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The Nordhaus-Gaddum problem for the k-defective chromatic number of a P_4-free graph

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Abstract

A graph is (m, k)-*colourable* if its vertices can be coloured with m colours such that the maximum degree of the subgraph induced on vertices receiving the same colour is at most k. The *k-defective chromatic number* $\chi_k(G)$ of a graph G is the least positive integer m for which G is (m, k)-colourable. The Nordhaus-Gaddum problem is to find sharp bounds for $\chi_k(G)+\chi_k(\overline{G})$ and $\chi_k(G).\chi_k(\overline{G})$ over the set of all graphs of order p where $\overline{G}$ is the complement of the graph G. In this paper we obtain a sharp upper bound for $\chi_k(G) + \chi_k(\overline{G})$, where G is a P_4-free graph of order p and $k = 1$ or 2.

1 Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part we follow the notation of Chartrand and Lesniak [4]. For a graph G, we denote the vertex set and the edge set by $V(G)$ and $E(G)$ respectively. P_n is a path of order n and $\overline{G}$ is the complement of G. For a subset U of $V(G)$, the subgraph of G induced on the set U is denoted by $G[U]$.

A graph is said to be P_4-free if it does not contain P_4 as an induced subgraph. A P_4-free graph is also known as a complement reducible graph (cograph). Cographs are extensively studied; see Corneil [5] and Seinsche [12]. Let k be a non-negative integer.

A subset U of $V(G)$ is said to be *k-independent* if the maximum degree of $G[U]$ is at most k. A graph is (m,k)-*colourable* if its vertices can be coloured with m colours such that the set of vertices receiving the same colour is *k-independent*. Sometimes we refer to an (m,k)-colouring of G as a *k-defective colouring* of G. Note that any (m,k)-colouring of a graph G partitions the vertex set of G into m subsets V_1, $V_2, \ldots, V_m$, such that every V_i is k-independent. The *k-defective chromatic number* $\chi_k(G)$ of G is the least positive integer m for which G is (m,k)-colourable. Note that $\chi_0(G)$ is the usual chromatic number of G. Clearly $\chi_k(G) \leq \lceil \frac{p}{k+1} \rceil$, where p is the order of G. If $\chi_k(G) = m$ then G is said to be an (m,k)-*chromatic* graph.

The concepts of k-independent sets and k-defective chromatic numbers have been studied by several authors under different names: see Hopkins and Staton [8], Maddox [10, 9], Andrews and Jacobson [3], Frick [6], Frick and Henning [7], Achuthan et al. [1], Simanihuruk et al. [13, 14].

The Nordhaus-Gaddum [11] problem associated with the parameter $\chi_k(G)$ is to find sharp bounds for $\chi_k(G) + \chi_k(\overline{G})$ and $\chi_k(G).\chi_k(\overline{G})$ as G ranges over the class $\mathcal{G}(p)$ of all graphs of order p. Maddox [10] investigated this problem and showed that if $G \in \mathcal{G}(p)$ and G is triangle-free then

$$\chi_k(G) + \chi_k(\overline{G}) \leq 5\lceil \frac{p}{3k+4} \rceil.$$

For $k = 1$, he improved the bound to $6\lceil \frac{p}{9} \rceil$. Achuthan et al. [1] proved that, for any graph $G \in \mathcal{G}(p)$,

$$\chi_1(G) + \chi_1(\overline{G}) \leq \frac{2p+4}{3}.$$

Maddox [10] suggested the conjecture: For a graph G of order p, $\chi_k(G) + \chi_k(\overline{G}) \leq \lceil \frac{p-1}{k+1} \rceil + 2$. Achuthan et al. [1] disproved the above conjecture by constructing graphs G of order p such that

$$\chi_k(G) + \chi_k(\overline{G}) = \lceil \frac{p-1}{k+1} \rceil + 3.$$

All these graphs have P_4 as an induced subgraph and hence Maddox's conjecture can be restated for the sum, $\chi_k(G) + \chi_k(\overline{G})$, as G ranges over the subclass $\mathcal{G}(p, P_4) = \{G \in \mathcal{G}(p) : G \text{ is a } P_4\text{-free graph}\}$. Achuthan et al. [1] established the weak upper bound $\frac{2p+2k+4}{k+2}$ for the sum, $\chi_k(G) + \chi_k(\overline{G})$, as G ranges over $\mathcal{G}(p)$. Furthermore, they proved Maddox's conjecture for $k = 1$, over the class of P_4-free graphs. In this paper we obtain a sharp bound for the sum $\chi_2(G) + \chi_2(\overline{G})$, where G is a P_4-free graph of order p. Some of these results appear in the form of an extended abstract (Achuthan et al. [2]).

In all the figures of this paper, we follow the convention that a solid line between two sets X and Y of vertices represents the existence of all possible edges between X and Y.

2 Upper bound for the sum

We first present a theorem due to Seinsche [12].

Theorem 1 *Let G be a graph. The following statements are equivalent.*

1. *G has no induced subgraph isomorphic to P_4.*
2. *For every subset U of $V(G)$ with more than one element, either $G[U]$ or $\overline{G}[U]$ is disconnected.*

We now present some lemmas.

Lemma 1 *Let G be a graph of order p and k a positive integer. Suppose that G contains m vertex disjoint complete bipartite graphs $B_i \cong K(t_i, k+1)$ with $1 \le t_i \le k+1$, for $1 \le i \le m$. Then*

$$\chi_k(\overline{G}) \;\le\; \left\lceil \frac{p - \sum_{i=1}^{m} t_i}{k+1} \right\rceil .$$

Proof: Using the fact that the vertex set of B_i is k-independent in $\overline{G}$ for each i, we can find a k-defective colouring of $\overline{G}$ which uses $\lceil (p - \sum_{i=1}^{m} t_i)/(k+1) \rceil$ colours. This proves the lemma.

Lemma 2 *Let $k = 1$ or 2. Suppose that F, H and L are graphs such that $\chi_k(F) \ge \chi_k(H) \ge 3$, $1 \le |V(L)| \le k$. Suppose that G is a graph of order p constructed by joining every vertex of H to every vertex of L. In addition, there are no edges between $V(F)$ and $V(H)$ in G, and there may be edges between the vertex sets of F and L. Let $\chi_k(I) + \chi_k(\overline{I}) \le 2 + \lceil \frac{n-1}{k+1} \rceil$ for every subgraph I of order $n < p$. Then*

$$\chi_k(G) + \chi_k(\overline{G}) \;\le\; \left\lceil \frac{p-1}{k+1} \right\rceil + 2.$$

Proof: Let a, b and c be the orders of F, H and L respectively. Since $\chi_k(H) \ge 3$, we can easily show that

$$\chi_k(\overline{H}) \le \left\lceil \frac{b-1}{k+1} \right\rceil - 1.$$

Clearly

$$\chi_k(G) \le \chi_k(F) + 1$$

and

$$\chi_k(\overline{G}) \;\le\; \chi_k(\overline{F}) + \chi_k(\overline{H} \cup \overline{L}) \;=\; \chi_k(\overline{F}) + \chi_k(\overline{H}).$$

Thus

$$\chi_k(G) + \chi_k(\overline{G}) \;\le\; \chi_k(F) + \chi_k(\overline{F}) + \chi_k(\overline{H}) + 1 \le 2 + \left\lceil \frac{a-1}{k+1} \right\rceil + \left\lceil \frac{b-1}{k+1} \right\rceil .$$

Now let m, s, q and r be integers such that $a-1 = m(k+1)+s$, $0 \leq s \leq k$ and $b-1 = q(k+1)+r$, $0 \leq r \leq k$. If $r = 0$ or $s = 0$ then it is easy to see that

$$\left\lceil \frac{a-1}{k+1} \right\rceil + \left\lceil \frac{b-1}{k+1} \right\rceil \leq m+q+1 \leq \left\lceil \frac{p-1}{k+1} \right\rceil.$$

Otherwise it can be shown that

$$\left\lceil \frac{a-1}{k+1} \right\rceil + \left\lceil \frac{b-1}{k+1} \right\rceil = m+q+2 \leq \left\lceil \frac{p-1}{k+1} \right\rceil.$$

Thus in both the cases we have the required inequality.

Lemma 3 *Let $k = 1$ or 2 and L be a graph with $1 \leq |V(L)| \leq k$. Let G be a graph constructed using graphs F, H and L such that every vertex of $V(F)$ is joined to every vertex of $V(L)$ and no vertex of $V(F)$ is joined to any vertex of $V(H)$. Suppose that $\chi_k(F+L) \geq \chi_k(H)+1$. Then $\chi_k(G) = \chi_k(F+L)$.*

Proof: Let $\chi_k(F+L) = m$. Consider an (m,k)-colouring of $F+L$. Let V_i be the i^{th} colour class from the above colouring of $F+L$. Without any loss of generality we assume that $V_1, V_2, \ldots, V_r$ are the colour classes that have at least one vertex of L. Note that $r = 1$ or 2.

Firstly let $r = 1$. Since $\chi_k(H) \leq m-1$, the vertices of H can be coloured using the colours $2, 3, \ldots, m$. Thus $\chi_k(G) \leq m$.

Next let $r = 2$. Note that $|V_i| \leq 3$, for $i = 1, 2$ and $|V(L)| = 2 = k$. Now partition $V_1 \cup V_2$ into sets U_1 and U_2 such that U_1 contains both the vertices of L and $|U_i| \leq 3$ for $i = 1, 2$. It is easy to see that this (m,k)-colouring of $F+L$ can be extended to the vertices of H. Thus in both the cases we have $\chi_k(G) \leq m = \chi_k(F+L)$. Since $F+L$ is a subgraph of G this implies that $\chi_k(G) = \chi_k(F+L)$.

Lemma 4 *Let H be a connected P_4-free graph with $\chi_2(H) = 2$. Suppose that for every partition (V_1, V_2) of $V(H)$ into 2-independent sets, we have $\min\{|V_1|, |V_2|\} \geq 2$. Then H contains $K(2,3)$ as a subgraph.*

Proof: Clearly $|V(H)| \geq 5$. It follows from Theorem 1 that $\overline{H}$ is disconnected. Let F be a component of $\overline{H}$ with the least order and I the union of all the other components of $\overline{H}$. Since $|V(H)| \geq 5$, we have $|V(I)| \geq 3$. Note that every vertex of F is adjacent to every vertex of I in H. Thus if $|V(F)| \geq 2$, then clearly H contains $K(2,3)$ as a subgraph. Now let $|V(F)| = 1$ and $F = \{x\}$. From the hypothesis it follows that $V(H) - \{x\}$ contains a vertex of degree at least 3. Since x is adjacent to every vertex in H, it follows that H contains $K(2,3)$ as a subgraph. This completes the proof of the lemma.

Lemma 5 *Let H and F be connected graphs and $G \cong (H \cup F) + K_1$ be a P_4-free graph. If $\chi_2(H) = \chi_2(F) = 2$ and $\chi_2(G) = 3$, then either H or F has a subgraph isomorphic to $K(2,3)$.*

Proof: This lemma follows from the fact that at least one of F and H satisfies the hypothesis of Lemma 4.

Theorem 2 *Let G be a P_4-free graph of order $p \geq k+2$, where $k = 1$ or 2. Then*

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{p-1}{k+1} \right\rceil + 2,$$

and this bound is sharp for all p.

Proof: We prove the theorem by induction on p. It is easy to check that the inequality holds for a graph of order $k+2$. Now let $p > k+2$. We make the induction hypothesis that if S is a P_4-free graph of order $n < p$ then $\chi_k(S)+\chi_k(\overline{S}) \leq \lceil \frac{n-1}{k+1} \rceil + 2$.

Let G be a graph of order p. If $\chi_k(G) \leq 2$ or $\chi_k(\overline{G}) \leq 2$ then the required inequality can easily be verified. Thus we assume that $\chi_k(G) \geq 3$ and $\chi_k(\overline{G}) \geq 3$.

Let $T \subseteq V(G)$ such that $|T| = k+1$. Clearly

$$\chi_k(G) \leq \chi_k(G-T) + 1 \tag{1}$$

and

$$\chi_k(\overline{G}) \leq \chi_k(\overline{G}-T) + 1. \tag{2}$$

If there is a subset T of $V(G)$ with $|T| = k+1$ such that strict inequality holds in (1) or (2) then the required inequality follows. Thus we will assume that equality holds in (1) and (2) for every subset T of size $k+1$. That is, for every subset T of $V(G)$ with $|T| = k+1$ we have

$$\chi_k(G) = \chi_k(G-T) + 1 \tag{3}$$

and

$$\chi_k(\overline{G}) = \chi_k(\overline{G}-T) + 1. \tag{4}$$

Since G is a P_4-free graph, by Theorem 1 it follows that either G or $\overline{G}$ is disconnected. Without loss of generality let us assume that G is disconnected. Let G_1 be the component of G with the largest value of the k-defective chromatic number and G_2 the union of all the other components of G. Note that $\chi_k(G) = \chi_k(G_1)$. Clearly $|V(G_2)| \leq k$, otherwise we have a contradiction to equation (3). Since G_1 is connected and P_4-free, it follows from Theorem 1 that $\overline{G_1}$ is disconnected. Let $F_1, F_2, \ldots, F_t$ be the components of $\overline{G_1}$ such that $\chi_k(F_1) \geq \chi_k(F_2) \geq \ldots \geq \chi_k(F_t)$. Clearly $\chi_k(F_1) \geq 2$, otherwise $\chi_k(\overline{G}) \leq 2$, a contradiction to our assumption. Let $U = F_2 \cup F_3 \cup \ldots \cup F_t$. The graph $\overline{G}$ is depicted in Figure 1.

If $\chi_k(U) \geq 3$, then by Lemma 2, we have the required inequality. Henceforth we assume that $\chi_k(U) \leq 2$.

Case 1: $\chi_k(U) = 2$. Clearly the order of U is at least $k+2$. It is easy to see that $\chi_k(\overline{G}-U) \leq \chi_k(\overline{G}) - 1$.

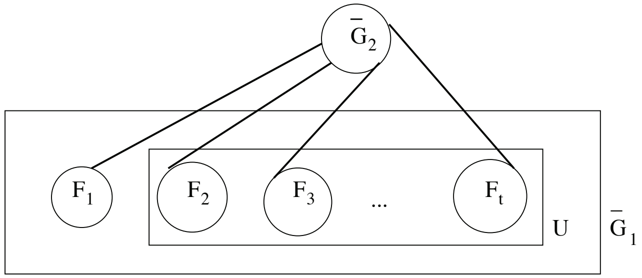

Figure 1: Graph $\overline{G}$

Applying Lemma 3 to $\overline{G}$, it is easy to see that $\chi_k(F_1 + \overline{G_2}) \leq \chi_k(U) = 2$. Now

$$\chi_k(\overline{G}) \leq \chi_k(F_1) + 1 \leq \chi_k(F_1 + \overline{G_2}) + 1 \leq 3.$$

Combining this with the assumption that $\chi_k(\overline{G}) \geq 3$, we have $\chi_k(\overline{G}) = 3$. Consequently $\chi_k(F_1) = 2$. Thus the theorem is proved in this case if we show that $\chi_k(G) \leq \lceil \frac{p-k-2}{k+1} \rceil$.

Now since $\chi_k(F_1) = \chi_k(U) = 2$, it follows that F_1 and U contain $K(1, k+1)$ as a subgraph. Since the order of $G_1 = p - |V(G_2)|$, by Lemma 1, we have $\chi_k(G_1) \leq \lceil \frac{p-|V(G_2)|-2}{k+1} \rceil$. Thus $\chi_k(G) = \chi_k(G_1) \leq \lceil \frac{p-k-2}{k+1} \rceil$, if $k = 1$ or if $k = 2$ and $|V(G_2)| = 2$.

Next let $k = 2$ and the order of G_2 be 1. Firstly if $t = 2$, then $\overline{G}$ satisfies the conditions of Lemma 5. Thus either F_1 or F_2 contains $K(2,3)$ as a subgraph. Without loss of generality, assume that $K(2,3) \subset F_1$. Since $\chi_2(F_2) = 2$, we have $K(2,3) \subset F_2 + \overline{G_2}$. Thus $\overline{G}$ has two vertex disjoint graphs B_1 and B_2, each isomorphic to $K(2,3)$. Thus by Lemma 1, $\chi_2(G) \leq \lceil \frac{p-4}{3} \rceil$.

Now let $t \geq 3$ and $\chi_2(F_3) = \chi_2(F_3 \cup \ldots \cup F_t) = 1$. Recall that $\overline{G} \cong ((F_1 \cup F_2) \cup (F_3 \cup \ldots \cup F_t)) + K_1$. Since $\chi_2(F_1 \cup F_2) = 2$, by Lemma 3, we have $\chi_2(\overline{G}) = \chi_2((F_1 \cup F_2) + K_1)$. Since $\chi_2(\overline{G}) = 3$, the graph $(F_1 \cup F_2) + K_1$ satisfies the conditions of Lemma 5. It is again easy to show that $\chi_2(G) \leq \lceil \frac{p-4}{3} \rceil$, as in the case $t = 2$.

Now let us assume that $t \geq 3$ and $\chi_2(F_3) = 2$. Clearly $\overline{G}$ contains three vertex disjoint graphs each isomorphic to $K(1,3)$. Thus by Lemma 1, $\chi_2(G) \leq \lceil \frac{p-4}{3} \rceil$. This completes the proof of the theorem in Case 1.

Case 2: $\chi_k(U) = 1$. Since $\chi_k(F_1 + \overline{G_2}) \geq 2 = \chi_k(U) + 1$, we have

$$\chi_k(\overline{G}) = \chi_k(F_1 + \overline{G_2}) \tag{5}$$

Now using equation (4) it is easy to show that the order of U is at most k.

Since F_1 is connected, $\overline{F_1}$ is disconnected. Let $H_1, H_2, \ldots, H_s$ be the components of $\overline{F_1}$ such that $\chi_k(H_1) \geq \chi_k(H_2) \geq \ldots \geq \chi_k(H_s)$. Figure 2 depicts the graph G.

Let $Y = H_2 \cup H_3 \cup \ldots \cup H_s$. Notice that $\chi_k(H_1) \geq 2$, otherwise $\chi_k(G) \leq 2$, a contradiction to our assumption that $\chi_k(G) \geq 3$. If $\chi_k(Y) \geq 3$ then $\chi_k(H_1) \geq 3$.

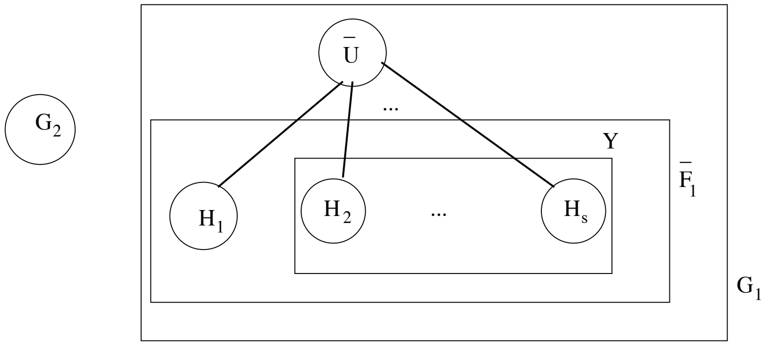

Figure 2: Graph G

Now $G \cong G_2 \cup ((H_1 \cup Y) \cup \overline{U})$. Since the order of $\overline{U}$ is at most k, invoking Lemma 2 we have the required inequality. Henceforth we will assume that $\chi_k(Y) \leq 2$. Note that

$$\chi_k(Y \cup G_2) = \chi_k(Y) = \chi_k(H_2) \leq \chi_k(H_1) \leq \chi_k(H_1 + \overline{U}).$$

Subcase 2.1: $\chi_k(H_1 + \overline{U}) = \chi_k(Y \cup G_2)$.

Clearly in this case, $\chi_k(H_1) = \chi_k(Y) = 2$. This in turn implies that $\chi_k(G) \leq 3$. Combining this with the assumption that $\chi_k(G) \geq 3$, we have

$$\chi_k(G) = 3. \tag{6}$$

To complete the proof of the theorem it is sufficient to establish that $\chi_k(\overline{G}) \leq \lceil \frac{p-k-2}{k+1} \rceil$. Firstly since $\chi_k(H_1) = \chi_k(Y) = 2$, it follows that both H_1 and Y contain subgraphs isomorphic to $K(1, k+1)$. Since H_1 and Y are both subgraphs of $\overline{F_1}$, this implies that $\overline{F_1} \cup G_2$ contains two vertex disjoint stars each of order $k+2$. Since the order of $\overline{F_1} \cup G_2$ is $p - |V(U)|$, it follows from Lemma 1 that $\chi_k(F_1 + \overline{G_2}) \leq \lceil \frac{p-|V(U)|-2}{k+1} \rceil$. From equation (5), we have $\chi_k(\overline{G}) \leq \lceil \frac{p-|V(U)|-2}{k+1} \rceil$. This implies that $\chi_k(\overline{G}) \leq \lceil \frac{p-k-2}{k+1} \rceil$ if $k = 1$ or if $k = 2$ and$|V(U)| = 2$. Thus for the remaining part of Subcase 2.1 we will assume that $k = 2$ and $|V(U)| = 1$. The theorem is proved in this subcase if we show that $\chi_2(\overline{G}) \leq \lceil \frac{p-4}{3} \rceil$.

Firstly let $s = 2$. Recall that $Y = H_2$ and $\chi_2(H_1) = \chi_2(H_2) = 2$. Since $\chi_2(G_1) = \chi_2(G) = 3$, G_1 satisfies the conditions of Lemma 5. Thus either H_1 or H_2 contains $K(2, 3)$ as a subgraph. Without any loss of generality we assume that $K(2, 3) \subset H_1$. Since $\chi_2(H_2) = 2$, we have $K(2, 3) \subset H_2 + \overline{U}$. Thus G_1 has two vertex disjoint subgraphs B_1 and B_2, each isomorphic to $K(2, 3)$. Hence by Lemma 1, $\chi_2(\overline{G}) \leq \lceil \frac{p-4}{3} \rceil$.

Next let $s \geq 3$ and $\chi_2(H_3) = \chi_2(H_3 \cup \ldots \cup H_s) = 1$. Recall that $G \cong ((H_1 \cup H_2) \cup (H_3 \cup \ldots \cup H_s)) + K_1$. Since $\chi_2(H_1 \cup H_2) = 2$, by Lemma 3, we have $\chi_2(G_1) =$

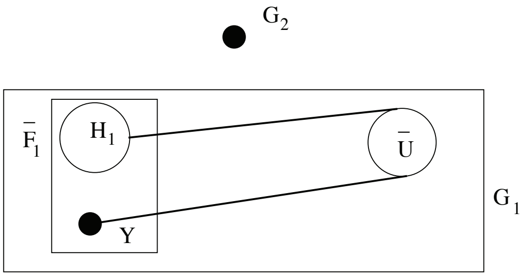

Figure 3: Graph G

$\chi_2((H_1 \cup H_2) + K_1)$. Since $\chi_2(G_1) = 3$, the graph $(H_1 \cup H_2) + K_1$ satisfies the conditions of Lemma 5. It is again easy to show that G_1 contains two vertex disjoint subgraphs each isomorphic to $K(2,3)$. Thus $\chi_2(\overline{G}) \leq \lceil \frac{p-4}{3} \rceil$.

Finally let $s \geq 3$ and $\chi_2(H_3) = 2$. Thus H_i contains a subgraph isomorphic to $K(1,3)$ for $i = 1, 2$ and 3. Thus the graph $H_1 \cup Y \cup G_2$ of order $p-1$ contains three vertex disjoint subgraphs each isomorphic to $K(1,3)$. Thus $\chi_2(\overline{H_1 \cup Y \cup G_2}) \leq \lceil \frac{p-4}{3} \rceil$. This together with equation (5) and the fact that $\overline{H_1 \cup Y \cup G_2} \cong F_1 + \overline{G_2}$ implies that $\chi_2(\overline{G}) \leq \lceil \frac{p-4}{3} \rceil$. This completes the proof of the theorem in Subcase 2.1.

Subcase 2.2: $\chi_k(H_1 + \overline{U}) \geq \chi_k(Y \cup G_2) + 1$.

From Lemma 3, it follows that

$$\chi_k(G) = \chi_k(H_1 + \overline{U}). \qquad (7)$$

Notice that (3) and (7)together imply that $|V(Y \cup G_2)| \leq k$. This in turn implies that $Y \cong G_2 \cong K_1$ and hence $k = 2$. Figure 3 depicts the graph G under this case.

Since H_1 is connected and P_4-free, it follows that $\overline{H_1}$ is disconnected. Let $I_1, I_2, \ldots, I_q$ be the components of $\overline{H_1}$ such that $\chi_2(I_1) \geq \chi_2(I_2) \geq \ldots \geq \chi_2(I_q)$. Figure 4 depicts the graph $\overline{G}$ under this case. Now $\chi_2(I_1) \geq 2$, for otherwise, $\chi_2(\overline{G}) \leq 2$, thus contradicting the assumption that $\chi_2(\overline{G}) \geq 3$. Let $Z \cong I_2 \cup I_3 \cup \ldots \cup I_q$.

Firstly let $\chi_2(Z) \geq 3$. Note that in $\overline{G}$, every vertex of Z is adjacent to every vertex of $\overline{G_2}$ and Y. Further there are no edges between Z and $I_1 \cup U$. Also it is easy to see that $\chi_2(I_1 \cup U) \geq \chi_2(Z)$. Thus $\overline{G}$ satisfies the conditions of Lemma 2, with $L \cong \overline{G_2} + Y$ and hence $\chi_2(G) + \chi_2(\overline{G}) \leq \lceil \frac{p-1}{3} \rceil + 2$.

Henceforth we will assume that $\chi_2(Z) \leq 2$. Note that

$$\chi_2(Z \cup U) = \chi_2(Z) = \chi_2(I_2) \leq \chi_2(I_1) \leq \chi_2(I_1 + \overline{G_2} + Y).$$

Subcase 2.2.1: $\chi_2(I_1 + \overline{G_2} + Y) = \chi_2(Z \cup U)$.

In this case, $2 \leq \chi_2(I_1) \leq \chi_2(I_1 + \overline{G_2} + Y) = \chi_2(Z \cup U) = \chi_2(Z) \leq 2$. Therefore $\chi_2(I_1) = \chi_2(Z) = 2$. Now

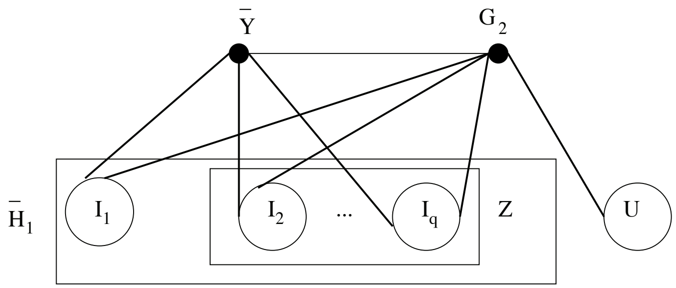

Figure 4: Graph $\overline{G}$

$$\chi_2(\overline{G}) \leq \chi_2(I_1 \cup Z \cup U) + \chi_2(Y + \overline{G_2}) = \chi_2(I_1) + 1 = 3.$$

Since $\chi_2(I_1) = \chi_2(Z) = 2$, both I_1 and Z contain $K(1,3)$ as a subgraph. Since the vertices of Y and $\overline{G_2}$ are adjacent to all the vertices of $I_1 \cup Z$ in $\overline{G}$, it follows that $\overline{G}$ has two vertex disjoint subgraphs, each isomorphic to $K(2,3)$. Applying Lemma 1 to $\overline{G}$ we have $\chi_2(G) \leq \lceil \frac{p-4}{3} \rceil$. This, together with the inequality $\chi_2(\overline{G}) \leq 3$, gives

$$\chi_2(G) + \chi_2(\overline{G}) \leq \lceil \frac{p-1}{3} \rceil + 2.$$

This proves the theorem in Subcase 2.2.1.

Subcase 2.2.2: $\chi_2(I_1 + \overline{G_2} + Y) \geq \chi_2(Z \cup U) + 1$.

Recall that every vertex of I_1 is adjacent to both the vertices of $Y + \overline{G_2}$ in $\overline{G}$ and that there are no edges between the vertex sets $Z \cup U$ and I_1. Applying Lemma 3 to the graph $\overline{G}$, we have

$$\chi_2(\overline{G}) = \chi_2(I_1 + \overline{G_2} + Y).$$

Now using (4) we can easily show that $|V(Z \cup U)| \leq 2$. Since the graphs Z and U have at least one vertex each, it follows that $|V(Z)| = |V(U)| = 1$. Figure 5 provides the graph $\overline{G}$ under this subcase. Since I_1 is P_4-free and connected, it follows from Theorem 1 that $\overline{I_1}$ is disconnected. Let $J_1, J_2, \ldots, J_r$ be the components of $\overline{I_1}$ such that $\chi_2(J_1) \geq \chi_2(J_2) \geq \ldots \geq \chi_2(J_r)$. Figure 6 depicts the graph G for this subcase. Clearly $\chi_2(J_1) \geq 2$, otherwise $\chi_2(G) \leq 2$, a contradiction to our assumption that $\chi_2(G) \geq 3$. Let $W \cong J_2 \cup J_3 \cup \ldots \cup Jr$.

Firstly let $\chi_2(W) \geq 3$. Consider the graphs W, $\overline{Z} + \overline{U}$, and $J_1 \cup Y \cup G_2$. Observe that every vertex of W is adjacent to all the vertices of $\overline{Z} + \overline{U}$; no vertex of $J_1 \cup Y \cup G_2$ is adjacent to a vertex of W. We also note that $\chi_2(J_1 \cup Y \cup G_2) \geq \chi_2(W) \geq 3$. Clearly G satisfies the conditions of Lemma 2 with $L \cong \overline{Z} + \overline{U}$. Therefore $\chi_2(G) + \chi_2(\overline{G}) \leq \lceil \frac{p-1}{3} \rceil + 2$.

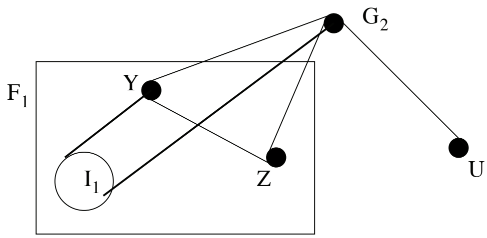

Figure 5: Graph $\overline{G}$

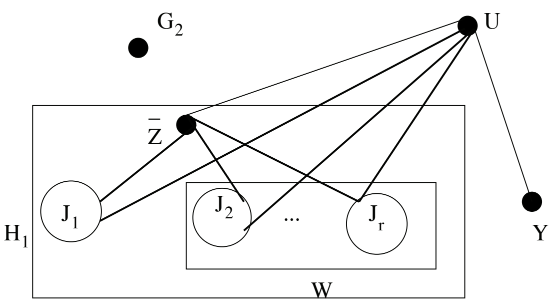

Figure 6: Graph G

Next let $\chi_2(W) \leq 2$. Firstly assume that $\chi_2(J_1 + \overline{Z} + \overline{U}) \geq \chi_2(W) + 1$. Now consider the subgraphs J_1, $\overline{Z} + \overline{U}$ and $W \cup Y \cup G_2$ of the graph G. Note that the order of $\overline{Z} + \overline{U}$ is 2. Clearly every vertex of J_1 is adjacent to both the vertices of $\overline{Z} + \overline{U}$ in G and there are no edges between J_1 and $W \cup Y \cup G_2$. Now applying Lemma 3 to G, we have

$$\chi_2(G) = \chi_2(J_1 + \overline{Z} + \overline{U}).$$

This is a contradiction to (3), since the order of $J_1 + \overline{Z} + \overline{U}$ is at most $p - 3$. Thus it follows that $\chi_2(J_1 + \overline{Z} + \overline{U}) \leq \chi_2(W)$. Since $\chi_2(J_1) \geq \chi_2(W)$, this implies that $\chi_2(J_1 + \overline{Z} + \overline{U}) = \chi_2(W)$. Since $\chi_2(J_1 + \overline{Z} + \overline{U}) \geq \chi_2(J_1) \geq 2$ and $\chi_2(W) \leq 2$, it follows that $\chi_2(J_1 + \overline{Z} + \overline{U}) = \chi_2(J_1) = \chi_2(W) = 2$. Now,

$$\chi_2(G) = \chi_2(G_1) \leq \chi_2(J_1) + 1 = 3.$$

Since $\chi_2(J_1) = \chi_2(W) = 2$, the graphs J_1 and W contain subgraphs isomorphic to $K(1, 3)$. Since the vertices of $\overline{Z} + \overline{U}$ are adjacent to all the vertices of J_1 and W in G, it follows that G contains two vertex disjoint $K(2, 3)$'s as subgraphs. Thus by Lemma 1, $\chi_2(\overline{G}) \leq \lceil \frac{p-4}{3} \rceil$. Combining this with the inequality $\chi_2(G) \leq 3$, we have the required inequality. The graph $K(1, p - 1)$ establishes the sharpness of the theorem.

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