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A direct encoding of Stoimenow's matchings as ascent sequences*

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Abstract

In connection with Vassiliev's knot invariants, Stoimenow (1998) introduced certain matchings, also called regular linearized chord diagrams. Bousquet-Mélou et al. (2008) gave a bijection from those matchings to unlabeled $(\mathbf{2}+\mathbf{2})$-free posets; they also showed how to encode the posets as so called ascent sequences. In this paper we present a direct encoding of Stoimenow's matchings as ascent sequences. In doing so we give the rules for recursively constructing and deconstructing such matchings.

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1 Introduction

To give upper bounds on the dimension of the space of Vassiliev's knot invariants of a given degree, Stoimenow [2] introduced what he calls regular linearized chord diagrams. We call them *Stoimenow matchings*. As an example, there are five Stoimenow matchings on the set $\{1,2,3,4,5,6\}$ as shown in Figure 1. In general, a *matching* of

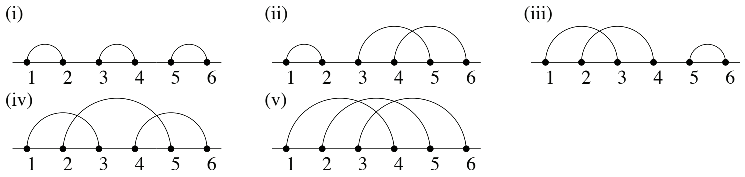

Figure 1. All five Stoimenow matchings on the set $\{1,2,3,4,5,6\}$.

the integers $\{1,2,\dots,2n\}$ is a partition of that set into blocks of size 2, often called *arcs*. We say that a matching is *Stoimenow* if there are no occurrences of Type 1 or Type 2 arcs that are defined in Figure 2.

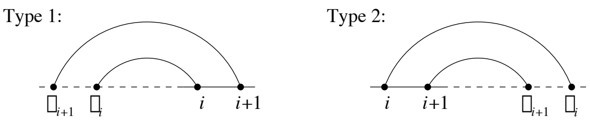

Figure 2. The two types of forbidden arcs.

In this paper we present a bijection between Stoimenow matchings on $\{1,2,\dots,2n\}$ and a collection of sequences of non-negative integers that we call *ascent sequences*.

Given a sequence of integers $x = (x_1,\dots,x_n)$, we say that the sequence x has an *ascent* at position i if $x_i < x_{i+1}$. The number of ascents of x is denoted by $\mathrm{asc}(x)$. Let $\mathcal{A}_n$ be the collection of *ascent sequences of length* n:

$$\mathcal{A}_n = \left\{ (x_1,\dots,x_n) : x_1 = 0 \text{ and } 0 \le x_i \le 1 + \mathrm{asc}(x_1,\dots,x_{i-1}) \text{ for } 1 < i \le n \right\}.$$

These sequences were introduced in a recent paper by Bousquet-Mélou et al. [1]. For example, $\mathcal{A}_3 = \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),(0,1,2)\}$.

Bousquet-Mélou et al. gave bijections between four classes of combinatorial objects, thus proving that they are equinumerous: Stoimenow matchings; unlabeled $(\mathbf{2}+\mathbf{2})$-free posets; permutations avoiding a specific pattern; and ascent sequences. Figure 3, in which solid arrows represents bijections given by Bousquet-Mélou et al., sums up the situation. In particular, $\Psi \circ \Omega$ is a bijection between Stoimenow matchings and ascent sequences. The dashed arrow is the contribution of this paper. That

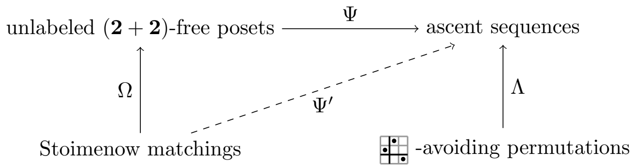

Figure 3.

is, we give a direct description of $\Psi' = \Psi \circ \Omega$. Ascent sequences have an obvious recursive structure. We unearth the corresponding recursive structure of Stoimenow matchings. It amounts to two functions, φ' and ψ', which act on matchings in an identical manner to the functions φ and ψ of [1, §3] acting on posets.

2 Stoimenow matchings and edge removal

Let $\mathcal{I}_n$ be the collection of Stoimenow matchings with n arcs. Let $\mathcal{S}_m$ be the collection of all permutations of the set $\{1, \ldots, m\}$. Any Stoimenow matchings may be written uniquely as a fixed point free involution $\pi \in \mathcal{S}_{2n}$ so that the number paired with i is π_i. We shall abuse notation ever so slightly by considering π to be dually a matching in $\mathcal{I}_n$ and an involution in $\mathcal{S}_{2n}$.

Given $\pi \in \mathcal{I}_n$ let $\mathsf{arcs}(\pi)$ be the collection of all n arcs $[i, \pi_i]$ of π. Let us introduce the following labelling scheme $\mathsf{label} : \mathsf{arcs}(\pi) \to \mathbb{N}$ of the arcs; for every arc in π, call the left endpoint the *opener* and the right endpoint the *closer*.

Label an arc with the number of runs of closers that precede it. For example, consider the matching $\{[1,3],[2,4],[5,6]\}$, or equivalently the involution $\pi = 341265$. The labels of the arcs are shown in Figure 4.

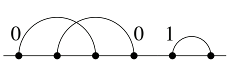

Figure 4.

To every Stoimenow matching we shall single out two (very important) arcs. Given $\pi \in \mathcal{I}_n$ call $\mathsf{maxarc}(\pi) = [\pi_{2n}, 2n]$ the *maximal arc* of π and call the arc $\mathsf{redarc}(\pi) = [\pi_{1+\pi_{2n}}, 1 + \pi_{2n}]$ the *reduction arc* of π.

To every Stoimenow matching we shall associate two statistics:

$$\mathsf{M}(\pi) = \mathsf{label}(\mathsf{maxarc}(\pi)) \quad \text{and} \quad \mathsf{m}(\pi) = \mathsf{label}(\mathsf{redarc}(\pi)).$$

For the matching $\{[1,3],[2,4],[5,6]\}$ we have $\mathsf{redarc}(\pi) = [5,6] = \mathsf{maxarc}(\pi)$ so that $\mathsf{M}(341265) = 1$ and $\mathsf{m}(341265) = 1$. In the diagrams that follow, vertices that are openers are marked with a $\bullet$ and closers are marked with a $\square$.

Example 1 *(i) Consider* $\pi = 3\,4\,1\,2\,7\,9\,5\,10\,6\,8 \in \mathcal{I}_5$*. We have* $\mathsf{redarc}(\pi) = [6, 9]$ *and* $\mathsf{maxarc}(\pi) = [8, 10]$*. This gives* $\mathsf{m}(\pi) = 1$ *and* $\mathsf{M}(\pi) = 2$*. See Figure 5(i).*

(ii) Consider $\pi = 4\,5\,7\,1\,2\,8\,3\,6\,10\,9 \in \mathcal{I}_5$*. The labels of the arcs are shown in Figure 5(ii). We have* $\mathsf{redarc}(\pi) = [9, 10] = \mathsf{maxarc}(\pi)$ *and so* $\mathsf{M}(\pi) = 2 = \mathsf{m}(\pi) = 2$*.*

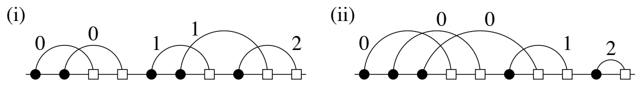

Figure 5.

We begin with the removal operations for Stoimenow matchings. Let $\pi \in \mathcal{I}_n$ where $n \geq 2$ and let $i = \mathsf{m}(\pi)$ be the label of the reduction arc $\mathsf{redarc}(\pi)$. In what follows we will remove the reduction arc in a very careful way so that we obtain $\sigma \in \mathcal{I}_{n-1}$.

Let $L_i(\pi) = \{x \in \mathsf{arcs}(\pi) : \mathsf{label}(x) = i\}$ be the set of arcs that have label i.

(Rem1) If $|L_i(\pi)| > 1$ then simply remove the reduction arc $\mathsf{redarc}(\pi)$.

(Rem2) If $|L_i(\pi)| = 1$ and $i = \mathsf{M}(\pi)$, then $\mathsf{maxarc}(\pi) = \mathsf{redarc}(\pi) = [2n-1, 2n]$ and we remove this arc from π.

(Rem3) If $|L_i(\pi)| = 1$ and $i < \mathsf{M}(\pi)$ then do as follows (these steps are illustrated in Figure 6;

(a) Let A be the collection of all closers between x and the next opener to its right. Move all points in A to between z and u while respecting their order relative to one-another.

(b) For all j with $0 \leq j < i$, partition the collection of openers with label j into three segments X_j, Y_j and Z_j where Y_j is the collection of openers that have closers in A. Swap each of the sets Y_j and Z_j while preserving their respective internal order.

(c) Remove the reduction arc.

Example 2 *Three examples corresponding to the above removal operations.*

(i) In Example 1(i) we had $i = \mathsf{m}(\pi) = 1$, $\mathsf{M}(\pi) = 2$ *and* $|L_2(\pi)| = 2 > 1$*. Thus rule (Rem1) applies and* σ *is given in Figure 7(i).*

(ii) In Example 1(ii) we had $i = \mathsf{m}(\pi) = 2 = \mathsf{M}(\pi)$ *and* $|L_2(\pi)| = 1$*. Thus rule (Rem2) applies and* σ *given in Figure 7(ii).*

(iii) See Figure 8.

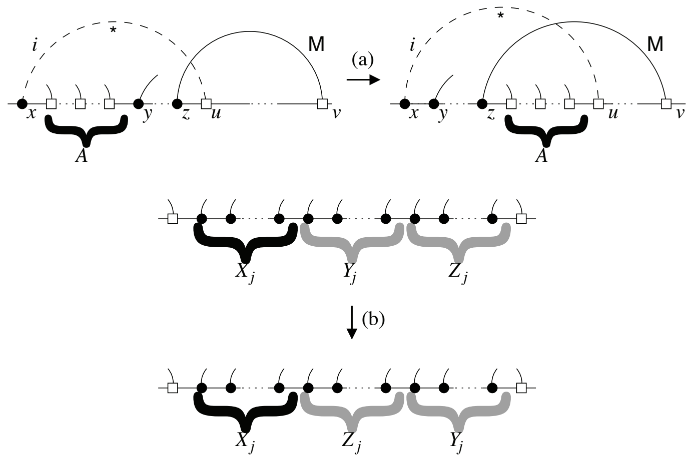

Figure 6. The removal rule Rem3.

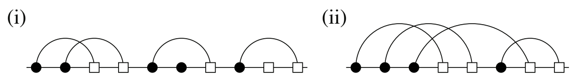

Figure 7.

Example 3 *See Figure 9 for an example of transforming a Stoimenow matching into an ascent sequence.*

We will now prove that the three types of removal operation give some $\sigma \in \mathcal{I}_{n-1}$. If $\mathsf{m}(\pi) = i$ and the removal operation, when applied to π gives σ, then define $\psi'(\pi) = (\sigma, i)$.

Lemma 4 *If $n \geq 2$, $\pi \in \mathcal{I}_n$ and $\psi'(\pi) = (\sigma, i)$ then $\sigma \in \mathcal{I}_{n-1}$ and $0 \leq i \leq 1 + \mathsf{M}(\pi)$. Also,*

$$\mathsf{M}(\sigma) = \begin{cases} \mathsf{M}(\pi) & \text{if } i \leq \mathsf{m}(\sigma), \\ \mathsf{M}(\pi) - 1 & \text{if } i > \mathsf{m}(\sigma). \end{cases}$$

Proof: In this proof we show that each of the three removal operations, when applied to a Stoimenow matching, produce another Stoimenow matching. The removal of an arc from a Stoimenow matching produces a matching, but it is necessary to show the matching is Stoimenow, i.e. does not contain type 1 or type 2 arcs.

In both Rem1 and Rem2 we are simply deleting the reduction arc. Thus the only neighbouring points to check the Stoimenow property (no type 1 or type 2 nestings) are the pairs of points adjacent to the left and right endpoints of $\mathsf{redarc}(\pi)$. However for Rem3 the situation is slightly more complicated.

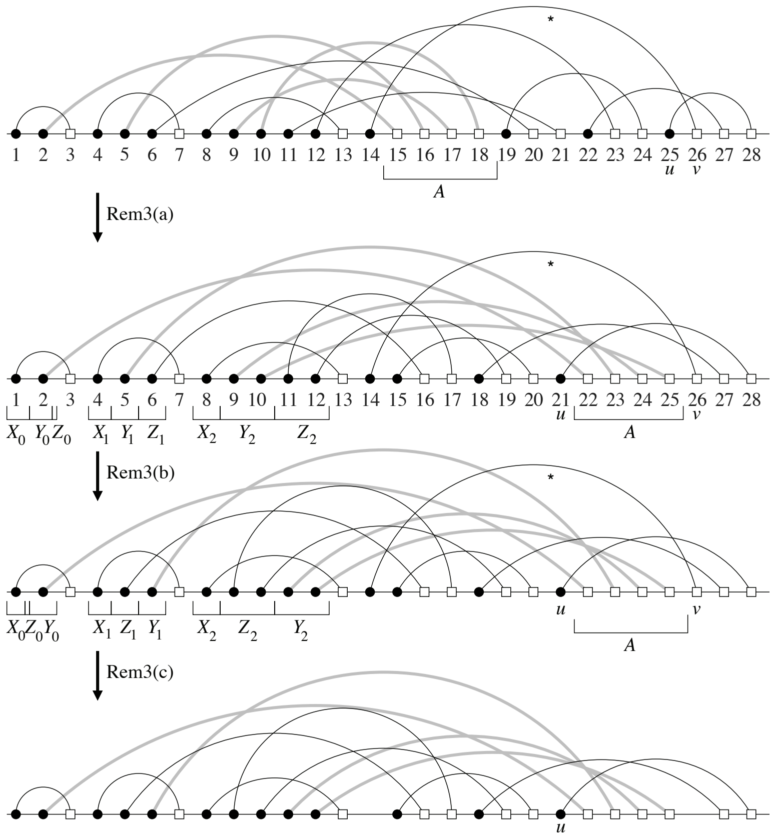

Figure 8. Illustration of the three steps for Rem3.

In the diagrams, lozenge vertices $\Diamond$ correspond to points which could be openers or closers and the reduction arc is indicated by $\star$.

(Rem1) In this case $|L_i(\pi)| > 1$. We must check that the removal of the reduction arc $\mathsf{redarc}(\pi)$ does not introduce a type 1 or 2 arc in σ. If $\mathsf{m}(\pi) = \mathsf{M}(\pi)$ then we have the situation as indicated in Figure 10(i).

If the set of points A is empty then the set B must be empty, for otherwise the arcs with endpoints y and z are type 1. By the same argument, if B is empty then so is A. Figure 10(ii) illustrates the situation for $A = B = \emptyset$. The point x must be a closer since there are no available points to its right. Removing the reduction arc preserves the Stoimenow property.

Alternatively, A is not empty iff B is not empty. In fact all arcs with opener in A have a closer in B. Similarly, all closers in B have openers in A (for otherwise a type 1 arc would appear). Also, x must be a closer, for otherwise a type 2 arc arises with x and w as endpoints. Figure 10(iii) illustrates the situation. It is straightforward

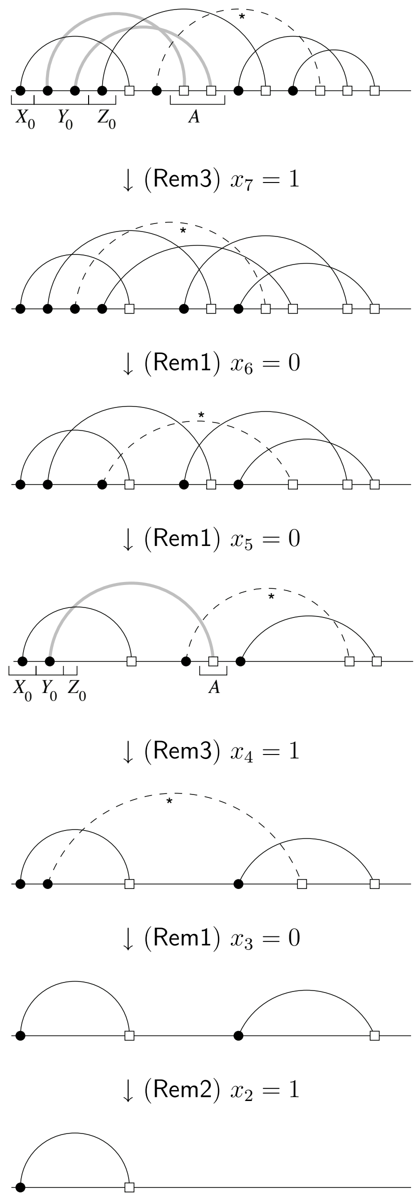

Figure 9. Using the removal operations to go from the Stoimenow matching $\pi = (5, 7, 8, 10, 1, 12, 2, 3, 13, 4, 14, 6, 9, 11) \in \mathcal{I}_7$ to the ascent sequence $x = (0, 1, 0, 1, 0, 0, 1)$.

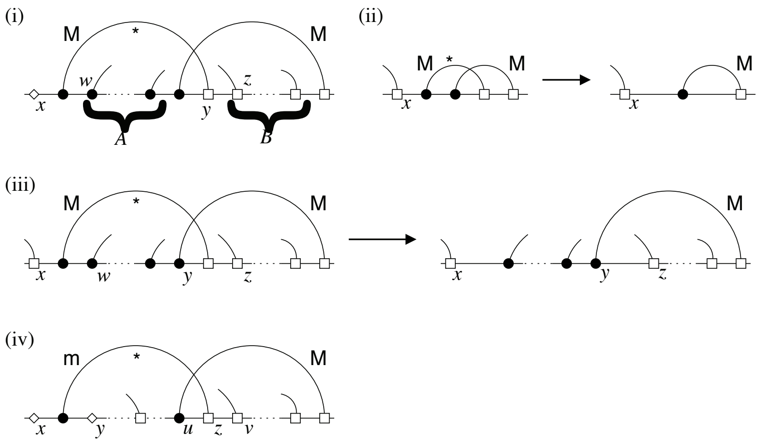

Figure 10.

to see that the removal of the opener of the reduction arc preserves the Stoimenow property.

If $\mathsf{m}(\pi) < \mathsf{M}(\pi)$ then there are at least 2 arcs with label $\mathsf{m}(\pi)$. Consequently, at least one of x and y in Figure 10(iv) must be an opener. Also, note that there must be a closer between the openers of $\mathsf{redarc}(\pi)$ and $\mathsf{maxarc}(\pi)$.

First note that the Stoimenow property is preserved at the newly adjacent points u and v once z is removed. Next, if x is a closer then y must be an opener. Thus removing the opener of $\mathsf{redarc}(\pi)$ preserves the Stoimenow property at (the newly adjacent points) x and y.

If x is an opener, then y can be an opener or a closer. In the case that y is a closer, then the Stoimenow property is preserved. If x and y are both openers then the endpoint of y must be to the left of z since it is a Stoimenow matching. Similarly, the endpoint of x must be to the left of u. Thus the Stoimenow property is preserved at the newly adjacent points x and y. In the arguments above, the label of the maximal arc remains unchanged, hence $\mathsf{M}(\pi) = \mathsf{M}(\sigma)$.

($\mathsf{Rem2}$) In this case $|L_i(\pi)| = 1$ and $i = \mathsf{M}(\pi)$. There is a unique arc $[2n-1, 2n]$ in π that has maximal label $\mathsf{M}(\pi)$. Removing this arc of course yields $\sigma \in \mathcal{I}_{n-1}$. Since this arc does not cross any other arcs in the diagram, its removal cannot induce a type 1 or type 2 arc. It was the only arc with label $\mathsf{M}(\pi)$ so we have $\mathsf{M}(\sigma) = \mathsf{M}(\pi) - 1$.

($\mathsf{Rem3}$) In this case $|L_i(\pi)| = 1$ and $i < \mathsf{M}(\pi)$. We must check that the matching obtained after performing operations (a), (b) and (c) is Stoimenow. It is not necessarily true that the matching is Stoimenow after performing (a). The combination of (a), (b) and (c) is needed to ensure the Stoimenow property. See Figure 11 for an illustration of $\mathsf{Rem3}$.

Note that $0 \leq j < i$. Let A be the run of closers between the opener of the reduction arc and the next opener to its right. Let B be the segment whose leftmost point is

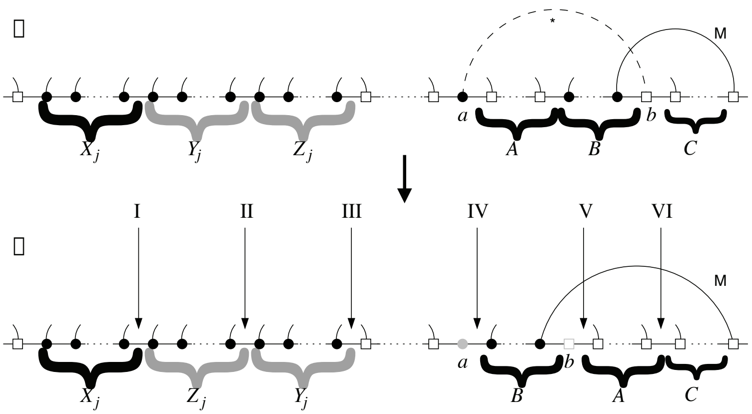

Figure 11.

the opener to the right of A and whose rightmost point is the opener of the maximal arc. Let C be the run of closers that is to the right of the closer of the reduction arc, and to the left of the rightmost closer.

There are only certain places in σ where the Stoimenow property may have been broken. The segments of openers $W_j = (X_j, Y_j, Z_j)$ in π are separated by closers. Thus no two arcs from two different W_k's (where $k < i$) can form a type 2 pair in σ. Hence we may restrict our attention to one segment of openers W_j and the action of steps (a), (b) and (c) on this segment and its interaction with A, B and C.

After Rem3 has been applied, the internal order of each of the X_j, Y_j and Z_j segments is unaltered. Thus the Stoimenow property cannot be broken within each of these segments. However, the order in which the segments Y_j and Z_j appear in σ has been transposed. Similarly, the segments A, B and C retain their internal order so that the Stoimenow property is not violated within each.

By this reasoning there are only six cases to consider where the Stoimenow property may be broken. These are indicated by roman numerals in Figure 11.

The adjacent points in cases III, IV and V are such that one point is an opener and the other is a closer, thereby preserving the Stoimenow property at these positions.

(I) If X_j is empty then there is no opener immediately to the left of Z_j in σ with which to form a type 2 arc. Otherwise X_j is not empty and in σ the closers corresponding to X_j are located to the left of a, whereas closers corresponding to Z_j are in B which is to the right of a. Hence the Stoimenow property is preserved.

(II) If both Z_j and X_j are empty then there is no opener immediately to the left of Y_j. If Z_j is empty and X_j is not empty then the closers of X_j are to the left of a and the closers of Y_j are to the right of a. If Z_j is not empty then the closers corresponding to Z_j are in B. The closers corresponding to Y_j are in A. Since

A is to the right of B in σ, the new neighbors do not form a prohibited type 2 arc.

(VI) If C is empty then we have two adjacent closers at the end of σ. The opener corresponding to the rightmost opener of A is to the left of b so the Stoimenow property is preserved. Otherwise C is non-empty and the openers corresponding to A are Y_j, whereas the openers corresponding to C are in B. Since Y_j is to the left of B, the new neighbors do not form the prohibited type I arc.

The arc that was removed was the only arc in π with label i, so $\mathsf{M}(\sigma) = \mathsf{M}(\pi) - 1$. □

3 Adding an edge to a Stoimenow matching

We now define the addition operation for Stoimenow matchings. Given $\sigma \in \mathcal{I}_{n-1}$ and $0 \leq i \leq 1 + \mathsf{M}(\sigma)$, let $\varphi'(\sigma, i)$ be the Stoimenow matching π obtained from σ according to the following addition rules

(Add1) If $i \leq \mathsf{m}(\sigma)$ then partition the segment of openers with label i into two (possibly empty) segments: let A be the contiguous segment of openers which do not intersect the maximal arc and let B be the contiguous segment of openers that do intersect the maximal arc. Note the A is always to the left of B. Insert an arc by introducing a new point between A and B, and another new point immediately to the right of π_{n-1}. (See Figure 12.)

(Add2) If $i = 1 + \mathsf{M}(\sigma)$ then introduce the arc $[2n-1, 2n]$ to σ.

(Add3) If $\mathsf{m}(\sigma) < i \leq \mathsf{M}(\sigma)$ then do as follows (each of these steps is illustrated in Figure 13.

- (a) Locate the first opener of σ with label i and call it d. Insert an imaginary vertical line L in the diagram just before d. Let A be the contiguous segment of closers immediately right of the opener of the maximal arc, c, whose openers lie to the right of L. Let C be the segment of points after A and before the rightmost point of σ. Insert two new points: one where the line L crosses the diagram, a, and another in-between A and C, b. Join these points by an arc.
- (b) For all $0 \leq j < i$, partition the segments of openers with label j into three segments X_j, Y_j and Z_j. The arcs from X_j have closers that lie to the left of L. The arcs from Y_j have closers that are in A. Z_j is what remains. Swap the segments Z_j and Y_j for each j while preserving the internal order of the openers.
- (c) Finally, move the segment of closers A in-between the points a and d.

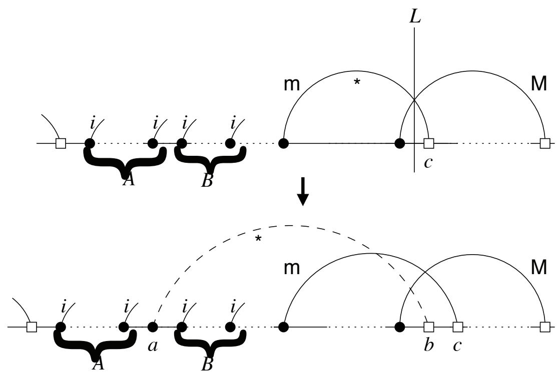

Figure 12. The addition rule $\mathsf{Add1}$.

Lemma 5 *If $n \geq 2$, $\sigma \in \mathcal{I}_{n-1}$, $0 \leq i \leq 1+\mathsf{M}(\sigma)$ and $\pi = \varphi'(\sigma, i)$ then $\pi \in \mathcal{I}_n$. Also,*

$$\mathsf{M}(\pi) = \begin{cases} \mathsf{M}(\sigma) & \text{if } i \leq \mathsf{m}(\sigma), \\ \mathsf{M}(\sigma) - 1 & \text{if } i > \mathsf{m}(\sigma). \end{cases}$$

Proof: The proof of this requires examining the three addition operations separately.

For the first case $i \leq \mathsf{m}(\sigma)$ and $\mathsf{Add1}$ is used. This is illustrated in Figure 12. It introduces a new arc (the end points of this arc are a and b in the figure) which has label i and serves as the new reduction arc. Since arcs with openers in B have closers to the right of c, and arcs with openers in A have closers to the left of c, the Stoimenow property is preserved. It is easy to see from the diagram that the Stoimenow property is preserved. Furthermore, since this new arc is essentially a copy of arcs with label i, $\mathsf{M}(\pi) = \mathsf{M}(\sigma)$.

If $i = 1 + \mathsf{M}(\sigma)$ then $\mathsf{Add2}$ is used. A new arc is added to the matching on the right hand side. This arc does not meet any other arcs so it retains the property of being Stoimenow. Also, the label of this arc will be one more than $\mathsf{M}(\sigma)$ so that $\mathsf{M}(\pi) = \mathsf{M}(\sigma) + 1$.

If $\mathsf{m}(\sigma) < i \leq \mathsf{M}(\sigma)$ then $\mathsf{Add3}$ is used. The details of this part of the proof are the same as the final part of [1, Lemma 4], re-written in the language of matchings as in Lemma 4. □

The machinery has now been set up so that we can see that the recursive structure of Stoimenow matchings is isomorphic to that of ascent sequences. We omit the formal proof by induction of the following result which gives the compatibility of the removal and addition operations for Stoimenow matchings.

Lemma 6 *For any Stoimenow matching σ and integer i such that $0 \leq i \leq 1+\mathsf{M}(Q)$ we have $\psi'(\varphi'(\sigma, i)) = (\sigma, i)$. If σ has more than one element then we also have $\varphi'(\psi'(\sigma)) = \sigma$.*

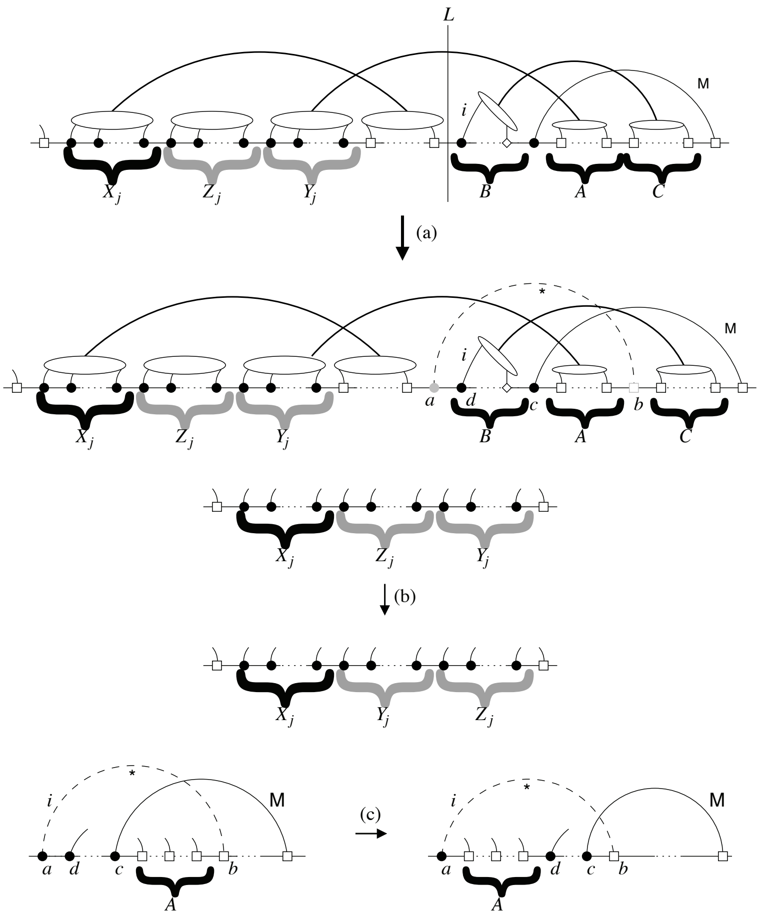

Figure 13. The addition rule Add3.

4 Stoimenow matchings to ascent sequences

Define the map $\Psi' : \mathcal{I}_n \to \mathcal{A}_n$ as follows. For $n = 1$ we associate the only Stoimenow matching in $\mathcal{I}_1$ with the sequence (0). Let $n \geq 2$ and suppose that the removal operation, applied to $\pi \in \mathcal{I}_n$ gives $\psi'(\pi) = (\sigma, i)$. Then the sequence associated with π is $\Psi'(\pi) := (x_1, \ldots, x_{n-1}, i)$ where $(x_1, \ldots, x_{n-1}) = \Psi'(\sigma)$. Combining the previous lemmas, we have the following theorem that is easily proved by induction.

Theorem 7 *The map Ψ' is a one-to-one correspondence between Stoimenow matchings with n arcs and ascent sequences of length n.*

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