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Double domination edge removal critical graphs

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Abstract

In a graph, a vertex is said to dominate itself and all its neighbors. A double dominating set of a graph G is a subset of vertices that dominates every vertex of G at least twice. The double domination number $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. A graph G without isolated vertices is called edge removal critical with respect to double domination, or just $\gamma_{\times 2}$-critical, if the removal of any edge increases the double domination number. We first give a necessary and sufficient condition for $\gamma_{\times 2}$-critical graphs. Then we give a characterization of $\gamma_{\times 2}$-critical graphs for some classes of graphs including trees, P_4-free and P_5-free graphs. Finally, we investigate $\gamma_{\times 2}$-critical graphs having double domination number 3 or 4.

1 Introduction

In a simple graph $G = (V(G), E(G)) = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is

$N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *pendent vertex* or a *leaf*, and its neighbor is called a *support vertex*. We let $S(G), L(G)$ be the set of support vertices and leaves of G, respectively. An edge incident with a leaf is called a *pendent edge*. If $A \subseteq V(G)$, then $G[A]$ is the subgraph induced by the vertices of A. The *diameter*, $\text{diam}(G)$, of a graph G is the maximum distance over all pairs of vertices of G. We denote by K_n the *complete graph* of order n, and by $K_{m,n}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = m$ and $|Y| = n$. A star of order $n+1$ is $K_{1,n}$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A subdivided star $K^*_{1,n}$ is a tree obtained from a star $K_{1,n}$ by replacing each edge uv of $K_{1,n}$ by a vertex w and edges uw and vw. A graph G is called *k-regular* if every vertex of G has degree k. The *path* and the *cycle* on n vertices are denoted by P_n and C_n, respectively.

A subset S of V is a *double dominating set*, abbreviated **DDS**, if every vertex in $V - S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. A double dominating set of G with minimum cardinality is called a *$\gamma_{\times 2}(G)$-set*. Double domination was introduced by Harary and Haynes [4] and is studied for example in [1, 3, 4]. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [5], [6].

Given a graph, a new graph can be obtained by removing or adding an edge. The study of the effects of such modifications have been considered for several domination parameters. Note that Sumner and Blitch [7] were the first to introduce *edge removal critical graphs* for the domination number. For a survey we cite [5] (Chapter 5). In this paper we study the effects on increasing double domination number when an edge is deleted.

2 Preliminary results

We begin with the following obvious observation.

Observation 1 *Every DDS of a graph contains all its leaves and support vertices.*

Next we show that the removal of a non-pendent edge of a graph G can increase the double domination number of G by at most two.

Theorem 2 *Let G be a graph without isolated vertices. Then $\gamma_{\times 2}(G) \le \gamma_{\times 2}(G-e) \le \gamma_{\times 2}(G) + 2$ for every non-pendent edge $e \in E(G)$.*

Proof. Let $e = xy$ be a non-pendent edge. Clearly every $\gamma_{\times 2}(G-e)$-set is a DDS of G and so $\gamma_{\times 2}(G) \le \gamma_{\times 2}(G-e)$. Now let S be a $\gamma_{\times 2}(G)$-set. If $S \cap \{x, y\} = \emptyset$, then S is a DDS of $G - e$ and hence $\gamma_{\times 2}(G-e) \le \gamma_{\times 2}(G)$. Assume now, without loss of generality, that $S \cap \{x, y\} = \{y\}$. Since x has two neighbors in S, $S \cup \{x\}$ is a DDS of $G - e$ implying that $\gamma_{\times 2}(G-e) \le \gamma_{\times 2}(G) + 1$. Finally assume that $\{x, y\} \subseteq S$. We examine three cases.

If each x and y has degree at least two in $G[S]$, then since e is a non pendent edge, S remains a DDS of $G-e$ and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)$. Assume that both x and y are pendent vertices in $G[S]$. Since $e = xy$ is a non-pendent edge, each of x and y has a neighbor in $V-S$. Let $x', y' \in V-S$ be the neighbors of x and y, respectively. Then $S \cup \{x', y'\}$ is a DDS of $G-e$ and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+2$. Finally, assume without loss of generality, that x is a vertex of degree one in $G[S]$, and y has degree at least two in $G[S]$. Since xy is a non-pendent edge, let $x' \in V-S$ be any neighbor of x. Then $S \cup \{x'\}$ is a DDS of $G-e$, and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+1$. ■

A graph G is said to be edge removal critical with respect to double domination, or just $\gamma_{\times 2}$-critical, if for every edge $e \in E(G)$, $\gamma_{\times 2}(G-e) > \gamma_{\times 2}(G)$. If e is a pendent edge, then $G-e$ contains an isolated vertex and we hence define $\gamma_{\times 2}(G-e) = +\infty$. Hence to show that a graph is $\gamma_{\times 2}$-critical it suffices to show that the removal of every non-pendent edge increases the double domination number. Let $X_G \subseteq E(G)$ be the set of non-pendent edges in G. Clearly, if $X_G = \emptyset$, then G is a nontrivial star and therefore G is a $\gamma_{\times 2}$-critical graph.

The following observation is straightforward.

Observation 3 *If G is a $\gamma_{\times 2}$-critical graph, then no two support vertices of G are adjacent.*

Observation 4 *Let G be a graph obtained from a subdivided star $K^*_{1,r}$ $(r \geq 2)$ with center vertex y by adding an edge from y to a vertex x of a nontrivial graph G'. Then G is not a $\gamma_{\times 2}$-critical graph.*

Proof. Assume that G is $\gamma_{\times 2}$-critical. Let $u_1, u_2, ..., u_r$ be the support vertices of the subdivided star $K^*_{1,r}$ and let S be a $\gamma_{\times 2}(G)$-set. By Observation 1 each u_i belongs to S. If $y \in S$, then removing any edge yu_i does not increase the double domination number. Thus $y \notin S$ and hence S is a DDS of $G-xy$ implying that $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)$, a contradiction. It follows that G is not a $\gamma_{\times 2}$-critical graph.

■

We now give a necessary and sufficient condition for $\gamma_{\times 2}$-critical graphs.

Theorem 5 *A graph G is a $\gamma_{\times 2}$-critical graph if and only if for every $\gamma_{\times 2}(G)$-set S the following conditions hold.*

i) Each component in $G[S]$ is a star.

ii) $V-S$ is an independent set.

iii) Every vertex of $V-S$ has degree two.

Proof. Assume that G is a $\gamma_{\times 2}$-critical graph and let S be any $\gamma_{\times 2}(G)$-set. Observe that $G[S]$ contains no cycle for otherwise removing any edge on the cycle does not increase the double domination number. Thus $G[S]$ is a forest. If $G[S]$ contains a component with diameter at least three, then there exists an edge on the diametrical path of such a component whose removal does not increase the double domination number, a contradiction. Since $G[S]$ does not contain isolated vertices, every component of $G[S]$ has diameter at most two, that is a star. Now assume that $V-S$ contains two adjacent vertices x, y. Then S remains a DDS for $G-xy$ and so $\gamma_{\times 2}(G-xy) \leq \gamma_{\times 2}(G)$, a contradiction. Therefore $V-S$ is an independent set. Finally assume that a vertex $x \in V-S$ has degree at least three. By item (ii) $N(x) \subset S$, and hence removing any edge incident with x does not increase the double domination number, a contradiction.

Conversely, suppose that for every $\gamma_{\times 2}(G)$-set (i), (ii) and (iii) are satisfied. Assume that G is not $\gamma_{\times 2}$-critical and let uv be an edge of X_G for which $\gamma_{\times 2}(G-uv) = \gamma_{\times 2}(G)$. Let D be a $\gamma_{\times 2}(G-uv)$-set. Clearly D is a DDS of G. Since $\gamma_{\times 2}(G-uv) = \gamma_{\times 2}(G)$, D is also $\gamma_{\times 2}(G)$-set. If $\{u,v\} \cap D = \emptyset$, then D is a $\gamma_{\times 2}(G)$-set, where $V-D$ is not independent in G. Thus D contains at least one of u or v. Assume that $\{u,v\} \subset D$. Then u has a neighbor, say $x \neq v$, in D and likewise, v has a neighbor, say $y \neq u$, in D, with possibly $x = y$. Then D is a $\gamma_{\times 2}(G)$-set such that $\{u,v,x,y\}$ induces either a cycle C_3, C_4 or a path P_4 in $G[D]$, a contradiction. Thus, without loss of generality, assume that $u \in D$ and $v \notin D$. Then v is dominated at least twice by D in $G-uv$. But then condition (iii) does not hold for D in G since v would have at least three neighbors. Consequently D is a $\gamma_{\times 2}(G)$-set for which conditions (i), (ii) and (iii) are not all satisfied, a contradiction. It follows that G is $\gamma_{\times 2}$-critical. ■

As an immediate consequence of Theorem 5 we have the following two corollaries.

Corollary 6 *If G is a $\gamma_{\times 2}$-critical graph, then every $\gamma_{\times 2}(G)$-set contains all vertices of degree at least three.*

Corollary 7 *If G is a graph with minimum degree at least three, then G is not $\gamma_{\times 2}$-critical.*

The following observation will be useful for the proof of the next result.

Observation 8 *1) If $n \geq 3$, then $\gamma_{\times 2}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.*

2) If $n \geq 2$, then $\gamma_{\times 2}(P_n) = \begin{cases} 2n/3+1 & \text{if } n \equiv 0 \pmod 3 \\ 2\lceil n/3 \rceil & \text{otherwise.} \end{cases}$

Proposition 9 *For $k \geq 2$, a connected k-regular graph G is $\gamma_{\times 2}$-critical if and only if $G = C_n$, where $n \equiv 0, 1 \pmod 3$.*

Proof. Assume that G is a connected $\gamma_{\times 2}$-critical k-regular graph. By Corollary 7, $k \leq 2$. Hence $k = 2$ and so G is a cycle. Using Observation 8 it is a simple exercise to see that the order of G must satisfy $n \equiv 0, 1 \pmod 3$.

The converse is obvious. ■

3 $\gamma_{\times 2}$-critical trees

For ease of presentation, we next consider rooted trees. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. Also, a vertex of degree at least three in T is called a *branch vertex*, and we denote by $B(T)$ the set of such vertices. We state some observations.

Observation 10 *If T is the tree obtained from a tree T' by attaching a vertex to a support vertex, then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$.*

Observation 11 *Let T be a tree obtained from a nontrivial tree T' by adding k ($k \geq 1$) paths $P_3 = a_ib_ic_i$ attached by edges c_ix for every i, at a vertex x of T' belonging to some $\gamma_{\times 2}(T')$-set. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$.*

Proof. Let S' be a $\gamma_{\times 2}(T')$-set that contains x. Then $S' \cup \{ a_i, b_i : 1 \leq i \leq n\}$ is a DDS of T and so $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2k$. Now let D be a $\gamma_{\times 2}(T)$-set. By Observation 1, D contains a_i, b_i for every i. If D contains three vertices from $\{c_1, c_2, ..., c_k\}$, say c_1, c_2, c_3, then $x \notin D$ (else $D - \{c_1\}$ is a DDS of T) and so $\{x\} \cup D - \{c_1, c_2\}$ is a DDS smaller than D, a contradiction. Thus every $\gamma_{\times 2}(T)$-set contains at most two vertices from $\{c_1, c_2, ..., c_k\}$. Now, without loss of generality, we can assume that $D \cap \{c_1, c_2, ..., c_k\} = \emptyset$ (else we replace such vertices by x or/and a neighbor of x in T'). Hence $x \in D$ to double dominate every c_i, implying that $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2k$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. ■

Observation 12 *Let T be a tree obtained from a nontrivial tree T' by attaching a new vertex x to a leaf u whose support vertex v is adjacent to at least one pendent path P_3. Then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$.*

Proof. Let $x_iy_iz_i$ with $1 \leq i \leq k$ be k pendent paths P_3 attached at v by edges vx_i. Clearly if S' is a $\gamma_{\times 2}(T')$-set, then $S' \cup \{x\}$ is a DDS of T, and hence $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 1$. Now let S be any $\gamma_{\times 2}(T)$-set. Then S contains x, u, y_i, z_i for every i, and without loss of generality, $v \in S$ (for otherwise the minimality of S implies that $k = 1$ and $x_1 \in S$, and so x_1 can be replaced by v in S). Hence $S \cap V(T')$ is a DDS of T' and so $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 1$ and the equality follows. ■

In order to characterize $\gamma_{\times 2}$-critical trees, we define the family of all trees $\mathcal{F}$ that can be obtained from a sequence $T_1, T_2, ..., T_j$ ($j \geqslant 1$) of trees such that T_1 is a star $K_{1,r}$ with $r \geq 1$, $T = T_j$, and if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the operations listed below.

- **Operation $\mathcal{O}_1$:** Add a new vertex and join it to a support vertex of T_i.
- **Operation $\mathcal{O}_2$:** Add a path P_3 and join one of its leaf to a support vertex of T_i.

- **Operation $\mathcal{O}_3$:** Add k ($k \geq 1$) paths P_3 and join one leaf of each path P_3 to a same leaf of T_i.

- **Operation $\mathcal{O}_4$:** Add a new vertex u and join it to a leaf v of T_i with the condition that the support vertex x adjacent to v in T_i has degree $k+2$ and is adjacent to $k \geq 1$ pendent paths P_3 such that every neighbor of x except v has degree two and does not belong to any $\gamma_{\times 2}(T_i)$-set.

Note that vertices that are in no minimum double dominating set of a tree can be determined in polynomial time (see [2]).

We state a lemma.

Lemma 13 *If $T \in \mathcal{F}$, then T is a $\gamma_{\times 2}$-critical tree.*

Proof. Let $T \in \mathcal{F}$. Then T can be obtained from a sequence $T_1, T_2, ..., T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,r}$ with $r \geq 1$ and $T = T_j$, and if $j \geq 2$, then T_{i+1} is obtained from T_i by one of the four operations defined above. We proceed by induction on j. If $j = 1$, then T is a star $K_{1,r}$ and so T is $\gamma_{\times 2}$-critical.

Assume that $j \geq 2$ and that the result holds for all trees T that can be constructed by a sequence of length at most $j-1$. Let T be a tree of $\mathcal{F}$ constructed from a tree $T' = T_{j-1}$. Note that by the inductive hypothesis T' is $\gamma_{\times 2}$-critical. Consider the following four cases.

Case 1. T is obtained from T' by Operation $\mathcal{O}_1$.

Let v be a support vertex in T' and u be the new vertex attached to v. Clearly, $X_T = X_{T'}$ and by Observation 10, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. We shall show that T is $\gamma_{\times 2}$-critical. Since T' is $\gamma_{\times 2}$-critical, the removal of any edge $e \in X_{T'}$ produces two subtrees T'_1 and T'_2 with $\gamma_{\times 2}(T'-e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$. Also removing e in T produces two subtrees T_1 and T'_2, where $T_1 = T'_1 \cup \{u\}$. By Observation 10, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$. Thus, $\gamma_{\times 2}(T-e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$. Therefore T is $\gamma_{\times 2}$-critical.

Case 2. T is obtained from T' by Operation $\mathcal{O}_2$.

Let xyz be the added path P_3 attached by an edge xv at a support vertex v of T'. Then by Observation 11, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. To show that T is $\gamma_{\times 2}$-critical, consider any edge e of X_T, where $X_T = X_{T'} \cup \{vx, xy\}$.

- If $e \in X_{T'}$, then removing e from T' results in two subtrees T'_1 and T'_2, where $v \in T'_1$ and such that $\gamma_{\times 2}(T'-e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$. Also removing e from T provides two subtrees T_1 and T'_2, where $T_1 = T'_1 \cup \{x, y, z\}$. Now by Observation 11, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2$. Hence, $\gamma_{\times 2}(T-e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 2 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$.

- If $e = vx$, then removing e from T produces the subtrees T_1 and $T_2 = xyz$, where $\gamma_{\times 2}(T_2) = 3$ and $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. Hence, $\gamma_{\times 2}(T - vx) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.

- Finally, if $e = xy$, then $T - e = T_1 \cup T_2$ where $T_2 = yz$. Clearly $\gamma_{\times 2}(T_2) = 2$ and $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2 + 1 = \gamma_{\times 2}(T) - 1$. Hence, $\gamma_{\times 2}(T - xy) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.

We conclude that for every edge $e \in X_T$, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ and therefore T is $\gamma_{\times 2}$-critical.

Case 3. T is obtained from T' by Operation $\mathcal{O}_3$.

Let $x_i y_i z_i$ with $1 \le i \le k$ be k paths P_3 attached by edges vx_i at a leaf v of a tree T' and let T be the resulting tree. By Observation 11, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. Let u be the support vertex of v in T'. We prove that u is in every $\gamma_{\times 2}(T)$-set. Clearly if u is a support vertex in T, then u belongs to every $\gamma_{\times 2}(T)$-set. Hence we assume that u is not a support vertex in T and let D be a $\gamma_{\times 2}(T)$-set that does contain u. Note that $y_i, z_i \in D$ for every i. Also to double dominate v, D contains a vertex $w \ne v$ neighbor of u and such a w is adjacent to some vertex w' in D. Now if $v \in D$, then the minimality of D implies that $|D \cap \{x_1, x_2, ..., x_k\}| = 1$, say $x_1 \in D$ but then $D_1 = (D - \{x_1\}) \cup \{u\}$ is a $\gamma_{\times 2}(T)$-set and so $D'_1 = D_1 \cap T'$ is a $\gamma_{\times 2}(T')$-set. If $v \notin D$ and since $u \notin D$, then $k \ge 2$ and $x_i \in D$ for every i. If $k \ge 3$, then $\{v, u\} \cup D - \{x_1, x_2, ..., x_k\}$ would be a DDS smaller than D, a contradiction. Hence $k = 2$, but then $D_2 = (D - \{x_1, x_2\}) \cup \{u, v\}$ is a $\gamma_{\times 2}(T)$-set and so $D'_2 = D_2 \cap T'$ is a $\gamma_{\times 2}(T')$-set. In both cases, D'_1 and D'_2 are two $\gamma_{\times 2}(T')$-sets containing an induced path $P_4 = uvww'$, contradicting the fact that T' is $\gamma_{\times 2}$-critical. Consequently u belongs to every $\gamma_{\times 2}(T)$-set.

Next we show that T is a $\gamma_{\times 2}$-critical tree. Consider any edge $e \in X_T = X_{T'} \cup \{uv, vx_i, x_i y_i \mid 1 \le i \le k\}$. Note that all edges vx_i and $x_i y_i$ play the same role as vx_1 and $x_1 y_1$, respectively.

- $e \in X_{T'}$. Let T'_1 and T'_2 be the subtrees resulting from removing e in T'. Clearly since T' is $\gamma_{\times 2}$-critical $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$. Also removing e from T provides the subtrees T_1 and T'_2, where $T_1 = T'_1 \cup \{x_i, y_i, z_i \mid 1 \le i \le k\}$. By Observation 11, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2k$. It follows that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 2k + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 2k = \gamma_{\times 2}(T)$.

- $e = uv$. Then $T - e$ is a forest with two components T_1 and T_2, where T_2 is a tree obtained from a star $K_{1,k}$ such that each edge is subdivided twice. It can be seen that $D_2 = \{v, x_1, y_i, z_i \mid 1 \le i \le k\}$ is a $\gamma_{\times 2}(T_2)$-set and so $\gamma_{\times 2}(T_2) = 2k + 2$. Also since every $\gamma_{\times 2}(T_1)$-set can be extended to a $\gamma_{\times 2}(T)$-set by adding the set D_2, it follows that $\gamma_{\times 2}(T_1) \ge \gamma_{\times 2}(T) - 2k - 2$. Now suppose that D_1 is a $\gamma_{\times 2}(T_1)$-set of $\gamma_{\times 2}(T) - 2k - 2$. Then $D = D_1 \cup D_2$ would be a $\gamma_{\times 2}(T)$-set. If $u \notin D_1$, then D is a $\gamma_{\times 2}(T)$-set that does not contain u, a contradiction with our assumption that u belongs to every $\gamma_{\times 2}(T)$-set. Hence $u \in D_1$, and so $D - \{x_1\}$ is a DDS smaller than D, a contradiction. Consequently $\gamma_{\times 2}(T_1) > \gamma_{\times 2}(T) - 2k - 2$ and so $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T) - 2k - 2 + 2k + 2 = \gamma_{\times 2}(T)$.

- $e = vx_1$. Then $T - e$ is a forest with two components T_1 and $T_2 = x_1 y_1 z_1$, where $\gamma_{\times 2}(T_2) = 3$. If $k = 1$, then v is a leaf in T_1 and by Observation 11,

$\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. If $k \geq 2$, then we can simply use the fact that there is some $\gamma_{\times 2}(T_1)$-set containing v and so by Observation 11, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. In each case, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.

- $e = x_1 y_1$. Then $T - e$ is a forest with two components T_1 and $T_2 = y_1 z_1$, where $\gamma_{\times 2}(T_2) = 2$. Since any $\gamma_{\times 2}(T_1)$-set can be extended to a DDS of T by adding $\{y_1, z_1\}$, $\gamma_{\times 2}(T_1) \geq \gamma_{\times 2}(T) - 2$. Assume that $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$, and let S_1 be a $\gamma_{\times 2}(T_1)$-set. Since v is a support vertex and x_1 is a leaf in T_1, by Observation 1, $\{v, x_1\} \subset S_1$. So $S = S_1 \cup \{y_1, z_1\}$ is a $\gamma_{\times 2}(T)$-set. Now if $u \in S_1$, then $S - \{x_1\}$ would be a DDS smaller than S, which is impossible. Hence $u \notin S_1$, but then S is a $\gamma_{\times 2}(T)$-set that does not contain u, contradicting the fact that u belongs to every $\gamma_{\times 2}(T)$-set. Therefore $\gamma_{\times 2}(T_1) > \gamma_{\times 2}(T) - 2$. It follows that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T) - 2 + 2 = \gamma_{\times 2}(T)$.

In all cases above, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ and hence T is a $\gamma_{\times 2}$-critical tree.

Case 4. T is obtained from T' by Operation $\mathcal{O}_4$.

Let us use the same terminology as given in Operation $\mathcal{O}_4$ and let $v_i u_i z_i$ with $1 \leq i \leq k$ be the k pendent paths P_3 attached by edges xv_i at x. By Observation 12, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. Now let e be any edge of $X_T = X_{T'} \cup \{xv\}$.

- $e \in X_{T'} - \{xv_i \mid 1 \leq i \leq k\}$. Then $T' - e$ is a forest with two components T_1' and T_2', where $x \in V(T_1')$, and since T' is $\gamma_{\times 2}$-critical $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T')$. Also $T - e$ is a forest with two components T_1 and T_2', where $V(T_1) = V(T_1') \cup \{u\}$. If $k \geq 2$ or $e \neq v_1 u_1$, then by Observation 12, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 1$. If $k = 1$ and $e = v_1 u_1$, then x is a support vertex in T_1' and T_1, and clearly we have $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 1$. In any case it follows that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2') = \gamma_{\times 2}(T_1') + 1 + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$.

- $e = xv_i$ for some i. Then $T' - e = T_1' \cup T_2'$, where $x \in V(T_1')$ and T_2' is a path P_3. Also $T - e = T_1 \cup T_2'$, where $V(T_1) = V(T_1') \cup \{u\}$. If $k \geq 2$, then by Observation 12, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 1$. Suppose now that $k = 1$ and so $e = xv_1$. Clearly $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T_1') + 1$ since every $\gamma_{\times 2}(T_1')$-set can be extended to a DDS of T_1 by adding u. We will prove that equality holds in the previous inequality. Suppose to the contrary that $\gamma_{\times 2}(T_1) < \gamma_{\times 2}(T_1') + 1$. We first claim that x does not belong to any $\gamma_{\times 2}(T_1)$-set. Assume that D is a $\gamma_{\times 2}(T_1)$-set containing x. Then by Observation 1, $u, v \in D$, but then the set $D - \{v\}$ is a DDS of T_1' implying that $\gamma_{\times 2}(T_1') \leq |D| - 1 = \gamma_{\times 2}(T_1) - 1$, a contradiction. Hence x does not belong to any $\gamma_{\times 2}(T_1)$-set. Now since any $\gamma_{\times 2}(T_1)$-set can be extended to a DDS of T by adding v_1, u_1, z_1, so we have $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T_1) + 3$. Suppose that $\gamma_{\times 2}(T) < \gamma_{\times 2}(T_1) + 3$, that is $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T_1) + 2$ and let S be any $\gamma_{\times 2}(T)$-set. Without loss of generality, we may assume that x belongs to S (if $x \notin S$, then replace v_1 by x in S), and so $S' = S \cap V(T_1)$ is a DDS of T_1. Since $v_1 \notin S$, $u_1, z_1 \in S$, it follows that $\gamma_{\times 2}(T_1) \leq |S'| = \gamma_{\times 2}(T) - 2$ and hence $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 2$. Therefore S is a $\gamma_{\times 2}(T_1)$-set that contains x, contradicting the fact that x belongs to no $\gamma_{\times 2}(T_1)$-set. Consequently, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$. Now let S_1 be any $\gamma_{\times 2}(T_1)$-set. By Observation 1, $u, v \in S_1$ and since $x \notin S_1$, the second neighbor of v, say $y \in S_1$. On the other hand y must have a neighbor, say z, in S_1, implying that $S = S_1 \cup \{x, u_1, z_1\}$ is a $\gamma_{\times 2}(T)$-set. Now by Observation 12, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ and using the fact that $u, v \in S$, it follows that $S'' = S - \{u\}$ is a $\gamma_{\times 2}(T')$-set containing vertices v, x, y and z, but then such vertices induce a path P_4 in $G[S']$, a contradiction, since T' is $\gamma_{\times 2}$-critical. Consequently, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$ for $k = 1$ and it is also valid for $k \geq 2$ as already shown. Therefore we have

$$\begin{aligned}\gamma_{\times 2}(T - e) &= \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) \\ &= \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T).\end{aligned}$$

- $e = xv$. Then $T - e = T_1 \cup T_2$, where T_2 is a path $P_2 = uv$. Then $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) + 2$ and $\gamma_{\times 2}(T - e) \geq \gamma_{\times 2}(T)$ since every DDS of $T - e$ is a DDS of T. Now suppose that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + 2 = \gamma_{\times 2}(T)$ and let S_1 be a $\gamma_{\times 2}(T_1)$-set. Without loss of generality, we may assume that x belongs to S_1. Note that to double dominate x in T_1, x has a neighbor, say w, in S_1. Then $S = S_1 \cup \{v, u\}$ is a $\gamma_{\times 2}(T)$-set, and so $S' = S - \{u\}$ would be a $\gamma_{\times 2}(T')$-set containing w, a contradiction with the construction since no neighbor of x except v belongs to a $\gamma_{\times 2}(T')$-set. Hence $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$.

Consequently, in all cases above, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ and hence T is a $\gamma_{\times 2}$-critical tree. ■

Now we are ready to characterize $\gamma_{\times 2}$-critical trees.

Theorem 14 *A nontrivial tree T is $\gamma_{\times 2}$-critical if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then by Lemma 13, T is a $\gamma_{\times 2}$-critical tree. To prove the converse we proceed by induction on the order n of T. Since stars are $\gamma_{\times 2}$-critical, and belong to $\mathcal{F}$, and double stars are not $\gamma_{\times 2}$-critical (by Observation 3), we may assume that T has diameter at least four. The smallest $\gamma_{\times 2}$-critical tree of diameter four is the path P_5 and can be obtained from a star $K_{1,1}$ by using Operation $\mathcal{O}_3$. Hence the path P_5 belongs to $\mathcal{F}$. Assume now that $\text{diam}(T) = 4$ and $T \neq P_5$. Let x_1-x_2-x_3-x_4-x_5 be a longest path in T. Clearly x_1, x_5 are leaves and x_2, x_4 are their support vertices, respectively. If $\deg_T(x_3) = 2$, then at least one of x_2 and x_4 is a strong support. Hence $T \in \mathcal{F}$ and is obtained from P_5 by using Operation $\mathcal{O}_1$ on x_2 and/or x_4. If $\deg_T(x_3) \geq 3$, then by Observation 3, x_3 cannot be a support vertex and hence every neighbor of x_3 is a support vertex, but then $\gamma_{\times 2}(T - x_2x_3) \leq \gamma_{\times 2}(T)$, contradicting the fact that T is $\gamma_{\times 2}$-critical. Thus we assume that $\text{diam}(T) \geq 5$. The smallest $\gamma_{\times 2}$-critical tree of diameter five is the path P_6 which belongs to $\mathcal{F}$ since it can be obtained from a star $K_{1,2}$ by using Operation $\mathcal{O}_3$.

Let $n \geq 7$ and assume that every $\gamma_{\times 2}$-critical tree T' of order $2 \leq n' < n$ is in $\mathcal{F}$. Let T be a $\gamma_{\times 2}$-critical tree of order n and let S be any $\gamma_{\times 2}(T)$-set. If a support vertex, say y, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to y. By Observation 10, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. Since

$X_{T'} = X_T$, we obtain T' is $\gamma_{\times 2}$-critical. By the inductive hypothesis on T', we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$, because it is obtained from T' by using Operation $\mathcal{O}_1$. Thus we may assume for the next that every support vertex is adjacent to exactly one leaf.

We now root T at vertex x of maximum eccentricity diam(T). Let u be a vertex at distance diam$(T)-1$ from x on a longest path starting at x, and let r be a child of u on this path. Let w_1, v be the parents of u and w_1, respectively. Clearly r is a leaf, and u is a support vertex with $\deg_T(u) = 2$. By Observation 3, w_1 is different from a support vertex. Now if $\deg_T(w_1) \geq 3$, then T_{w_1} is a subdivided star and hence T is a tree obtained from a tree $T' = T - T_{w_1}$ and a subdivided star T_{w_1} of center w_1 by adding the edge $w_1 v$. In that case T is not $\gamma_{\times 2}$-critical by Observation 4, a contradiction. It follows that $\deg_T(w_1) = 2$. We consider the following two cases.

Case 1. v is a support vertex. Let $T' = T - \{r, u, w_1\}$. Then $X_{T'} = X_T - \{vw_1, w_1 u\}$ and by Observation 11, $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$. Now let e be any edge of $X_{T'}$. Then $T - e = T_1 \cup T_2$ and since T is $\gamma_{\times 2}$-critical, $\gamma_{\times 2}(T-e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$. Note that $e \in E(T')$. Without loss of generality, assume that $\{r, u, w_1\} \subset V(T_1)$. Then T_1 can be seen as a tree obtained from a nontrivial tree T_1' by adding a path $P_3 = ruw_1$ attached by the edge $w_1 v$ at v. Using Observation 11, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 2$. Note that $e \in E(T')$ and $T' - e = T_1' \cup T_2$. It follows that

$$\begin{aligned}\gamma_{\times 2}(T'-e) &= \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T_1) - 2 + \gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T-e) - 2 > \gamma_{\times 2}(T) - 2 = \gamma_{\times 2}(T').\end{aligned}$$

Consequently, removing any edge $e \in X_{T'}$ increases the double domination number of T' and therefore T' is $\gamma_{\times 2}$-critical. By induction on T', we have $T' \in \mathcal{F}$ and hence $T \in \mathcal{F}$ since it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. v is not a support vertex. Let $C(v) = \{w_1, w_2, ..., w_k\}$ be the set of children of v. Assume first that no vertex of $C(v)$ is a support vertex. Observe in this case that every w_i plays the same role as w_1 and so $\deg_T(w_i) = 2$ for every i. If $\deg_T(v) = 2$, then, without loss of generality, we may assume that $v \in S$ (else $w_1 \in S$ and w_1 can be replaced by v). If $\deg_T(v) \geq 3$, then by Corollary 6, $v \in S$ and S contains no vertex of $C(v)$, for otherwise $\{r, u, w_1, v\}$ induces a path P_4 in S, contradicting Theorem 5. Thus in all cases S contains no w_i. Let y be the parent of v and $c(w_i)$ the unique child of w_i for every i. Let $T' = T - \bigcup_{i=1}^{k} T_{w_i}$. Clearly v is a leaf in T' and $X_{T'} = X_T - \{vw_i, w_i c(w_i), yv \mid 1 \leq i \leq k\}$. By using the same argument to that used in Case 1, it can be seen that T' is $\gamma_{\times 2}$-critical. By the inductive hypothesis on T', we have $T' \in \mathcal{F}$. Hence $T \in \mathcal{F}$ and is is obtained from T' by Operation $\mathcal{O}_3$.

We now assume that a vertex of $C(v) - \{w_1\}$, say w, is a support vertex. Recall that since $|C(v)| \geq 2$, then $\deg_T(v) \geq 3$ and so by Corollary 6, $v \in S$. If $w'' \in C(v) - \{w_1, w\}$ is a second support vertex, then S remains a DDS for $T - vw''$, and so $\gamma_{\times 2}(T - vw'') \leq \gamma_{\times 2}(T)$, a contradiction. Thus w is the unique support vertex in $C(v) - \{w_1\}$. Let w' be the leaf neighbor of w. Note that $w', w, v \in S$. By item (i) of Theorem 5, every component of $G[S]$ is a star implying that $y \notin S$ and no vertex of

$C(v)$, except w, is in S. It follows that $\deg_T(y) = 2$ for otherwise $y \in S$ (by Corollary 6). Let us redefine the vertices of $C(v)$ by $C(v) = \{w, w_1, w_2, ..., w_k\}$, where $k \geq 1$. As mentioned above $\deg_T(w_i) = 2$ for every i. Now let T' be the tree obtained from T by removing the leaf w'. Then w is a leaf in T' and v is a vertex of T' adjacent to at least one pendent path P_3. Hence, by Observation 12, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. Next we shall show that y belongs to no $\gamma_{\times 2}(T')$-set. Suppose to the contrary that there is a $\gamma_{\times 2}(T')$-set S' that contains y. Since $v, w \in S'$, $S' \cup \{w'\}$ would be a $\gamma_{\times 2}(T)$-set for which $\{y, v, w, w'\}$ induces a path P_4, a contradiction. Moreover, since v, w are in every $\gamma_{\times 2}(T')$-set, it follows also that for every i, w_i belongs to no $\gamma_{\times 2}(T')$-set (else such a set minus w_i remains a DDS of T').

We prove next that T' is $\gamma_{\times 2}$-critical. Let e be any edge of $X_{T'} = X_T - \{vw\}$. Then $T - e = T_1 \cup T_2$ and since T is $\gamma_{\times 2}$-critical, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$. We examine the following three subcases.

Subcase 1. $e = vw_i$ for some i, where $1 \leq i \leq k$. Assume that $T_2 = T_{w_i}$ and T_1 is the subtree containing v. Since $e \in E(T')$, $T' - e = T_1' \cup T_2$. We claim that $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$. To prove this claim, we consider separately the cases $k \geq 2$ and $k = 1$.

If $k \geq 2$, then by Observation 12, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$. Suppose now that $k = 1$. Clearly if Y is a $\gamma_{\times 2}(T_1')$-set, then $w \in Y$ and hence $Y \cup \{w'\}$ is a DDS of T_1 implying that $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T_1') + 1$, that is $\gamma_{\times 2}(T_1') \geq \gamma_{\times 2}(T_1) - 1$. Assume that $\gamma_{\times 2}(T_1') > \gamma_{\times 2}(T_1) - 1$. If v belongs to a $\gamma_{\times 2}(T_1)$-set S, then since $w, w' \in S$, $S - \{w'\}$ would be a DDS of T_1' implying that $\gamma_{\times 2}(T_1') \leq \gamma_{\times 2}(T_1) - 1$, a contradiction. Hence v does not belong to any $\gamma_{\times 2}(T_1)$-set. Also $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T_1) + 3$ since every $\gamma_{\times 2}(T_1)$-set can be extended to a DDS of T by adding $\{w_1, u, r\}$. Suppose that $\gamma_{\times 2}(T) < \gamma_{\times 2}(T_1) + 3$ and let D be any $\gamma_{\times 2}(T)$-set. Then by Corollary 6 and Observation 1, $v, w, w' \in D$, and so $D_1 = D \cap V(T_1)$ is a DDS of T_1 implying that $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T) - 2$ and the equality holds, that is $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 2$. Therefore D_1 would be a $\gamma_{\times 2}(T_1)$-set containing v, contradicting the fact that v does not belong to any $\gamma_{\times 2}(T_1)$-set. Hence $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$. Now let X_1 be any $\gamma_{\times 2}(T_1)$-set. Since $v \notin X_1$, and $\deg_T(y) = \deg_{T_1}(y) = 2$, then $y \in X_1$. Hence $X_2 = X_1 \cup \{w_1, u, r\}$ is a $\gamma_{\times 2}(T)$-set and $(X_2 - w_1) \cup \{v\}$ is also a $\gamma_{\times 2}(T)$-set, but its induced subgraph contains a path $P_4 = w'wvy$, contradicting Theorem 5-(i). Consequently $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$, and the proof of our claim is complete.

Recall that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. Therefore for every edge $e = vw_i$ we have

$$\begin{aligned}\gamma_{\times 2}(T' - e) &= \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T'),\end{aligned}$$

and hence $\gamma_{\times 2}(T' - e) > \gamma_{\times 2}(T')$.

Subcase 2. $e = w_i c(w_i)$ for some i, where $1 \leq i \leq k$. Assume that $T_2 = T_{c(w_i)}$ is a path P_2 and T_1 is the subtree containing v. Since $e \in E(T')$, $T' - e = T_1' \cup T_2$. Note that v is a support vertex for w_i in T_1' and T_1. Clearly then, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$.

It follows that

$$\begin{aligned}\gamma_{\times 2}(T'-e) &= \gamma_{\times 2}(T'_1)+\gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T_1)-1+\gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T-e)-1 > \gamma_{\times 2}(T)-1=\gamma_{\times 2}(T'),\end{aligned}$$

and hence $\gamma_{\times 2}(T'-e) > \gamma_{\times 2}(T')$.

Subcase 3. $e \notin \{vw_i, w_ic(w_i) \mid 1 \leq i \leq k\}$. Without loss of generality, suppose that T_1 is the subtree containing v. Since $e \in E(T')$, $T'-e = T'_1 \cup T_2$. By Observation 12, $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$ and hence

$$\begin{aligned}\gamma_{\times 2}(T'-e) &= \gamma_{\times 2}(T'_1)+\gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T_1)-1+\gamma_{\times 2}(T_2)\\ &= \gamma_{\times 2}(T-e)-1 > \gamma_{\times 2}(T)-1=\gamma_{\times 2}(T'),\end{aligned}$$

and so $\gamma_{\times 2}(T'-e) > \gamma_{\times 2}(T')$.

Therefore for all cases, removing any edge from $X_{T'}$ increases the double domination number of T'. Hence T' is $\gamma_{\times 2}$-critical and by the inductive hypothesis, $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ since it is obtained from T' by Operation $\mathcal{O}_4$. This completes the proof of Theorem 14. ■

Corollary 15 *A path P_n is $\gamma_{\times 2}$-critical if and only if $n \equiv 0, 2 \pmod 3$.*

4 $\gamma_{\times 2}$-critical P_4-free graphs

Our aim in this section is to characterize $\gamma_{\times 2}$-critical P_4-free graphs. A graph is said to be P_4-free if it does not contain an induced path P_4. We define the family $\mathcal{H}_1$ of all graphs obtained from a star $K_{1,r}$ with $r \geq 1$ by adding k ($k \geq 0$) new vertices, each one joined to two adjacent vertices of the star $K_{1,r}$. Clearly if $G \in \mathcal{H}_1$, then G is P_4-free. We begin by the following lemma.

Lemma 16 *If G is a connected $\gamma_{\times 2}$-critical P_4-free graph, then $G[S]$ is connected for every $\gamma_{\times 2}(G)$-set S.*

Proof. Let S be any $\gamma_{\times 2}(G)$-set and assume that $G[S]$ is not connected. Using Theorem 5 and the fact that G is connected, there exists a vertex v in $V-S$ adjacent to two components of $G[S]$. Since $\deg_G(v) = 2$, v and two adjacent vertices from each component neighbor of v would induce a path P_4, a contradiction. ■

Theorem 17 *A P_4-free graph G is $\gamma_{\times 2}$-critical if and only if every component of G belongs to $\mathcal{H}_1$ or is isomorphic to a complete bipartite graph $K_{2,p}$ with $p \geq 2$.*

Proof. Let G be a $\gamma_{\times 2}$-critical P_4-free graph having m components $G_1, G_2, ..., G_m$. Let S be any $\gamma_{\times 2}(G)$-set and $S_i = S \cap V(G_i)$. It suffices to check the theorem for some component G_i. By Lemma 16, $G[S_i]$ is connected. Also by Theorem 5, $G[S_i]$

is a star $K_{1,r}$ with $r \geq 1$ and $V(G_i) - S_i$ is an independent set, where every vertex in $V(G_i) - S_i$ is adjacent to exactly two vertices of S_i. Let x be the center and $x_1, x_2, ..., x_r$ be the leaves of the star. Clearly if $|V(G_i) - S_i| = 0$, then G_i is a star and belongs to $\mathcal{H}_1$. Hence we assume that $|V(G_i) - S_i| \geq 1$.

If $r = 1$, then $G[S_i] = P_2$ and so G_i is obtained by attaching $t \geq 1$ vertices to both x and x_1. Hence $G_i \in \mathcal{H}_1$.

If $r = 2$, then $G[S_i]$ is a path $P_3 = x_2xx_1$. Assume there is a vertex u adjacent to x_1 and x_2. Then all vertices of $V(G_i) - S_i$ are adjacent to only x_1 and x_2, for otherwise if a vertex $v \in V(G_i) - S_i$ is adjacent, say to x and x_1, then $\{v, x, u, x_2\}$ induces a path P_4. Then clearly $G_i = K_{2,p}$ with $p \geq 2$. Now if no vertex of $V(G_i) - S_i$ is adjacent to both x_1 and x_2, then every vertex of $V(G_i) - S_i$ is adjacent to two adjacent vertices of the star, and hence $G_i \in \mathcal{H}_1$.

Finally, if $r \geq 3$, then no vertex of $V(G_i) - S_i$ has its two neighbors in $\{x_1, x_2, ..., x_r\}$ for otherwise G_i would contain an induced path P_4. Therefore $G_i \in \mathcal{H}_1$.

The converse is obvious and the proof is omitted. ■

5 $\gamma_{\times 2}$-critical P_5-free graphs

We shall next characterize all $\gamma_{\times 2}$-critical P_5-free graphs. A P_5-free graph is a graph that does not contain an induced path P_5. Note that a P_4-free graph is a P_5-free graph and hence the class of $\gamma_{\times 2}$-critical P_4-free graphs is included in the class of $\gamma_{\times 2}$-critical P_5-free graphs. Let $\mathcal{H}_2$ be the family of graphs obtained from a star $K_{1,r}$ $(r \geq 1)$ of center vertex x and leaves $x_1, x_2, ..., x_r$, by adding first j $(j \geq 0)$ new vertices, all joined to the same pair of non adjacent vertices of the star $K_{1,r}$, say x_l, $x_{l'}$, and then add k $(k \geq 0)$ new vertices, each one joined to two adjacent vertices of the star $K_{1,r}$, with the condition that if $j \geq 1$, then all these k vertices have the same neighborhood in the star. Clearly if $G \in \mathcal{H}_2$, then G is P_5-free. Also, if $G = K_{2,p}$ with $p \geq 2$ or $G \in \mathcal{H}_1$, then $G \in \mathcal{H}_2$.

Theorem 18 *A connected P_5-free graph G is $\gamma_{\times 2}$-critical if and only if $G \in \mathcal{H}_2$.*

Proof. Let G be a connected $\gamma_{\times 2}$-critical P_5-free graph and D be any $\gamma_{\times 2}(G)$-set. By Theorem 5, every vertex of $V(G) - D$ has degree two. Using the same argument to that used in the proof of Lemma 16, it can be seen that $G[D]$ is connected. Hence by Theorem 5, $G[D]$ is a star $K_{1,r}$, with $r \geq 1$, and $V(G) - D$ is an independent set. Let x be the center vertex of the star in $G[D]$ with leaves $x_1, x_2, ..., x_r$. If G is P_4-free, then by Theorem 17, $G \in \mathcal{H}_1$ or $G = K_{2,p}$ with $p \geq 2$, and so G belongs to the family $\mathcal{H}_2$. Thus we assume that G contains at least an induced P_4 and so $r \geq 2$. If $|V(G) - D| = 1$, then $r \geq 3$ (else G would be P_4-free) and hence $G \in \mathcal{H}_2$. Thus we suppose that $|V(G) - D| \geq 2$. If $r = 2$, then it can be easily seen that $G \in \mathcal{H}_2$. Hence we assume that $r \geq 3$. Let u, v be two vertices of $V(G) - D$, each adjacent to two leaves of $G[D]$. Let $N(u) \cap D = \{x_i, x_{i'}\}$ and $N(v) \cap D = \{x_s, x_{s'}\}$. If $N(u) \cap D \neq N(v) \cap D$, then vertices x, u, v, x_i together with x_s or $x_{s'}$ induce a path P_5. Thus all vertices of $V(G) - D$ not adjacent to x must be adjacent to the same pair of leaves of $G[D]$. Assume now that there is a vertex $z \in V(G) - D$ adjacent to two leaves x_l, $x_{l'}$ of $G[D]$, and suppose that there are two vertices y_1 and y_2 in $V(G) - D$ such that $N_G(y_1) = \{x, x_l\}$ and $N_G(y_2) = \{x, x_{l'}\}$. Then $\{y_1, x_l, z, x_{l'}, y_2\}$ induces a path P_5. Hence, if a vertex in $V(G) - D$ is adjacent to two leaves x_l, $x_{l'}$ of $G[D]$, then all remaining vertices of $V(G) - D$ are either x_l, $x_{l'}$ or to x, x_l. Therefore $G \in \mathcal{H}_2$.

The converse is easy to check and the proof is omitted. ■

Corollary 19 *A P_5-free graph G is $\gamma_{\times 2}$-critical if and only if every component of G belongs to the family $\mathcal{H}_2$.*

6 $\gamma_{\times 2}$-critical graphs for $\gamma_{\times 2} = 2, 3$ or 4

In this section, we investigate $\gamma_{\times 2}$-critical graphs G with small double domination number, more precisely when $\gamma_{\times 2}(G) = 2, 3$ or 4. For short, will call such graphs t-critical graphs, where $t \in \{2, 3, 4\}$. Using Theorem 5 we have:

Theorem 20 *A connected graph G is 2-critical if and only if G is a path P_2 or G is formed by k $(k \geq 1)$ triangles sharing the same edge.*

Let $\mathcal{H}_3$ be the family of graphs obtained from a path P_3 by adding i $(i \geq 0)$ new vertices and joining each one to two vertices of the path P_3.

Theorem 21 *A connected graph G is 3-critical if and only if $G \in \mathcal{H}_3$.*

Proof. Let G be a connected 3-critical graph and S any $\gamma_{\times 2}(G)$-set. By Theorem 5, $G[S] = P_3$, $V - S$ is independent set, and every vertex (if any) of $V - S$ is adjacent to exactly two arbitrary vertices of the path P_3. Hence $G \in \mathcal{H}_3$.

The proof for the converse is straightforward and therefore is omitted. ■

Let $\mathcal{H}_4$ be the family of all graphs obtained from two paths P_2 by adding i $(i \geq 0)$ new vertices, each one joined to two vertices of $P_2 \cup P_2$. Let $\mathcal{H}_5$ be the family of all graphs obtained from a star $K_{1,3}$ by adding j $(j \geq 0)$ new vertices, each one joined to two vertices of the star $K_{1,3}$. Let $\mathcal{H}_6$ be the family of graphs obtained from a star $K_{1,3}$ of center x and leaves x_1, x_2, x_3 by adding a vertex u joining to x_1, x_2, r $(r \geq 0)$ new vertices joined to x_1, x_3, s $(s \geq 0)$ new vertices joined to x, x_3, and t $(t \geq 0)$ new vertices joined to x and x_1. Observe that $\mathcal{H}_6$ is a subfamily of $\mathcal{H}_5$, but for every graph $G \in \mathcal{H}_6$ the set $\{x_3, x, x_1, u\}$ is a $\gamma_{\times 2}(G)$-set whose vertices induces a path P_4. By Theorem 5, no graph in $\mathcal{H}_6$ is 4-critical.

Theorem 22 *A connected graph G is 4-critical if and only if $G \in \mathcal{H}_4$ or $G \in \mathcal{H}_5 - \mathcal{H}_6$.*

Proof. Let G be a connected 4-critical graph and S any $\gamma_{\times 2}(G)$-set. By Theorem 5, every component of $G[S]$ is a star. Hence $G[S] = K_{1,3}$ if $G[S]$ is connected and $G[S] = P_2 \cup P_2$ for otherwise. Clearly if $G[S] = P_2 \cup P_2$, then by conditions (ii) and (iii) of Theorem 5, $G \in \mathcal{H}_4$. Now if $G[S] = K_{1,3}$, then again by conditions (ii) and (iii) of Theorem 5, $G \in \mathcal{H}_5$. But the graphs of $\mathcal{H}_6$ are not 4-critical. It follows that $G \in \mathcal{H}_5 - \mathcal{H}_6$.

The converse can be easily checked. ■

References

[1] M. Blidia, M. Chellali and T.W. Haynes, Characterizations of trees with equal paired and double domination numbers, *Discrete Math.* 306 (2006), 1840–1845.

[2] M. Blidia, M. Chellali and S. Khelifi, Vertices belonging to all or to no minimum double domination sets of trees, *AKCE Int. J. Graphs Combin.* 2 No. 1 (2005), 1–9.

[3] M. Chellali and T. W. Haynes, On paired and double domination in graphs, *Utilitas Math.* 67 (2005), 161–171.

[4] F. Harary and T. W. Haynes, Double domination in graphs, *Ars Combin.* 55 (2000), 201–213.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[7] D. P. Sumner and P. Blitch, Domination critical graphs. *J. Combin. Theory Ser. B* 34 (1983), 65–76.

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