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Even factor of a graph with a bounded number of components

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Abstract

Let G be a connected simple graph of order n, k a positive integer and n sufficiently large relative to k. An even factor of G is a spanning subgraph of G in which every vertex has even positive degree. In this paper, we prove that if $\delta(G) \geq \lfloor n/k \rfloor - 1$, then the (collapsible) reduction G' of G satisfies $|V(G')| \leq k$, and the preimage of each vertex of G' is nontrivial. We use this result to prove that if $\delta(G) \geq \lfloor n/k \rfloor - 1$, then G has an even factor with at most k components. Moreover, if G is 2-edge-connected and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \lfloor n/(3k+1) \rfloor - 1$, then G has an even factor with at most k components, which extends a theorem of Catlin [*J. Graph Theory* 12 (1988), 29–44]. Finally, we show that every 2-edge-connected reduced graph of order $n \leq 3k + 1 \leq 10$ has a spanning even subgraph with at most k components. All results are best possible.

1 Introduction

We use [4] for terminology and notation not defined here and consider only finite simple graphs. A graph is called *trivial* if it has only one vertex. Let $O(G)$ denote the set of all odd degree vertices of G. A graph G is called *even* if $O(G) = \emptyset$. A graph G is *eulerian* if G is connected and even; and G is *supereulerian* if G has a spanning eulerian subgraph. Regard K_1 as supereulerian.

A spanning subgraph of a graph is called a *factor*. An *even factor* of G is a spanning subgraph of G in which every vertex has even positive degree. A 2-*factor* of G is a spanning subgraph in which every vertex has degree 2. A graph G is called

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k-supereulerian if G has a spanning even subgraph with at most k components. Obviously, if G has an even factor with at most k components, then G is k-supereulerian, whereas the converse is not true in general, see [9].

There exist many minimum degree conditions guaranteeing the existence of certain factors of a graph, such as hamiltonian cycles and spanning eulerian subgraphs; see, e.g., [1], [2], [3]. The following are two prior famous results.

Theorem 1. (Dirac, [5]) *Every simple graph of order $n \geq 3$ with $\delta(G) \geq n/2$ is hamiltonian.*

Theorem 2. (Catlin, [1]) *Every 2-edge-connected simple graph of order $n > 16$ with $\delta(G) \geq n/4 - 1$ is supereulerian.*

In this paper, we obtain several minimum degree conditions for a graph to have an even factor with a bounded number of components.

Let H be a connected subgraph of G. The *contraction* G/H is the graph obtained from G by replacing H by a new vertex v_H, such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H (H is called the *preimage* of v_H).

A graph H is called *collapsible* if for every even set $X \subseteq V(H)$, there is a spanning connected subgraph H_X of H such that $O(H_X) = X$. In [1], Catlin showed that any graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_q$ such that $\bigcup_{i=1}^{q} V(H_i) = V(G)$. The *reduction* of G, denoted by G', is the graph obtained from G by contracting each H_i ($1 \leq i \leq q$) to a single vertex. A graph G is *reduced* if $G = G'$ (see [1]). A vertex v_H in G' is *nontrivial* if v_H is the contraction image of a nontrivial connected subgraph H of G. For more results, see the survey paper of Catlin [2] and its update [3].

In Section 2, we will use a refinement of Catlin's reduction method given by Veldman [10] to obtain the following result.

Theorem 3. *Let p be a positive integer and G a connected simple graph of order n such that*
$$\delta(G) \geq \lfloor n/p \rfloor - 1. \tag{1.1}$$
If n is sufficiently large relative to p, then the reduction G' of G satisfies $|V(G')| \leq p$, and each vertex of G' is nontrivial.

Let $k \geq 2$ be an integer, G a connected simple graph and $V(G) = \{v_1, \ldots, v_n\}$. The *$k$-enlarging* of G is the graph obtained from G and n pairwise-disjoint complete graphs $K_{k-1}^1, \ldots, K_{k-1}^n$ of order $k-1$ by joining each $u \in V(K_{k-1}^i)$ to $v_i \in V(G)$ ($1 \leq i \leq n$).

Theorem 3 is best possible: for an integer $k \geq 3$, obtain the graph $G_{p,k}$ from an arbitrary large connected reduced graph G_p of order p by k-enlarging G_p. Then $|V(G'_{p,k})| = |V(G_p)| = p$ while $\delta(G_{p,k}) = k - 1 = \lfloor \frac{1}{p}|V(G_{p,k})| \rfloor - 1$.

In Sections 2 and 4, respectively, we use Theorem 3 to prove Theorems 4 and 9.

Theorem 4. *Let G be a connected simple graph of order n and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{k} \rfloor - 1$. If n is sufficiently large relative to k, then G has an even factor with at most k components.*

This result is best possible: for a integer $p \geq 3$, obtain the graph $T_{k,p}$ from an arbitrary tree T_k of order k by p-enlarging T_k. The reduction of $T_{k,p}$ is T_k. Then $T_{k,p}$ has an even factor with exactly k components, while $\delta(T_{k,p}) = p-1 = \lfloor \frac{1}{k}|V(T_{k,p})| \rfloor - 1$.

Let $G = (V(G), E(G))$ be a graph. The *line graph* $L(G)$ of G is the graph on $E(G)$ in which $x, y \in E(G)$ are adjacent as vertices if and only if they are adjacent as edges in G. Let G be a simple graph with $\delta(G) \geq 3$ and let S be a set of mutually edge-disjoint connected even nontrivial subgraphs and stars. If each star has at least three edges and every edge in $E(G) \setminus \cup_{L \in S} E(L)$ is incident to an even subgraph in S, then S is called a *system that dominates* G.

Theorem 5. (Gould and Hynds, [7]) *Let G be a simple graph. Then $L(G)$ has a 2-factor with c components if and only if there is a system that dominates G with c elements.*

Theorem 5 shows a closed relationship between a system that dominates G with c elements and a 2-factor of $L(G)$ with the same number of components. From Theorem 5, one can easily obtain the following observation (see [11]).

Observation 6. If G has an even factor with at most k components, then $L(G)$ has a 2-factor with at most k components.

By Observation 6, Theorem 4 has the following consequence, which is best possible in some sense; see Section 5.

Corollary 7. *Let G be a connected simple graph of order n and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{k} \rfloor - 1$. If n is sufficiently large relative to k, then $L(G)$ has a 2-factor with at most k components.*

Corollary 7 uses minimum degree condition to deal with the number of components of a 2-factor in $L(G)$, and so does the following conjecture, which is true for $\delta = 3$ (see [8]).

Conjecture 8. (Fujisawa, Xiong, Yoshimoto and Zhang, [6]) *Let G be a simple graph with $\delta(G) \geq 3$. Then $L(G)$ has a 2-factor with at most $\dfrac{(2\delta(G)-3)n}{2(\delta(G)^2 - \delta(G) - 1)}$ components.*

For 2-edge-connected simple graphs, we improve Theorem 4 in the case when $k \leq 3$. The example in Section 4 shows that Theorem 9 is also best possible.

Theorem 9. *Let G be a 2-edge-connected simple graph of order n and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \lfloor \frac{n}{3k+1} \rfloor - 1$. If n is sufficiently large relative to k, then G has an even factor with at most k components.*

By Observation 6, Theorem 9 has the following consequence, which is best possible in some sense; see Section 5.

Corollary 10. *Let G be a 2-edge-connected simple graph of order n and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \lfloor \frac{n}{3k+1} \rfloor - 1$. If n is sufficiently large relative to k, then $L(G)$ has a 2-factor with at most k components.*

In Section 2, we will prove Theorems 3 and 4. In Section 3, we present some auxiliary results, which are then applied to the proof of Theorem 9 in Section 4.

2 Proofs of Theorems 3 and 4

Before presenting the proofs of the main results, we start with the following well-known result and a refinement of Catlin's reduction method given by Veldman [10].

Theorem 11. (Catlin, [1]) *If G is reduced, then G is simple and triangle-free.*

Let G be a simple graph and define $D(G) = \{v \in V(G) \mid d(v) \in \{1, 2\}\}$. For an independent subset X of $D(G)$, define $I_X(G)$ as the graph obtained from G by deleting the vertices in X of degree 1 and replacing by an edge each path of length 2 whose internal vertex is a vertex in X of degree 2. Note that $I_X(G)$ may not be simple. We call G *X-collapsible* if $I_X(G)$ is collapsible. A subgraph H of G is an *X-subgraph* of G if $d_H(x) = d_G(x)$ for all $x \in X \cap V(H)$. An X-subgraph H of G is called X-collapsible if H is $(X \cap V(H))$-collapsible. Let $R(X)$ be the set of vertices in X that are not contained in an X-collapsible X-subgraph of G. Since $I_X(G)$ has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $F_1, \ldots, F_k$ such that $\cup_{i=1}^{k} V(F_i) = V(I_X(G))$, the graph G has a unique collection of pairwise vertex-disjoint maximal X-collapsible X-subgraphs $H_1, \ldots, H_k$ such that $(\cup_{i=1}^{k} V(H_i)) \cup R(X) = V(G)$. The *$X$-reduction* of G is the graph obtained from G by contracting $H_1, \ldots, H_k$. The graph G is *X-reduced* if there exists a graph G_1 and an independent subset X_1 of $D(G_1)$ such that $X = R(X_1)$ and G is the X_1-reduction of G_1. An X-subgraph H of G is called X-reducted if H is $(X \cap V(H))$-reduced.

Note that if $X = \emptyset$, the refinement method is just Catlin's reduction method.

For a graph G with $|E(G)| > 0$, let $\overline{\sigma}_2(G) = \min\{d(u) + d(v) | uv \in E(G)\}$. Veldman [10] used the above refinement method to prove the following result.

Theorem 12. (Veldman, [10]) *Let G be a connected simple graph of order n and $p \geq 2$ an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{2.1}$$

If n is sufficiently large relative to p, then

$$|V(G')| \leq \max\{p, \frac{3}{2}p - 4\}, \tag{2.2}$$

where G' is the $D(G)$-reduction of G. Moreover, for $p \leq 7$, (2.2) *holds with equality only if* (2.1) *holds with equality.*

Now we prove Theorem 3.

Proof of Theorem 3. Let G be a connected simple graph of order n and p a positive integer such that (1.1) holds. If $p = 1$, then G is a complete graph of order n. Since n is sufficiently large relative to p, Theorem 3 holds.

Now let $p \geq 2$. By (1.1), $D(G) = \emptyset$ when $n \geq 4p$. So the $D(G)$-reduction of G equals the reduction G' of G. By (1.1), $\overline{\sigma}_2(G) \geq 2\delta(G) \geq 2(\lfloor n/p \rfloor - 1)$. Hence by Theorem 12, $|V(G')| \leq \max\{p\,, \frac{3}{2}p - 4\}$.

Suppose $v \in V(G')$ is a trivial vertex, then $v \in V(G)$. Since G' is reduced, for each $u \in V(G') \setminus \{v\}$, there is at most one edge between v and u in G', which implies that there is also at most one edge between v and the preimage of u in G. Then $d_G(v) = d_{G'}(v) \leq \max\{p\,, \frac{3}{2}p - 4\} - 1$, which yields a contradiction to (1.1) when $n > \max\{p^2 + p\,, \frac{3}{2}p^2 - 3p\}$. So each vertex of G' is nontrivial.

It remains to prove $|V(G')| \leq p$. Suppose, to the contrary, that $|V(G')| \geq p + 1$. Then there exist a vertex $v_i \in V(G')$ whose preimage H_i satisfies $|V(H_i)| \leq n/(p+1)$. Since G' is reduced, for $v_j \in V(G')$ and $v_j \neq v_i$, there is at most one edge between v_i and v_j in G', i.e., there is at most one edge (in G) between the preimage of v_i and the preimage of v_j. Note that $|V(G')| \leq \max\{p\,, \frac{3}{2}p - 4\}$ and $|V(G')| \geq p + 1$. Hence, we have $|V(G')| \leq \frac{3}{2}p - 4$. Then for each vertex v in H_i,

$$d_G(v) = d_{H_i}(v) + d_{G'}(v_i) \leq (\frac{n}{p+1} - 1) + ((\frac{3}{2}p - 4) - 1),$$

which also yields a contradiction to (1.1) when $n > p(p+1)(\frac{3}{2}p - 4)$.

Hence $|V(G')| \leq p$. This completes the proof of Theorem 3. □

Now we prove Theorem 4.

Proof of Theorem 4. By Theorem 3, $|V(G')| \leq k$ and each vertex of G' is nontrivial. Then the preimage of each vertex in G' is supereulerain and nontrivial. Hence G has an even factor with at most k components. □

3 Auxiliary results on 2-edge-connected reduced graphs

For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S. The path and cycle with n vertices are denoted by P_n and C_n, respectively. Let G be a 2-edge-connected reduced graph of order n, $C = v_0v_1 \ldots v_{c-1}v_0$ a longest cycle of G with length $c = c(G)$, and let $G_1 = G[V(G) \backslash V(C)]$.

The following lemma is in fact Claim 1 of the proof of Theorem 15 in [12].

Lemma 13. *If $4 \le c \le 7$, then there is no pair of paths P_1 with ends u_{P_1} and u'_{P_1} and P_2 with ends u_{P_2} and u'_{P_2} in G_1 such that u_{P_1} and u'_{P_1} are adjacent to two nonadjacent vertices v_i, v_j of C, respectively, and u_{P_2} and u'_{P_2} are adjacent to different components of $C - \{v_i, v_j\}$, respectively.*

Lemma 14. *Each of the following holds:*

(a) *If $4 \le c \le 5$, then G_1 has no nontrivial path whose ends are adjacent to two distinct vertices in C respectively. In particular, G_1 has no edge as a component;*

(b) *Each isolated vertex in G_1 is adjacent to a pair of nonadjacent vertices in C.*

Proof. By Theorem 11 and by the fact that G is a 2-edge-connected reduced graph and has no longer cycle than C, Lemma 14 holds. $\square$

A graph G is called a *circuit* if it is connected and even. The following is the main result of this section, which will be used in the proof of Theorem 9 in Section 4.

Lemma 15. *Let G be a 2-edge-connected reduced graph of order n and $k \in \{1, 2, 3\}$ such that $n \le 3k + 1$. Then G is k-supereulerian.*

Proof. It suffices to prove the cases when $n = 3k+1$: if there exist a graph G, which is not k-supereulerian but $n < 3k+1$, then we can obtain a non-k-supereulerian graph G^* with $n(G^*) = 3k + 1$ by replacing an edge of H by a $P_{3k+3-n(G)}$, where H is a nontrivial eulerian component of G.

Case 1. $k = 3$. It suffices to prove the case when $n = 10$.

Suppose, to the contrary, that G is not 3-supereulerian. Hence by the fact that any circuit is eulerian and $n = 10$,

$$\text{each circuit of } G \text{ contains at most 7 vertices.} \tag{3.1}$$

Since G is reduced, by Theorem 11, G is triangle-free. Hence by (3.1), $4 \le c \le 7$.

Claim 1. *If $5 \le c \le 7$, then each of the following holds:*

(a) *G_1 is a forest;*

(b) *G_1 contains no path P_i with endvertices u, v satisfying $i \ge 3$ and $d_{G_1}(u) = d_{G_1}(v) = 1$.*

Proof of Claim 1. (a) Otherwise by the fact that G is triangle-free and $n = 10$, G is 3-supereulerian, a contradiction.

(b) Otherwise by the fact that G is 2-edge-connected, either both u and v are adjacent to the same vertex in C, which will yield a circuit with at least 8 vertices, contrary to (3.1); or u, v are adjacent to the different vertices in C, then G contains a longer cycle than C, a contradiction. $\square$

Let $V(G_1) = \{u_1, u_2, \ldots, u_{10-c}\}$. For $u \in V(G_1)$ and $e \in E(G_1)$, let $N_C(u) = N_G(u) \cap V(C)$ and $N_C(e) = \{v : v \in V(C)$ and v is adjacent to an endvertex of $e\}$, respectively. Since G is 2-edge-connected, if $d_{G_1}(u) = 1$, then $|N_C(u)| \geq 1$; if $d_{G_1}(u) = 0$, then $|N_C(u)| \geq 2$. Now we distinguish the following cases according to c.

Subcase 1.1. $c = 4$. Then by the assumption and by Lemmas 13 and 14 (a), either G is isomorphic to the graph H_1 depicted in Fig. 1 or G has the circuit H_2 depicted in Fig. 1.

First suppose $G \cong H_1$. Then G is a circuit with 10 vertices, contrary to (3.1).

Next suppose $H_2 \subset G$. Let $G_2 = G[V(G)\backslash V(H_2)]$. Then $|V(G_2)| = 3$. Since G is triangle-free, G_2 is composed of 3 isolated vertices, or one edge and one isolated vertex, or $G_2 \cong P_3$. In the first case, by $c(G) = 4$, Lemma 13 and since G is 2-edge-connected, at least 2 isolated vertices of G_2 have the same pair of neighbors in H_2, which will yield a circuit with 9 vertices, contrary to (3.1). In the second case, since G is 2-edge-connected and triangle-free, we can find a longer cycle than C, a contradiction. In the last case, by $c(G) = 4$ and since G is 2-edge-connected, the two vertices of degree 1 in G_2 are adjacent to the same vertex in H_2. Hence G is supereulerian, a contradiction.

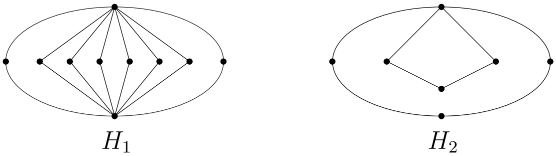


Fig. 1.

Subcase 1.2. $c = 5$. Then by Claim 1 (a), (b) and Lemma 14 (a), G_1 is composed of 5 isolated vertices. Hence by Lemmas 13 and 14 (b), s isolated vertices of G_1 have neighbors v_i and v_{i+2}, and t isolated vertices of G_1 have neighbors v_i and v_{i+3}, where subscript takes modules 5. Since $s + t = 5$, we can assume s is odd. Then we will obtain a circuit with 9 vertices by deleting one of the s vertices in G, contrary to (3.1).

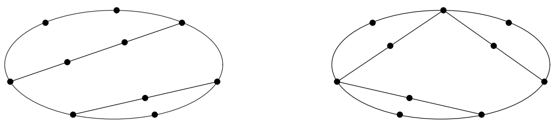
Subcase 1.3. $c = 6$. Then by Claim 1 (a) and (b), G_1 is composed of 2 edges e_1, e_2, or one edge e and 2 isolated vertices u_1, u_2, or 4 isolated vertices.

In the first case, by $c(G) = 6$ and Lemma 13, we have $N_C(e_1) = N_C(e_2)$, which will yield a circuit with 10 vertices, contrary to (3.1). In the second case, by $c(G) = 6$, we may assume that $N_C(e) = \{v_0, v_3\}$. Then by Lemma 13 and since G is triangle-free, we have $N_C(u_1) \cap \{v_0, v_3\} \neq \emptyset$, which will yield a circuit with 9 vertices, contrary to (3.1). In the last case, by Lemmas 13 and 14 (b), at least 2 isolated vertices have the same pair of neighbors in C, which will yield a circuit with 8 vertices, contrary to (3.1).

Subcase 1.4. $c = 7$. Then by Claim 1 (a) and (b), G_1 is composed of either one edge e and one isolated vertex u, or 3 isolated vertices u_1, u_2, u_3.

In the first case, since $c(G) = 7$, we may assume that $N_C(e) = \{v_0, v_3\}$. First suppose $N_C(u) \cap \{v_0, v_3\} \neq \emptyset$. Then G contains a circuit with at least 9 vertices, contrary to (3.1). Next suppose $N_C(u) \cap \{v_0, v_3\} = \emptyset$. Then by Lemma 13, G is isomorphic to the graph depicted in Fig. 2 (a), which is 2-supereulerian, a contradiction.

In the second case, by (3.1), u_1, u_2, u_3 have pairwise different neighbors in C. Hence by Lemmas 13 and 14 (b), G is isomorphic to the graph depicted in Fig. 2 (b), which is supereulerian, a contradiction.



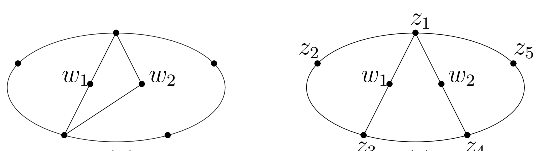
(a) (b)

Fig. 2.

Case 2. $k = 1, 2$.

For $k = 1$, note that G is a 2-edge-connected reduced graph with $n \leq 4$. G is trivial or isomorphic to C_4. Lemma 15 holds.

For $k = 2$, it suffices to prove the case when $n = 7$. Suppose, to the contrary, that G is not 2-supereulerian. Then $c \leq 5$. Since $n = 7 < 10$, G is 3-supereulerian by the proof of Case 1. Thus G has a spanning even subgraph with exactly 3 components. Note that G is reduced and $n = 7$. The 3 components are two isolated vertices w_1, w_2 and a $C_5 = z_1 z_2 z_3 z_4 z_5 z_1$. By Lemma 14 (a), $w_1 w_2 \notin E(G)$. Hence by Lemmas 13 and 14 (b), either w_1, w_2 have the same pair of neighbors in C_5, depicted in Fig. 3 (a), or w_1, w_2 have a common neighbor z_1(say) in C_5, and hence the other neighbors in C_5 are z_3 and z_4, respectively, depicted in Fig. 3 (b). In each case, G is supereulerian, a contradiction.



(a) (b)

Fig. 3.

This completes the proof of Lemma 15. □

Let s_1, s_2, s_3 be positive integers, u, v the vertices of $K_{2,3}$ with degree 3, and $K_{2,3}(s_1, s_2, s_3)$ a graph obtained from $K_{2,3}$ by replacing each $u - v$ path by a path of length $s_i + 1$, depicted in Fig. 4. Obviously, $K_{2,3}(1, 1, 1) = K_{2,3}$, and $K_{2,3}(k, k, k)$ is $(k + 1)$-supereulerian, but not k-supereulerian. Lemma 15 just deals with the case when $n \leq 10$. For large n, we propose the following conjecture.

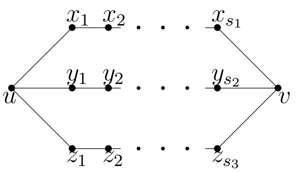


Fig. 4: $K_{2,3}(s_1, s_2, s_3)$

Conjecture 16. *Let G be a 2-edge-connected reduced graph of order n and k a positive integer such that $n \leq 3k+2$. Then G is either k-supereulerian or the graph $K_{2,3}(k,k,k)$.*

4 Proof of Theorem 9

We now prove Theorem 9.

Proof of Theorem 9. By Theorem 3, $|V(G')| \leq 3k+1$, and each vertex of G' is nontrivial. If $k \in \{1,2,3\}$, by Lemma 15, then G' has a spanning even subgraph with at most k components $F_1, F_2, \ldots, F_l$, where $l \leq k$. For each F_i, let $F_i^* = G[V(H_1) \cup V(H_2) \cup \cdots \cup V(H_{n(F_i)})]$, where H_j is the preimage of $v_j \in V(F_i)$. Since each vertex of G' is nontrivial, then each F_i^* is supereulerian and nontrivial. By the definition of collapsible and contraction, $\bigcup_{1 \leq i \leq l} V(F_i^*) = V(G)$ and $V(F_i^*) \cap V(F_j^*) = \emptyset$ for $i \neq j$. Hence G has an even factor with $l\,(\leq k)$ components. $\square$

Theorem 9 is best possible in the following two senses.

First, there is a simple graph G satisfying $\delta(G) = \lfloor \frac{n}{3k+1} \rfloor - 1$ which has an even factor with exactly k components: for an integer $p \geq 3$, obtain the graph G by p-enlarging $K_{2,3}(k,k,k-1)$ ($k \in \{1,2,3\}, d(u) = d(v) = 3$). Then $G' = K_{2,3}(k,k,k-1)$, $n = (3k+1) \cdot p$, $\delta(G) = n/(3k+1) - 1$, and G has an even factor with exactly k components (the $k-1$ spanning cycles of the $k-1$ complete graph of order p, whose contraction images are the $k-1$ inner vertices in the shortest $u-v$ path in G', and a spanning eulerian subgraph of the subgraph induced by the rest vertices in G).

Second, we cannot improve the minimum degree condition to get a better result. For an integer $p \geq 3$, obtain the graph G by p-enlarging $K_{2,3}(k,k,k)$ ($k \in \{1,2,3\}$). Then $G' = K_{2,3}(k,k,k)$, $n = (3k+2) \cdot p$, and hence $\delta(G) = n/(3k+2) - 1 < \lfloor n/(3k+1) \rfloor - 1$, but each even factor of G has at least $k+1$ components. (Otherwise if G has an even factor with $l(\leq k)$ components, then G is l-supereulerian. By a result of [9], we know that G is k-supereulerian if and only if G' is k-supereulerian. Hence $G' = K_{2,3}(k,k,k)$ is l-supereulerian, which is a contradiction).

If Conjecture 16 holds, we can extend Theorem 9 for arbitrarily large k.

By the proof of Theorem 9, we can get the following corollary immediately, which extends Theorem 2 and improves a theorem in [9].

Corollary 17. *Let G be a 2-edge-connected simple graph of order n and $k \in \{1, 2, 3\}$ such that $\delta(G) \geq \lfloor \frac{n}{3k+1} \rfloor - 1$. If n is sufficiently large relative to k, then G is k-supereulerian.*

5 Sharpness and conclusions

We show the sharpness of Corollaries 7 and 10.

Let $k > 1, s \geq 4$ be two integers, obtain the graph G_1 from a complete bipartite graph $K_{1,k-1}$ by s-enlarging $K_{1,k-1}$. Then $n(G_1) = sk$ and $\delta(G_1) = s-1 = \frac{n(G_1)}{k} - 1$. By the fact that G_1 has no system that dominates G with less than k elements, and by Theorem 5, each 2-factor of $L(G_1)$ has at least k components. Hence Corollary 7 is best possible.

Let $k = 1, 2, 3$, $s \geq 4$ be integers. Let G_2^k be the graph depicted in Fig. 5, where each solid point represents a complete graph of order s. Then $n(G_2^k) = (3k+1)s$ and $\delta(G_1) = s - 1 = \frac{n(G_1)}{3k+1} - 1$. By the fact that G_2^k has no system that dominates G with less than k elements, and by Theorem 5, each 2-factor of $L(G_2^k)$ has at least k components. Hence Corollary 10 is best possible.

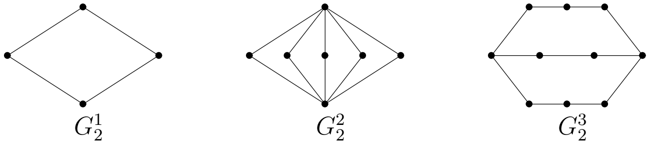


Fig. 5.

Now we consider the degree sum condition of two nonadjacent vertices. We start with two graphs G_3 and G_4, depicted in Fig. 6.

In graph G_3, the number of hollow vertices is k and the solid point represents a complete graph of order $n-k$. Then for constants $k \geq p \geq 3$ and n sufficiently large, $\overline{\sigma}_2(G_3) = 1+n-k = (2n/p-2)+((p-2)n/p+3-k) \geq 2(\lfloor n/p \rfloor -1)+((p-2)n/p+3-k)$, but $G_3' \cong K_{1,k}$, $V(G_3') = k+1 > p$ and G_3 has no even factor.

In graph G_4, $n(G_4) = 4k+4$ and each of the solid point represents a complete graph of order k. Then for n sufficiently large, $\overline{\sigma}_2(G_4) = 1 + k = 1 + (n-4)/4 = n/4 > n/5 - 2 + n/20 \geq 2(\lfloor n/(3 \cdot 3 + 1) \rfloor - 1) + n/20$, but G_4 has no even factor.

G_3 and G_4 show Theorems 3, 4 and 9 cannot be extended by replacing the $\delta(G)$ condition by the $\overline{\sigma}_2(G)$ condition which is just $2\delta(G)$. It is still open whether the degree sum condition of two nonadjacent vertices is two times of minimum degree.

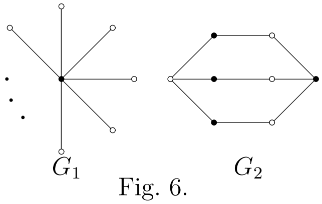


Fig. 6.

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References

[1] P. A. Catlin, A reduction method to find spanning eulerian subgraphs, *J. Graph Theory* 12 (1988), 29–44.

[2] P. A. Catlin, Supereulerian graphs: a survey, *J. Graph Theory* 16 (1992), 177–196.

[3] Z.-H. Chen and H.-J. Lai, Reduction techniques for supereulerian graphs and related topics—a survey, *Combinatorics and graph theory* 95, Vol. 1 (Hefei), 53–69, World Sci. Publishing, Singapore, 1995.

[4] Reinhard Diestel, *Graph Theory*, Springer-Verlag, New York (2000).

[5] G. A. Dirac, Some theorems on abstract graphs, *Proc. Lond. Math. Soc.* 2 (1952), 69–81.

[6] J. Fujisawa, L. Xiong, K. Yoshimoto and S. Zhang, The upper bound of the number of cycles in a 2-factor of a line graph, *J. Graph Theory* 55 (2007), 72–82.

[7] R. Gould and E. Hynds, A note on cycles in 2-factors of line graphs, *Bull. ICA.* 26 (1999), 46–48.

[8] B. Jackson and K. Yoshimoto, Even subgraphs of bridgeless and 2-factors of line graphs, *Discrete Math.* 22 (2007), 2775–2785.

[9] Z. Niu, H.-J. Lai and L. Xiong, Spanning subgraph with eulerian components, (preprint).

[10] H. J. Veldman, On dominating and spanning circuits in graphs, *Discrete Math.* 124 (1994), 229–239.

[11] L. Xiong, M. Lu and L. Han, The structure of even factors in claw-free graphs, *Discrete Math.* 309 (2009), 2417–2423.

[12] L. Xiong and M. Zong, Traceability of line graphs, *Discrete Math.* 309 (2009), 3779–3785.

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