \ldots, n\}$. The partial transformation semigroup $\mathcal{PT}_n$ is the semigroup of all partial transformations on $\mathbf{n}$; i.e. all functions between subsets of $\mathbf{n}$. (Note that the use of the word "partial" does not imply that the domain is necessarily a *proper* subset of $\mathbf{n}$. In this way, $\mathcal{PT}_n$ also includes all *full* transformations of $\mathbf{n}$; i.e. all functions $\mathbf{n} \to \mathbf{n}$.) A partial transformation α moves a point i of its domain to a (possibly) new point j in its image. If the elements of $\mathbf{n}$ are thought of as points, equally spaced along a line, then the point i has been moved a distance of $|i-j|$ units. Summing these values, as i varies over the domain of α, gives the *(total) work* performed by α, denoted by $w(\alpha)$. We may also consider the total and average work performed by a collection S of partial transformations, being the quantities $w(S) = \sum_{\alpha\in S} w(\alpha)$ and $\overline{w}(S) = \frac{1}{|S|}w(S)$ respectively. It is the purpose of the current article to calculate $w(S)$ and $\overline{w}(S)$ when S is either $\mathcal{PT}_n$ itself, or one of its six key subsemigroups:

- $\mathcal{T}_n = \{\alpha \in \mathcal{PT}_n \mid \mathrm{dom}(\alpha) = \mathbf{n}\}$, the (full) transformation semigroup;
- $\mathcal{I}_n = \{\alpha \in \mathcal{PT}_n \mid \alpha \text{ is injective}\}$, the symmetric inverse semigroup;
- $\mathcal{S}_n = \mathcal{T}_n \cap \mathcal{I}_n$, the symmetric group;
- $\mathcal{PO}_n = \{\alpha \in \mathcal{PT}_n \mid \alpha \text{ is order-preserving}\}$;
- $\mathcal{O}_n = \mathcal{T}_n \cap \mathcal{PO}_n$; and
- $\mathcal{POI}_n = \mathcal{I}_n \cap \mathcal{PO}_n$.

For the above definitions, recall that a partial transformation $\alpha \in \mathcal{PT}_n$ is order-preserving if $i\alpha \le j\alpha$ whenever $i,j \in \mathrm{dom}(\alpha)$ and $i \le j$.[1] Figure 1 illustrates the various inclusions; for a more comprehensive picture, see [2].

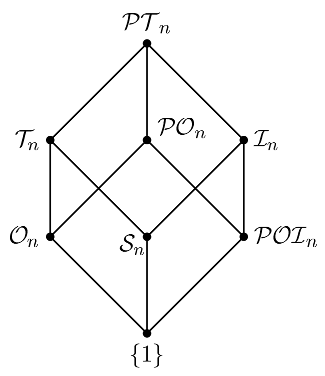

Figure 1: The relevant part of the subsemigroup lattice of $\mathcal{PT}_n$.

Our interest in this topic began after attending a talk by Tim Lavers at a Semigroups Special Interest meeting in Sydney 2004, in which the formula $w(\mathcal{O}_n) = (n-1)2^{2n-3}$

[1] We write $i\alpha$ for the image of $i \in \mathbf{n}$ under $\alpha \in \mathcal{PT}_n$, although the semigroup operation on $\mathcal{PT}_n$ (composition as binary relations) does not feature in the current work.

was conjectured. We also note that the quantity $\frac{1}{n}\overline{w}(\mathcal{S}_n) = \frac{n^2-1}{3n}$ has been calculated previously in relation to turbo coding [1] although, in the absence of such "external" considerations, we feel that $w(S)$ and $\overline{w}(S)$ are the more intrinsic quantities.

Our results are summarized in Tables 1 and 2 below, where the reader will notice some interesting relationships such as $w(\mathcal{O}_n) = w(\mathcal{POI}_n)$ and $\overline{w}(\mathcal{S}_n) = \overline{w}(\mathcal{T}_n)$. Table 3 catalogues the calculated values of $w(S)$ for $n = 1, \ldots, 10$.

S	Formula for $w(S)$
$\mathcal{S}_n$	$\frac{n!(n^2-1)}{3}$
$\mathcal{T}_n$	$\frac{n^n(n^2-1)}{3}$
$\mathcal{PT}_n$	$\frac{(n+1)^n(n^2-n)}{3}$
$\mathcal{I}_n$	$\frac{n^3-n}{3}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 k!$
$\mathcal{POI}_n$	$(n-1)2^{2n-3}$
$\mathcal{O}_n$	$(n-1)2^{2n-3}$
$\mathcal{PO}_n$	$\sum_{i,j=1}^{n}\sum_{k,\ell=0}^{n}\lvert i-j\rvert\binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}$

Table 1: Formulae for the total work $w(S)$ performed by a semigroup $S \subseteq \mathcal{PT}_n$.

S	Formula for $\overline{w}(S)$
$\mathcal{S}_n$	$\frac{n^2-1}{3}$
$\mathcal{T}_n$	$\frac{n^2-1}{3}$
$\mathcal{PT}_n$	$\frac{n^2-n}{3}$
$\mathcal{I}_n$	$\frac{n^3-n}{3}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}^2\ell!\right)^{-1}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 k!$
$\mathcal{POI}_n$	$\binom{2n}{n}^{-1}(n-1)2^{2n-3}$
$\mathcal{O}_n$	$\binom{2n-1}{n}^{-1}(n-1)2^{2n-3}$
$\mathcal{PO}_n$	$\left(\sum_{m=0}^{n}\binom{n}{m}\binom{n+m-1}{m}\right)^{-1}\sum_{i,j=1}^{n}\sum_{k,\ell=0}^{n}\lvert i-j\rvert\binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}$

Table 2: Formulae for the average work $\overline{w}(S)$ performed by an element of a semigroup $S \subseteq \mathcal{PT}_n$.

n	1	2	3	4	5	6	7	8	9	10
$w(\mathcal{S}_n)$	0	2	16	120	960	8400	80640	846720	9676800	119750400
$w(\mathcal{T}_n)$	0	4	72	1280	25000	544320	13176688	352321536	10331213040	330000000000
$w(\mathcal{PT}_n)$	0	6	128	2500	51840	1176490	29360128	803538792	24000000000	778122738030
$w(\mathcal{I}_n)$	0	4	56	680	8360	108220	1492624	21994896	346014960	5798797620
$w(\mathcal{POI}_n)$	0	2	16	96	512	2560	12288	57344	262144	1179648
$w(\mathcal{O}_n)$	0	2	16	96	512	2560	12288	57344	262144	1179648
$w(\mathcal{PO}_n)$	0	4	48	424	3312	24204	169632	1155152	7702944	50550932

Table 3: Calculated values of $w(S)$ for small values of n.

The article is organized as follows. In Section 2 we obtain a general formula for $w(S)$ involving the cardinalities of certain subsets $M_{i,j}(S)$ of S. In Section 3 we consider separately the seven semigroups described above, calculating the cardinalities $|M_{i,j}(S)|$, and thereby obtaining explicit formulae for $w(S)$ in each case. The formulae obtained in this way are in a closed form when S is one of $\mathcal{S}_n$, $\mathcal{T}_n$, $\mathcal{PT}_n$, but expressed as a sum involving binomial coefficients in the remaining four cases. In Section 4 we prove Lavers' conjecture, which essentially boils down to a proof of the identity

$$\sum_{p,q=0}^{n} |p-q| \binom{p+q}{p} \binom{2n-p-q}{n-p} = n2^{2n-1},$$

giving rise to the postulated closed form for $w(\mathcal{O}_n) = w(\mathcal{POI}_n)$. By contrast, the expression $w(\mathcal{I}_n) = \frac{n^3-n}{3}|\mathcal{I}_{n-1}|$ may not be simplified further, since no closed form is known for $|\mathcal{I}_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k!$. It is not known to the authors whether a closed form for $w(\mathcal{PO}_n)$ exists, but the presence of large prime factors suggests that the situation could not be as simple as that of $w(\mathcal{O}_n)$; for example, $w(\mathcal{PO}_9) = 2^5 \cdot 3 \cdot 80239$.

Unless specified otherwise, all numbers we consider are integers, so a statement such as "let $1 \leq i \leq 5$" should be read as "let i be an integer such that $1 \leq i \leq 5$". It will also be convenient to interpret a binomial coefficient $\binom{p}{q}$ to be 0 if $p < q$.

2 General Calculations

We now make precise our definitions and notation. The work performed by a partial transformation $\alpha \in \mathcal{PT}_n$ in moving a point $i \in \mathbf{n}$ is defined to be

$$w_i(\alpha) = \begin{cases} |i - i\alpha| & \text{if } i \in \text{dom}(\alpha) \\ 0 & \text{otherwise,} \end{cases}$$

and the (total) work performed by α is

$$w(\alpha) = \sum_{i \in \mathbf{n}} w_i(\alpha).$$

For $S \subseteq \mathcal{PT}_n$, we write

$$w(S) = \sum_{\alpha \in S} w(\alpha) \qquad \text{and} \qquad \overline{w}(S) = \frac{1}{|S|} w(S)$$

for the total and average work performed by the elements of S, respectively.

For the remainder of this section, we fix a subset $S \subseteq \mathcal{PT}_n$. For $i \in \mathbf{n}$, put

$$w_i(S) = \sum_{\alpha \in S} w_i(\alpha),$$

which may be interpreted as the total work performed by S in moving just the point i. Rearranging the defining sum for $w(S)$ gives

$$w(S) = \sum_{\alpha \in S} w(\alpha) = \sum_{\alpha \in S} \sum_{i \in \mathbf{n}} w_i(\alpha) = \sum_{i \in \mathbf{n}} \sum_{\alpha \in S} w_i(\alpha) = \sum_{i \in \mathbf{n}} w_i(S).$$

For $i, j \in \mathbf{n}$, we consider the set

$$M_{i,j}(S) = \big\{\alpha \in S \,\big|\, i \in \operatorname{dom}(\alpha) \ \text{ and } \ i\alpha = j\big\},$$

and write

$$m_{i,j}(S) = \big|M_{i,j}(S)\big|$$

for the cardinality of $M_{i,j}(S)$. Note that $w_i(\alpha) = |i - j|$ for all $\alpha \in M_{i,j}(S)$, so that

$$w_i(S) = \sum_{j \in \mathbf{n}} |i - j| m_{i,j}(S).$$

These preliminaries give the following result.

Lemma 1 *Let $S \subseteq \mathcal{PT}_n$. Then* $w(S) = \sum_{i,j \in \mathbf{n}} |i - j| m_{i,j}(S)$. □

3 Specific Calculations

We now use Lemma 1 as the starting point to derive explicit formulae for $w(S)$ for each of the semigroups S defined in Section 1. We consider each case separately, covering them roughly in order of difficulty. When S is one of $\mathcal{S}_n$, $\mathcal{T}_n$, $\mathcal{PT}_n$, or $\mathcal{I}_n$, we will see that $m_{i,j}(S)$ is independent of $i, j \in \mathbf{n}$, and so $w(S)$ turns out to be rather easy to calculate, relying only on Lemma 1 and the well-known identity

$$\sum_{i,j \in \mathbf{n}} |i - j| = 2\binom{n+1}{3} = \frac{n^3 - n}{3}.$$

(The reader is reminded of the convention that $\binom{n+1}{3} = 0$ if $n = 1$.) In each of the remaining three cases, the formulae we derive for the quantities $m_{i,j}(S)$ yields an expression for $w(S)$ as a sum involving binomial coefficients. We defer further investigation of the $\mathcal{O}_n$ and $\mathcal{POI}_n$ cases until Section 4, where we show that this so-obtained expression may be simplified.

It may be that some of the intermediate results of this section are already known (for example Lemmas 2, 6 and 9) but the proofs, which are believed to be original, are included for completeness; the reader is referred to the introduction of [4] for a review of related studies.

Note that the proofs we give are largely geometrically motivated. A partial transformation $\alpha \in \mathcal{PT}_n$ may be represented diagrammatically by drawing an upper and lower row of n dots, representing the elements of $\mathbf{n}$ (in increasing order from left to right), and drawing a line from upper vertex i to lower vertex j whenever $i \in \mathrm{dom}(\alpha)$ and $i\alpha = j$. In this way, the quantity $m_{i,j}(S)$ may be interpreted as the number of ways to "extend" the partial map $\pi_{i,j}$, pictured in Figure 2, to an element of S.

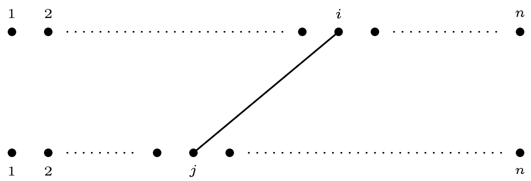

Figure 2: The partial map $\pi_{i,j} \in \mathcal{PT}_n$ with domain $\{i\}$ and image $\{j\}$.

3.1 The Symmetric Group $\mathcal{S}_n$

To extend the partial map $\pi_{i,j}$ to a permutation of $\mathbf{n}$, we must add $n-1$ lines, ensuring that they correspond to a bijection from $\mathbf{n} \setminus \{i\}$ to $\mathbf{n} \setminus \{j\}$. It follows then that $m_{i,j}(\mathcal{S}_n) = (n-1)!$ for all $i, j \in \mathbf{n}$, and so Lemma 1 gives

$$w(\mathcal{S}_n) = \sum_{i,j \in \mathbf{n}} |i-j|(n-1)! = \frac{n^3-n}{3} \cdot (n-1)! = \frac{n!(n^2-1)}{3}.$$

The average work is given by

$$\overline{w}(\mathcal{S}_n) = \frac{w(\mathcal{S}_n)}{n!} = \frac{n^2-1}{3}.^2$$

[2]This result may be found in [1], in a slightly different form.

3.2 The Transformation Semigroup $\mathcal{T}_n$

To extend $\pi_{i,j}$ to a full transformation of $\mathbf{n}$, each upper vertex must be connected by a line to a lower vertex. Since the lower vertex of such a line is not constrained in any way, we see that $m_{i,j}(\mathcal{T}_n) = n^{n-1}$ for all $i, j \in \mathbf{n}$. By Lemma 1, we therefore have

$$w(\mathcal{T}_n) = \sum_{i,j\in\mathbf{n}} |i-j|n^{n-1} = \frac{n^3-n}{3} \cdot n^{n-1} = \frac{n^n(n^2-1)}{3},$$

and

$$\overline{w}(\mathcal{T}_n) = \frac{w(\mathcal{T}_n)}{n^n} = \frac{n^2-1}{3},$$

giving rise to the first interesting (and coincidental, as far as we can tell) identity $\overline{w}(\mathcal{T}_n) = \overline{w}(\mathcal{S}_n)$.

3.3 The Partial Transformation Semigroup $\mathcal{PT}_n$

To extend $\pi_{i,j}$ to a partial transformation, each upper vertex may be connected to any lower vertex or else left unconnected. It follows that $m_{i,j}(\mathcal{PT}_n) = (n+1)^{n-1}$ for all $i, j \in \mathbf{n}$ and so, by Lemma 1, we have

$$w(\mathcal{PT}_n) = \sum_{i,j\in\mathbf{n}} |i-j|(n+1)^{n-1} = \frac{n^3-n}{3} \cdot (n+1)^{n-1} = \frac{(n+1)^n(n^2-n)}{3},$$

and

$$\overline{w}(\mathcal{PT}_n) = \frac{w(\mathcal{PT}_n)}{(n+1)^n} = \frac{n^2-n}{3}.$$

Although $\overline{w}(\mathcal{PT}_n) \neq \overline{w}(\mathcal{S}_n) = \overline{w}(\mathcal{T}_n)$, all three sequences are of course asymptotic to $n^2/3$.

3.4 The Symmetric Inverse Semigroup $\mathcal{I}_n$

To extend $\pi_{i,j}$ to an injective partial transformation, we must add at most $n-1$ more lines, ensuring that they correspond to an injective partially defined map from $\mathbf{n} \setminus \{i\}$ to $\mathbf{n} \setminus \{j\}$. Since such a partial map obviously corresponds to an injective partial transformation on $\{1, \ldots, n-1\}$, we see that $m_{i,j}(\mathcal{PT}_n) = |\mathcal{I}_{n-1}|$ for all $i, j \in \mathbf{n}$. Using the well-known formula $|\mathcal{I}_m| = \sum_{k=0}^{m} \binom{m}{k}^2 k!$, it then follows that

$$w(\mathcal{I}_n) = \sum_{i,j\in\mathbf{n}} |i-j| \cdot |\mathcal{I}_{n-1}| = \frac{n^3-n}{3} \cdot |\mathcal{I}_{n-1}| = \frac{n^3-n}{3} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 k!,$$

and

$$\overline{w}(\mathcal{I}_n) = \frac{w(\mathcal{I}_n)}{|\mathcal{I}_n|} = \frac{n^3-n}{3} \cdot \frac{|\mathcal{I}_{n-1}|}{|\mathcal{I}_n|}.$$

3.5 The Semigroup $\mathcal{POI}_n$

From this point onward, calculation of the quantities $m_{i,j}(S)$ is not as straightforward. For $0 \le p, q \le n$ let $\mathcal{POI}_{p,q}$ denote the set of all order-preserving injective partial maps from $\mathbf{p}$ to $\mathbf{q}$. (Note that we interpret $\mathbf{k} = \{1, \ldots, k\}$ to be empty if $k = 0$.)

Lemma 2 *Let $0 \le p, q \le n$. Then $|\mathcal{POI}_{p,q}| = \binom{p+q}{p} = \binom{p+q}{q}$.*

Proof Let $\mathbf{q}' = \{1', \ldots, q'\}$ be a set in one-one correspondence with $\mathbf{q}$. Denote also by $\,' : \mathbf{q}' \to \mathbf{q}$ the inverse bijection, so that we write $i'' = i$ for all $i \in \mathbf{q}$. Consider the set

$$\Sigma = \big\{ A \subseteq \mathbf{p} \cup \mathbf{q}' \,\big|\, |A| = q \big\}.$$

For $A \in \Sigma$, define $\phi_A \in \mathcal{POI}_{p,q}$ by

$$\mathrm{dom}(\phi_A) = A \cap \mathbf{p} \qquad \text{and} \qquad \mathrm{im}(\phi_A) = \mathbf{q} \setminus (A \cap \mathbf{q}')',$$

noting that $|A \cap \mathbf{p}| = \big|\mathbf{q} \setminus (A \cap \mathbf{q}')'\big|$, and that an element of $\mathcal{POI}_{p,q}$ is completely determined by its domain and image. It is then easy to check that the maps determined by

$$A \mapsto \phi_A \qquad \text{and} \qquad \phi \mapsto \mathrm{dom}(\phi) \cup \big(\mathbf{q} \setminus \mathrm{im}(\phi)\big)'$$

are mutually inverse bijections between Σ and $\mathcal{POI}_{p,q}$. The result follows since we clearly have $|\Sigma| = \binom{p+q}{q}$. □

Remark 3 Geometrically, this proof corresponds to the fact that, given p upper vertices and q lower vertices, an element of $\mathcal{POI}_{p,q}$ is determined by choosing q vertices, and then joining the selected upper vertices to the unselected lower vertices. An alternative proof begins by noting that $\mathcal{POI}_{p,q}$ contains $\binom{p}{k}\binom{q}{k}$ maps of rank k, and then concludes by applying the identity $\sum_{k=0}^{\infty} \binom{p}{k}\binom{q}{k} = \binom{p+q}{q}$.

Lemma 4 *Let $i, j \in \mathbf{n}$. Then $m_{i,j}(\mathcal{POI}_n) = \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$.*

Proof Let $\alpha \in M_{i,j}(\mathcal{POI}_n)$. Then since $i\alpha = j$ and α is order-preserving, we see that $k\alpha < j$ whenever $k \in \mathrm{dom}(\alpha)$ and $k < i$. Thus, we may define a map $\lambda_\alpha \in \mathcal{POI}_{i-1,j-1}$ by

$$\mathrm{dom}(\lambda_\alpha) = \mathrm{dom}(\alpha) \cap \{1, \ldots, i-1\} \qquad \text{and} \qquad \mathrm{im}(\lambda_\alpha) = \mathrm{im}(\alpha) \cap \{1, \ldots, j-1\}.$$

Similarly, we have $k\alpha > j$ whenever $k \in \mathrm{dom}(\alpha)$ and $k > i$, and so we may also define a map $\rho_\alpha \in \mathcal{POI}_{n-i,n-j}$ by

$$\mathrm{dom}(\rho_\alpha) = \big\{ k - i \,\big|\, k \in \mathrm{dom}(\alpha),\ k > i \big\} \ \text{ and } \ \mathrm{im}(\rho_\alpha) = \big\{ k - j \,\big|\, k \in \mathrm{im}(\alpha),\ k > j \big\}.$$

It is then easy to check that the map $\alpha \mapsto (\lambda_\alpha, \rho_\alpha)$ defines a bijection from $M_{i,j}(\mathcal{POI}_n)$ to $\mathcal{POI}_{i-1,j-1} \times \mathcal{POI}_{n-i,n-j}$. The result now follows from Lemma 2. □

Remark 5 The idea of the above proof is summed up in the schematic picture of a typical element of $M_{i,j}(\mathcal{POI}_n)$ illustrated in Figure 3.

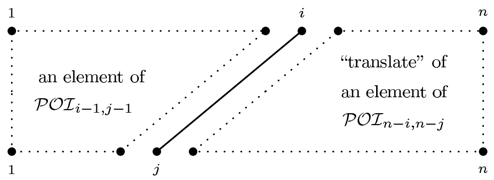

Figure 3: A schematic picture of an element of $M_{i,j}(\mathcal{POI}_n)$.

It follows by Lemmas 1 and 4 that the total work performed by $\mathcal{POI}_n$ is given by

$$w(\mathcal{POI}_n) = \sum_{i,j\in\mathbf{n}} |i-j| \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}.$$

In Section 4 we revisit this formula, and show that in fact $w(\mathcal{POI}_n) = (n-1)2^{2n-3}$. An expression for $\overline{w}(\mathcal{POI}_n)$ may be found by dividing through by $|\mathcal{POI}_n| = |\mathcal{POI}_{n,n}| = \binom{2n}{n}$.

3.6 The Semigroup $\mathcal{O}_n$

For $0 \leq p \leq n$ and $q \in \mathbf{n}$ let $\mathcal{O}_{p,q}$ denote the set of all order-preserving maps from $\mathbf{p}$ to $\mathbf{q}$.

Lemma 6 *Let* $0 \leq p \leq n$ *and* $q \in \mathbf{n}$. *Then* $|\mathcal{O}_{p,q}| = \binom{p+q-1}{p} = \binom{p+q-1}{q-1}$.

Proof Consider the set

$$\Omega = \big\{\alpha \in \mathcal{POI}_{p,q} \,\big|\, p \in \mathrm{dom}(\alpha)\big\}.$$

There is an obvious bijection $\mathcal{O}_{p,q} \to \Omega$ determined geometrically by removing all but the right-most lines from the connected components in the picture of $\alpha \in \mathcal{O}_{p,q}$; see Figure 4. For $i \in \mathbf{q}$, put

$$\Omega_i = \{\alpha \in \Omega \,|\, p\alpha = i\},$$

so that we have the disjoint union $\Omega = \Omega_1 \sqcup \cdots \sqcup \Omega_q$. Clearly, the operation of removing the right-most line gives a bijection between Ω_i and $\mathcal{POI}_{p-1,i-1}$ for each $i \in \mathbf{q}$ so that, by Lemma 2, we have

$$|\mathcal{O}_{p,q}| = |\Omega| = \sum_{i\in\mathbf{q}} \binom{p+i-2}{p-1}.$$

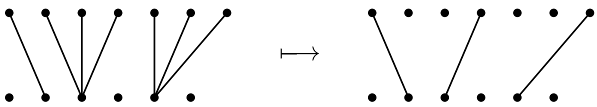

Figure 4: The bijection $\mathcal{O}_{p,q} \to \Omega$; see the proof of Lemma 6 for an explanation of the notation.

The result now follows from the identity $\displaystyle\sum_{k=0}^{s} \binom{r+k}{r} = \binom{r+s+1}{s}$. □

Remark 7 An argument similar to that used in the proof of Lemma 2 may also be used here. An element $\alpha \in \Omega$ is completely determined by the sets $\mathrm{dom}(\alpha) \setminus \{p\} \subseteq \{1, \ldots, p-1\}$ and $\mathbf{q} \setminus \mathrm{im}(\alpha) \subseteq \mathbf{q}$. This gives rise to a bijection between Ω and the set

$$\big\{A \subseteq \{1, \ldots, p-1, 1', \ldots, q'\} \,\big|\, |A| = q-1\big\},$$

which has cardinality $\binom{p+q-1}{q-1}$.

Lemma 8 *Let $i, j \in \mathbf{n}$. Then $m_{i,j}(\mathcal{O}_n) = \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$.*

Proof The proof follows a similar pattern to the proof of Lemma 4. Rather than include all the details, we simply refer to Figure 5 which gives a schematic picture of an element of $M_{i,j}(\mathcal{O}_n)$, indicating a bijection between $M_{i,j}(\mathcal{O}_n)$ and $\mathcal{O}_{i-1,j} \times \mathcal{O}_{n-i,n-j+1}$. □

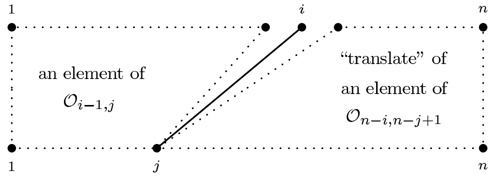

Figure 5: A schematic picture of an element of $M_{i,j}(\mathcal{O}_n)$.

In particular, we have $m_{i,j}(\mathcal{O}_n) = m_{i,j}(\mathcal{POI}_n)$ for all $i, j \in \mathbf{n}$, so that $w(\mathcal{O}_n) = w(\mathcal{POI}_n)$. The differing cardinalities of $\mathcal{O}_n$ and $\mathcal{POI}_n$ mean that $\overline{w}(\mathcal{O}_n) \neq \overline{w}(\mathcal{POI}_n)$. However, since $|\mathcal{POI}_n| = \binom{2n}{n} = 2\binom{2n-1}{n} = 2|\mathcal{O}_{n,n}| = 2|\mathcal{O}_n|$, we do have the relationship $\overline{w}(\mathcal{O}_n) = 2\overline{w}(\mathcal{POI}_n)$.

3.7 The Semigroup $\mathcal{PO}_n$

For $0 \le p \le n$ and $q \in \mathbf{n}$ let $\mathcal{PO}_{p,q}$ denote the set of all order-preserving partial maps from $\mathbf{p}$ to $\mathbf{q}$.

Lemma 9 *Let* $0 \le p \le n$ *and* $q \in \mathbf{n}$*. Then* $|\mathcal{PO}_{p,q}| = \sum_{k=0}^{n} \binom{p}{k}\binom{q+k-1}{k}$.

Proof For $A \subseteq \mathbf{p}$ write $\mathcal{PO}_{p,q}^{A} = \left\{\alpha \in \mathcal{PO}_{p,q} \,\middle|\, \mathrm{dom}(\alpha) = A\right\}$. We then have the disjoint union

$$\mathcal{PO}_{p,q} = \bigsqcup_{A \subseteq \mathbf{p}} \mathcal{PO}_{p,q}^{A}.$$

Now, for any $0 \le k \le p$, there are $\binom{p}{k}$ subsets $A \subseteq \mathbf{p}$ for which $|A| = k$ and, for each such subset A, we have $\left|\mathcal{PO}_{p,q}^{A}\right| = |\mathcal{O}_{k,q}| = \binom{q+k-1}{k}$, the last equality following by Lemma 6. This shows that

$$|\mathcal{PO}_{p,q}| = \sum_{A \subseteq \mathbf{p}} \left|\mathcal{PO}_{p,q}^{A}\right| = \sum_{k=0}^{p} \binom{p}{k}\binom{q+k-1}{k}.$$

The upper limit may be changed to n, in light of the convention that $\binom{p}{k} = 0$ if $k > p$. □

Lemma 10 *Let* $i, j \in \mathbf{n}$*. Then*

$$m_{i,j}(\mathcal{PO}_n) = \sum_{k,\ell=0}^{n} \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}.$$

Proof Again we find that there is a bijection between $M_{i,j}(\mathcal{PO}_n)$ and $\mathcal{PO}_{i-1,j} \times \mathcal{PO}_{n-i,n-j+1}$. The result now follows from Lemma 9. □

It follows, by Lemmas 1 and 10, that

$$w(\mathcal{PO}_n) = \sum_{i,j=1}^{n} \sum_{k,\ell=0}^{n} |i-j| \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}.$$

An expression for $\overline{w}(\mathcal{PO}_n)$ is then found by dividing through by

$$|\mathcal{PO}_n| = |\mathcal{PO}_{n,n}| = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k-1}{k}.$$

4 The Proof of Lavers' Conjecture

We now turn to the task of proving the conjectured result of Lavers that $w(\mathcal{O}_n) = (n-1)2^{2n-3}$. In light of Section 3.6, this amounts to a proof of the identity

$$\sum_{i,j\in\mathbf{n}} |i-j|\binom{i+j-2}{i-1}\binom{2n-i-j}{n-i} = (n-1)2^{2n-3}.$$

Replacing n by $n+1$, and introducing the new parameters $p = i-1$ and $q = j-1$, the identity takes on the more pleasing form:

$$\sum_{p,q=0}^{n} |p-q|\binom{p+q}{p}\binom{2n-p-q}{n-p} = n2^{2n-1}.$$

The remainder of this section is devoted to a proof of this identity and, hence, a proof of the conjecture.

For $0 \leq m \leq 2n$, define

$$f(n,m) = \sum_{p=0}^{m} |m-2p|\binom{m}{p}\binom{2n-m}{n-p},$$

noting first that $f(n,0) = f(n,2n) = 0$. We now find a closed form for the remaining values of m. In what follows, for a real number x and non-negative integer m, we use the standard falling factorial notation, writing $(x)_m = x(x-1)\cdots(x-m+1)$.

Lemma 11 *We have the following identities*:

$$f(n,2k+1) = \frac{2}{n}\frac{(2n-2k-1)!}{(n-k-1)!(n-k-1)!}\frac{(2k+1)!}{k!k!} \qquad \textit{for } 0 \leq k \leq n-1 \qquad (12)$$

$$f(n,2k) = \frac{2}{n}\frac{(2n-2k)!}{(n-k)!(n-k-1)!}\frac{(2k)!}{k!(k-1)!} \qquad \textit{for } 1 \leq k \leq n-1. \qquad (13)$$

Proof We apply two different methods of proof, one to each identity, and each of which may be adapted to treat the other case.

We first present a purely human-discovered proof of (12). Let $0 \leq k \leq n-1$. Consider the degree k polynomial

$$P_k(x) = \sum_{p=0}^{k} |2k+1-2p|\binom{2k+1}{p} x_{(p)}(x-k-1)_{(k-p)}$$

in an indeterminate x. In the defining sum for $f(n,m)$, the terms with $p = i$ and $p = m-i$ are equal. In this way, we calculate

$$f(n,2k+1) = \frac{2(2n-2k-1)!}{n!(n-k-1)!}P_k(n).$$

So it suffices to prove the polynomial identity

$$P_k(x) = \frac{(2k+1)!}{k!k!}(x-1)_{(k)}. \tag{14}$$

We do this by induction on k, noting first that when $k = 0$ both sides of (14) are identically equal to 1. Suppose now that $1 \le k \le n-1$ and that $0 \le \ell < k$. We consider $P_k(k+\ell+1)$. In the defining sum, terms with $p \le k-\ell-1$ will be zero and so, replacing the index of summation by $r = p-k+\ell$, we have

$$P_k(k+\ell+1) = \sum_{r=0}^{\ell} |2\ell+1-2r| \binom{2k+1}{k-\ell+r} (k+\ell+1)_{(k-\ell+r)} \ell_{(\ell-r)},$$

which is readily checked to be equal to

$$\frac{(2k+1)!}{k!} \frac{\ell!}{(2\ell+1)!} P_\ell(k+\ell+1).$$

By an inductive hypothesis,

$$P_\ell(x) = \frac{(2\ell+1)!}{\ell!\ell!}(x-1)_{(\ell)},$$

and it quickly follows that (14) holds for the k distinct x-values $x = k+1, k+2, \ldots, 2k$. Since the identity (14) involves polynomials of degree k, it suffices to verify it for one more value of x. Noting that the defining sum for $P_k(0)$ has only one non-zero term (namely the $p = 0$ term), it is easy to check that both sides of (14) are equal to $(-1)^k \frac{(2k+1)!}{k!}$ when $x = 0$. So (14) holds, and the proof of (12) is complete.

We now present a computer-aided proof of (13) using the WZ method [5]. Let $1 \le k \le n-1$. Define

$$F(n,k,p) = \frac{2n(k-p)\binom{2k}{p}\binom{2n-2k}{n-p}}{k(n-k)\binom{2k}{k}\binom{2n-2k}{n-k}},$$

noting that the desired result is equivalent to

$$\sum_{p=0}^{k-1} F(n,k,p) = 1. \tag{15}$$

A computer implementation[3] of Gosper's algorithm [3] gives us the identity

$$F(n+1,k,p) - F(n,k,p) = G(n,k,p+1) - G(n,k,p),$$

where the function G is defined by

$$G(n,k,p) = \frac{p(k+1-p)}{2n(n+1-p)} F(n,k,p).$$

[3] Available at `http://www.cis.upenn.edu/~wilf/progs.html`

Summing over $p \in \{0, \dots, k-1\}$, and noting that $G(n,k,k) = G(n,k,0) = 0$, we obtain the equality

$$\sum_{p=0}^{k-1} F(n+1,k,p) = \sum_{p=0}^{k-1} F(n,k,p).$$

So it suffices to prove (15) in the case $n = k+1$ only. This however is a triviality as only the $p = k-1$ term in the sum is non-zero. This completes the proof of (13). □

Proposition 16 *The following identity holds*:

$$\sum_{p,q=0}^{n} |p-q| \binom{p+q}{p} \binom{2n-p-q}{n-p} = n2^{2n-1}. \tag{17}$$

Proof Consider the two generating functions

$$\sum_{k=0}^{\infty} \frac{(2k+1)!}{k!k!} x^k = (1-4x)^{-3/2} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \binom{k+2}{2} 4^k x^k = (1-4x)^{-3}.$$

After squaring the first, and equating coefficients of x^{n-1}, Lemma 11 gives

$$\frac{n}{2} \sum_{k=0}^{n-1} f(n, 2k+1) = \binom{n+1}{2} 4^{n-1}.$$

Similarly, starting from

$$\sum_{k=1}^{\infty} \frac{(2k)!}{k!(k-1)!} x^k = 2x(1-4x)^{-3/2} \qquad \text{and} \qquad \sum_{k=2}^{\infty} \binom{k}{2} 4^{k-1} x^k = 4x^2(1-4x)^{-3},$$

squaring the first, and looking at the coefficient of x^n, we obtain

$$\frac{n}{2} \sum_{k=1}^{n-1} f(n, 2k) = \binom{n}{2} 4^{n-1}.$$

Returning to the original sum in (17), we rewrite it to first sum over those p and q for which $p + q = m$ is fixed, and get

$$\begin{aligned} &\sum_{p,q=0}^{n} |p-q| \binom{p+q}{p} \binom{2n-p-q}{n-p} \\ &\quad = \sum_{m=0}^{2n} f(n,m) \\ &\quad = f(n,0) + \sum_{k=0}^{n-1} f(n,2k+1) + \sum_{k=1}^{n-1} f(n,2k) + f(n,2n) \\ &\quad = \frac{2}{n} \left[\binom{n+1}{2} 4^{n-1} + \binom{n}{2} 4^{n-1} \right] \\ &\quad = n\, 2^{2n-1}. \end{aligned}$$

□

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