AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **48** (2010), Pages 231–241

Bounds on the domination number in oriented graphs

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Abstract

A dominating set of an oriented graph D is a set S of vertices of D such that every vertex not in S is a successor of some vertex of S. The minimum cardinality of a dominating set of D, denoted $\gamma(D)$, is the domination number of D. An irredundant set of an oriented graph D is a set S of vertices of D such that every vertex of S has a private successor, that is, for all $x \in S$, $|O[x] - O[S - x]| \geq 1$. The irredundance number of an oriented graph, denoted $ir(D)$, is the least number of vertices in a maximal irredundant set. We denote by $\beta_1(D)$ and $s(D)$, the number of edges in a maximum matching and support vertices of the underlying graph of an oriented graph D, respectively. In this paper, we show that for every oriented graph D, $s(D) \leq ir(D) \leq \gamma(D) \leqslant n(D) - \beta_1(D)$. We also give characterizations of oriented trees satisfying $\gamma(T) = n(T) - \beta_1(T)$ and oriented graphs satisfying $\gamma(D) = s(D)$ and $s(D) = n(D) - \beta_1(D)$, respectively.

1 Introduction

An oriented graph (or digraph) D is a finite nonempty set of points called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of D called arcs or oriented edges. An oriented graph D can be obtained from a simple graph G by assigning a direction (possibly both) to each edge of G. We say that G is the underlying graph of D and that D is an orientation of G. As with graphs, the vertex set of D is denoted by $V(D)$ and the arc set is denoted by $A(D)$. The oriented graph $D = (V, A)$ considered here has no loops and no multiple arcs (but pairs of opposite arcs are allowed). If $(x, y) \in A$, then the arc is oriented from x to y. The vertex x is called a predecessor of y and y is called a successor of x. If the reversal (y, x) of an arc (x, y) of D is also present in D, we say that (x, y) is a reversible (symmetrical) arc. If $(x, y) \in A$ but $(y, x) \notin A$, then (x, y) is an asymmetrical arc.

The sets $O(u) = \{v : (u, v) \in A\}$ and $I(u) = \{v : (v, u) \in A\}$ are called the outset and inset of the vertex u, respectively. Likewise, $O[u] = O(u) \cup \{u\}$ and $I[u] = I(u) \cup \{u\}$. If $S \subseteq V$ then $O(S) = \bigcup_{s \in S} O(s)$ and $I(S) = \bigcup_{s \in S} I(s)$. Similarly $O[S] = \bigcup_{s \in S} O[s]$ and $I[S] = \bigcup_{s \in S} I[s]$. The indegree of a vertex u is given by $id(u) = |I(u)|$ and the outdegree of a vertex u is $od(u) = |O(u)|$. The maximum outdegree of a vertex in D is denoted by $\Delta_+(D)$

Let G be the underlying graph of an oriented graph D. If $e = uv$ is an edge of G, then u and v are adjacent vertices, while u and e are incident, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are adjacent edges. The *degree* of a vertex v of G is the number of vertices adjacent to v. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* If u is a support vertex, then L_u will denote the set of leaves attached at u. If $|L_u| = 1$, then u is called a weak support. An edge incident with a leaf is called a *pendant edge.* A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with p and q leaves attached at each support vertex, respectively, is denoted by $S_{p,q}$. Denote by T_x the subtree induced by a vertex x and its descendants in a rooted tree T. The *diameter* $\text{diam}(G)$ of a graph G is the maximum distance over all pairs of vertices of G. The *corona* $G \circ K_1$ of a graph G is obtained from G by adding a leaf at each of its vertices. For the underlying graph G of an oriented graph D, we denote by $n(D) = n(G)$, $\ell(D) = \ell(G)$, $s(D) = s(G)$, $L(D) = L(G)$ and $S(D) = S(G)$ the number of vertices, leaves, support vertices and the set of leaves and support vertices of G, respectively.

A set of pairwise independent edges of G is called a matching in G. The number of edges in a maximum matching of G is the edge independence number $\beta_1(G)$ ($= \beta_1(D)$ if there is no ambiguity). If M is a specified matching in graph G, then every vertex of G is incident with at most one edge of M. A vertex that is incident with no edges of M is called an $\overline{M}$-vertex.

A set $S \subseteq V$ of an oriented graph D is independent if and only if for all $x, y \in S$, $x \notin O(y)$. The size of the largest independent set in D is denoted by $\beta(D)$.

A set $S \subseteq V$ of an oriented graph D is a dominating set of D if, for all $v \notin S$, v is a successor of some vertex $s \in S$ or $O[S] = V(D)$. We use the notation $\gamma(D)$ to represent the domination number of an oriented graph, i.e., the minimum cardinality of a set $S \subseteq V$ which is dominating. A set $S \subseteq V$ is irredundant if, for all $x \in S$, $|O[x] - O[S - x]| \geq 1$. If $y \in O[x] - O[S - x]$, then we say that y is a private successor of x with respect to S. Observe that x may be its own private successor. The irredundance number of an oriented graph, denoted $ir(D)$, is the least number of vertices in a maximal irredundant set. It is clear that $ir(D) \leq \gamma(D)$. A dominating set of D with minimum cardinality is called a $\gamma(D)$-set. For more details on domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4, 5].

In general, domination in oriented graphs has not been as intensively studied as that in graphs without orientation. In [3], Ghoshal, Lasker, and Pillone consider related topics in oriented graphs and suggest further avenues of study. Gallai-type results have been considered in [7]. In [1], Albertson et al. characterize oriented trees satisfying $\gamma(D) + \Delta_+(D) = n$ and thus satisfying $ir(D) + \Delta_+(D) = n$.

In this paper, we show that for every oriented graph D, $s(D) \leq ir(D) \leq \gamma(D) \leqslant n(D) - \beta_1(D)$. We also give characterizations of oriented trees satisfying $\gamma(T) = n(T) - \beta_1(T)$ and oriented graphs satisfying $\gamma(D) = s(D)$ and $s(D) = n(D) - \beta_1(D)$, respectively.

2 Bounds

Before presenting our results, we recall some know bounds of a dominating number in oriented graphs.

Theorem 1 *[5] For any oriented graph D on n vertices,* $\dfrac{n(D)}{1 + \Delta_+(D)} \leq \gamma(D) \leqslant n(D) - \Delta_+(D)$.

Theorem 2 *[6] For a strongly connected oriented graph D on n vertices,* $\gamma(D) \leqslant \left\lceil \dfrac{n(D)}{2} \right\rceil$.

Observation 3 *Let D be an oriented graph.*

1. *Let x be a vertex of D such that $I(x) = \varnothing$. Then every $\gamma(D)$-set contains x.*
2. *Let v be a support vertex of D. Then every $\gamma(D)$-set contains at least one vertex of $L_v \cup \{v\}$.*

Recall that the number $\beta_1(D)$ can be computed for any graph in polynomial time [2]. Therefore, the following bounds can also be computed in polynomial time.

Theorem 4 *For any oriented graph D on n vertices,* $s(D) \leq ir(D) \leq \gamma(D) \leqslant n(D) - \beta_1(D)$.

Proof. Let S be an $ir(D)$-set of D. For every support vertex v such that $S \cap (L_v \cup \{v\}) = \varnothing$, there corresponds at least one vertex $z \in S$ with v its unique private successor (this is possible for otherwise $S \cup L_v$ is an irredundant set which contradicts the maximality of S). If z is a support vertex, then $L_z \in S$. Indeed, all pendant edges attached at v are oriented from $y \in L_v$ to v (may be symmetrically). So, $ir(D) = |S| \geq s(D)$.

Let $M = \{x_i y_i : 1 \leq i \leq \beta_1\}$ be a set of edges of a maximum matching in the underlying graph G of D with Z_M the set of all $\overline{M}$-vertices of G (which are incident with no edges of M). Without loss of generality, we suppose that (x_i, y_i) is an arc of $D; 1 \leq i \leq \beta_1$. It is clear that $S = \{x_1, x_2, \ldots x_{\beta_1}\} \cup Z_M$ is a dominating set of D. So, $\gamma(D) \leq |S| = |\{x_1, x_2, \ldots x_{\beta_1}\}| + |Z_M| = \beta_1 + n - 2\beta_1 = n - \beta_1$, which implies the upper bound $\gamma(D) \leqslant n(D) - \beta_1(D)$. ■

Note that the difference between $\gamma(D)$ and $ir(D)$ can be arbitrarily large even for oriented trees. To see this, consider the oriented tree of Figure 1, where $\gamma(T) = p+2$ and $ir(T) = 2 = s(D)$.

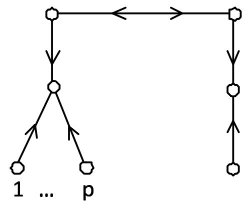

Figure 1

Next, in Sections 3 and 4, we present characterizations of special oriented graphs achieving equality in each bound of $s(D) \leq \gamma(D) \leqslant n(D) - \beta_1(D)$.

3 Characterization of directed trees achieving the upper bound

We begin by giving somme useful results:

Lemma 5 *Let D be a nontrivial oriented graph. If $\gamma(D) = n(D) - \beta_1(D)$, then every maximum matching $M = \{x_i y_i : 1 \leq i \leq \beta_1\}$ in the underlying graph G of D with corresponding arcs (x_i, y_i) ; $1 \leq i \leq \beta_1$ and Z_M the set of all $\overline{M}$-vertices of G, satisfies:*

1. *$\forall z \in Z_M$, $I(z) \cap \{x_1, \ldots, x_{\beta_1}\} = \varnothing$.*
2. *$\forall e = xy$ an edge of M and (x, y) a corresponding arc in D. If one end-vertex z of e satisfies $I(z) \cap ((\{x_1, \ldots, x_{\beta_1}\} - \{x\}) \cup Z_M) \neq \varnothing$, then the other end-vertex z' of e verifies $I(z') \cap ((\{x_1, \ldots, x_{\beta_1}\} - \{x\}) \cup Z_M) = \varnothing$.*

Proof. Let $M = \{x_i y_i : 1 \leq i \leq \beta_1\}$ be a maximum matching in the underlying graph G of D with corresponding arcs (x_i, y_i) ; $1 \leq i \leq \beta_1$ and Z_M the set of all $\overline{M}$-vertices of G. First, suppose that there exists $z \in Z_M$ such that $I(z) \cap \{x_1, \ldots, x_{\beta_1}\} \neq \varnothing$. It is clear that $S = \{x_1, \ldots, x_{\beta_1}\} \cup (Z_M - \{z\})$ is a dominating set of D and $|S| = |\{x_1, x_2, \ldots x_{\beta_1}\}| + |Z_M - \{z\}| = \beta_1 + n - 2\beta_1 - 1 = n - \beta_1 - 1$. Then S is a dominating set of D of size less than $n - \beta_1$, a contradiction. Now assume that there exists an edge $e = xy$ of M with a corresponding arc (x, y) in D, which do not satisfy Part 2 of Lemma 5. Without loss of generality, suppose that $I(y) \cap ((\{x_1, \ldots, x_{\beta_1}\} - \{x\}) \cup Z_M) \neq \varnothing$ and $I(x) \cap ((\{x_1, \ldots, x_{\beta_1}\} - \{x\}) \cup Z_M) \neq \varnothing$. Consider now $S = ((\{x_1, \ldots, x_{\beta_1}\} - \{x\}) \cup Z_M)$, it is clear that S is a dominating set of D of size less than $n - \beta_1$, a contradiction. ■

Observation 6 *Let T be a tree.*

1. *If T is a tree obtained from a tree T' by attaching a vertex to a support vertex of T', then $\beta_1(T) = \beta_1(T')$.*

2. *For every support vertex v of a nontrivial tree, there exists a maximum matching M which contains a pendant edge with end-vertex v.*

3. *If T is a tree obtained from a tree T' by attaching an end-vertex of P_2 to a vertex of T', then $\beta_1(T) = \beta_1(T') + 1$.*

We call the oriented graph of Figure 2 the obstruction. The arcs shown may be symmetrical.

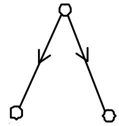

Figure 2: The obstruction

Let $\overrightarrow{K_{1,p}}$ be the oriented star (the underlying graph is a star) without the obstruction as a subdigraph, that is, the oriented star with center x such that $|O(x) \cap L_x| \leq 1$.

Observation 7 *Let T be a nontrivial oriented tree. If $\gamma(T) = n(T) - \beta_1(T)$, then for every support vertex x of T, the subdigraph induced by $L_x \cup \{x\}$ is an oriented star $\overrightarrow{K_{1,p}}$ where $p \geq 1$.*

Proof. Assume that there exists a support vertex x of T such that $L_x \cup \{x\}$ is an oriented star $\overrightarrow{K_{1,p}}$; $p \geq 2$ with the obstruction as a subdigraph. By Part 2 of Observation 6, we consider a maximum matching M which contains a pendant edge with end-vertex x. Then Part 1 of Lemma 5 is not satisfied, so $\gamma(T) < n(T) - \beta_1(T)$, a contradiction. ■

We denote by $\overrightarrow{S_{p,q}}$ the oriented tree obtained from two oriented stars $\overrightarrow{K_{1,p}}$ and $\overrightarrow{K_{1,q}}$ by attaching the center x of $\overrightarrow{K_{1,p}}$ to the center y of $\overrightarrow{K_{1,q}}$ where the edge xy is arbitrary oriented (possibly symmetrically). In Figure 3 and elsewhere the arc with two arrowheads pointing in the same direction indicates an asymmetrically oriented arc.

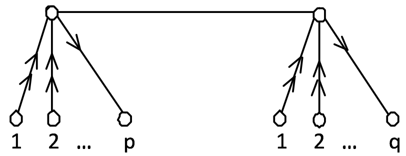

Figure 3: The oriented tree $\overrightarrow{S_{p,q}}$

We also denote by $\overrightarrow{P_2^1(x,y)}$ the oriented chain obtained from $P_2 = xy$ where the edge xy is asymmetrically oriented from y to x, that is, (x,y) is not present. And denote by $\overrightarrow{P_2^2(x,y)}$ the oriented chain obtained from $P_2 = xy$ where the edge xy is oriented from x to y, possibly the arc (y,x) is also present.

Also denote by $H_k(z)$ the oriented tree obtained from oriented chains $\overrightarrow{P_2^2(x_i,y_i)}$; $1 \leq i \leq k$ and join every vertex x_i ; $1 \leq i \leq k$ by an edge to vertex z, where at least one edge x_iz is oriented from x_i to z (possibly symmetrically) and all others are arbitrary oriented. (For all these oriented graphs see Figure 4 and Figure 5.)

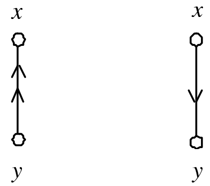

Figure 4: $\overrightarrow{P_2^1(x,y)}$ and $\overrightarrow{P_2^2(x,y)}$

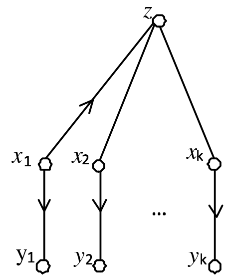

Figure 5: $H_k(z)$

In order to characterize the oriented trees with $\gamma(T) = n(T) - \beta_1(T)$, we introduce the family $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ $(m \geq 1)$ of oriented trees, where T_1 is $\overrightarrow{P_2^1(x,y)}$ or $\overrightarrow{P_2^2(x,y)}$, $T = T_m$, and, if $m \geq 2$, T_{i+1} is obtained recursively from T_i by one of the five operations defined below.

- **Operation** $\mathcal{O}_1$: Add a vertex y and join y by an edge to a support vertex x of T_i, where the edge xy is asymmetrically oriented from y to x.

- **Operation** $\mathcal{O}_2$: Add an oriented chain $\overrightarrow{P_2^1(x,y)}$ and join x by an edge to a vertex z of T_i, where the edge xz is arbitrary oriented.

- **Operation** $\mathcal{O}_3$: Add an oriented chain $\overrightarrow{P_2^2(x,y)}$ and join x by an edge to a support vertex z of T_i, where the edge xz is arbitrary oriented.

- **Operation** $\mathcal{O}_4$: Add oriented chains $\overrightarrow{P_2^2(x_i, y_i)}$; $i = 1, \dots k$ and join every vertex x_i by an edge to a pendent vertex z of T_i, where the edge $x_i z$ is asymmetrically oriented from z to x_i for $i = 1, \dots k$.

- **Operation** $\mathcal{O}_5$: Add an oriented tree $H_k(z)$ and join z by an edge to a vertex w of T_i such that there exists a maximum matching M where w is an $\overline{M}$-vertex and where the edge zw is arbitrary oriented.

Lemma 8 *If a nontrivial oriented tree* T *is in* $\mathcal{F}$, *then* $\gamma(T) = n(T) - \beta_1(T)$.

Proof. Let T be a nontrivial oriented tree of $\mathcal{F}$. To show that $\gamma(T) = n(T) - \beta_1(T)$, we proceed by induction on m where $m - 1$ is the number of operations performed to construct T from T_1. If $m = 1$, then $T = \overrightarrow{P_2^1(x, y)}$ or $\overrightarrow{P_2^2(x, y)}$ and since $\beta_1(T) = 1$, $\gamma(T) = 1$ and $n(T_1) = 2$, $\gamma(T) = n(T) - \beta_1(T)$. This establishes the basis case. Assume now that $m \geq 2$ and the result holds for all trees of $\mathcal{F}$ that can be constructed from a sequence of at most $m - 2$ operations. Let $T = T_m$ be a nontrivial oriented tree of $\mathcal{F}$ constructed by $m-1$ operations, $T' = T_{m-1}$ and assume that T' has order $n(T'), \beta_1(T')$ and $\gamma(T')$. By induction hypothesis applied to T', we know that $\gamma(T') = n(T') - \beta_1(T')$. We consider five cases depending on whether T is obtained from T' by using $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ or $\mathcal{O}_5$.

Case 1. Suppose that T was obtained from T' by operation $\mathcal{O}_1$. Let S' be $\gamma(T')$-set. Then $S' \cup \{y\}$ is a dominating set of T, so $\gamma(T) \leq |S' \cup \{y\}| \leq \gamma(T') + 1$. Let now S be a $\gamma(T)$-set of T. By Part 1 of Observation 3, S contains y. Without loss of generality since x is a support vertex of T', either x is contained in S or x is dominated by one vertex of $L_x - \{y\}$, so $S' = S - \{y\}$ is dominating set of T'. So, $\gamma(T') \leq |S'| = |S - \{y\}| = \gamma(T) - 1$. Thus, $\gamma(T) = \gamma(T') + 1$. By induction $\gamma(T') = n(T') - \beta_1(T')$ and by Part 1 Observation 6 $\beta_1(T) = \beta_1(T')$, so $\gamma(T) = n(T') - \beta_1(T') + 1 = n(T) - \beta_1(T)$.

Case 2. Suppose that T was obtained from T' by performing operation $\mathcal{O}_2$. Let S' be $\gamma(T')$-set. Then $S' \cup \{y\}$ is a dominating set of T, so $\gamma(T) \leq |S' \cup \{y\}| \leq \gamma(T') + 1$. Let now S be a $\gamma(T)$-set of T. By Part 1 of Observation 3, S contains y. Without loss of generality, we suppose that $x \notin S$ (otherwise replace x by z). So $S' = S - \{y\}$ is a dominating set of T'. So, $\gamma(T') \leq |S'| = |S - \{y\}| = \gamma(T) - 1$. Thus, $\gamma(T) = \gamma(T') + 1$. By induction $\gamma(T') = n(T') - \beta_1(T')$ and by Part 3 Observation 6, $\beta_1(T) = \beta_1(T') + 1$, so $\gamma(T) = n(T') - \beta_1(T) + 2 = n(T) - \beta_1(T)$.

Case 3. Suppose that T was obtained from T' by performing operation $\mathcal{O}_3$. Let S' be $\gamma(T')$-set. Then $S' \cup \{x\}$ is a dominating set of T, so $\gamma(T) \leq |S' \cup \{x\}| \leq \gamma(T') + 1$. Let now S be a $\gamma(T)$-set of T. Without loss of generality, we suppose that $x \in S$ and $y \notin S$ and since z is a support vertex of T', either z is contained in S or z is dominated by one vertex of L_z, so $S' = S - \{x\}$ is a dominating set of T'. So, $\gamma(T) \leq |S'| = |S - \{x\}| = \gamma(T) - 1$. Thus, $\gamma(T) = \gamma(T') + 1$. By induction $\gamma(T') = n(T') - \beta_1(T')$ and by Part 3 of Observation 6, $\beta_1(T) = \beta_1(T') + 1$, so $\gamma(T) = n(T') - \beta_1(T) + 2 = n(T) - \beta_1(T)$.

Case 4. Suppose that T was obtained from T' by performing operation $\mathcal{O}_4$. Let S' be $\gamma(T')$-set. Then $S' \cup \{x_1, \dots, x_k\}$ is a dominating set of T, so $\gamma(T) \leq$

$|S' \cup \{x_1, \ldots, x_k\}| \leq \gamma(T') + k$. Let now S be a $\gamma(T)$-set of T. Without loss of generality, we suppose that $x_i \in S$ and $y_i \notin S$ for $i = 1, \ldots, k$ and since every edge $x_i z$ is asymmetrically oriented from z to x_i for $i = 1, \ldots, k$, $S' = S - \{x_1, \ldots, x_k\}$ is a dominating set of T'. So, $\gamma(T') \leq |S'| = |S - \{x_1, \ldots, x_k\}| = \gamma(T) - k$. Thus, $\gamma(T) = \gamma(T') + k$. By induction $\gamma(T') = n(T') - \beta_1(T')$ and by Part 3 of Observation 6, $\beta_1(T) = \beta_1(T') + k$, so $\gamma(T) = n(T') - \beta_1(T) + 2k = n(T) - \beta_1(T)$.

Case 5. Suppose that T was obtained from T' by performing operation $\mathcal{O}_5$. Let S' be a $\gamma(T')$-set. Since there exists at least one edge $x_i z$ which is oriented from x_i to z, $S' \cup \{x_1, \ldots, x_k\}$ is a dominating set of T, so $\gamma(T) \leq |S' \cup \{x_1, \ldots, x_k\}| \leq \gamma(T') + k$. Let S be a $\gamma(T)$-set of T. Without loss of generality, we suppose that $x_i \in S$ and $y_i \notin S$ for $i = 1, \ldots, k$ and $z \notin S$ (otherwise replace w by z). So $S' = S - \{x_1, \ldots, x_k\}$ is a dominating set of T'. So, $\gamma(T') \leq |S'| = |S - \{x_1, \ldots, x_k\}| = \gamma(T) - k$. Thus, $\gamma(T) = \gamma(T') + k$. By induction $\gamma(T') = n(T') - \beta_1(T')$ and since there exists a maximum matching M with w is a $\overline{M}$-vertex, it is clear that $\beta_1(T) = \beta_1(T') + k + 1$, so $\gamma(T) = n(T') - \beta_1(T) + 2k + 1 = n(T) - \beta_1(T)$. ■

Theorem 9 *If T is a nontrivial oriented tree of order $n(T)$, then $\gamma(T) = n(T) - \beta_1(T)$ if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then by Lemma 8, $\gamma(T) = n(T) - \beta_1(T)$. To prove that if T is a nontrivial oriented tree of order $n \geq 2$, then $\gamma(T) = n(T) - \beta_1(T)$ only if $T \in \mathcal{F}$, we process by induction on the order of T. If $\text{diam}(T) = 1$ (the diameter of the underlying tree of the oriented tree), then $T = \overrightarrow{P_2^1(x,y)}$ or $\overrightarrow{P_2^2(x,y)}$ which belongs to $\mathcal{F}$. If $\text{diam}(T) = 2$, then $T = \overrightarrow{K_{1,p}}$ (see Observation 7) which is obtained from $\overrightarrow{P_2^1(x,y)}$ or $\overrightarrow{P_2^2(x,y)}$ by applying $p-2$ times $\mathcal{O}_1$. If $\text{diam}(T) = 3$, then $T = \overrightarrow{S_{p,q}}$ which is obtained by applying operations $\mathcal{O}_2$ or $\mathcal{O}_3$ followed by zero or more repetitions of Operation $\mathcal{O}_1$. This establishes the base cases.

So we suppose that $\text{diam}(T) \geq 4$, and that every nontrivial oriented tree T' of order less than n satisfying $\gamma(T') = n(T') - \beta_1(T')$ is in $\mathcal{F}$. Let T be a nontrivial oriented tree of order n satisfying $\gamma(T) = n(T) - \beta_1(T)$. Consider a $\gamma(T)$-set S of T. We consider the underlying tree of the oriented tree and we root T at a vertex r of maximum eccentricity. Let x be a support vertex at maximum distance from r in the rooted tree. Let T_u denote the subtree induced by a vertex u and its descendants in the rooted tree T. We consider three cases.

Case 1. x is a support vertex with $|L_x| \geq 2$. By Observation 7, the subdigraph induced by $L_x \cup \{x\}$ is an oriented star $\overrightarrow{K_{1,p}}$ where $p \geq 1$ without the obstruction as a subdigraph. So, there exists y attached to x with the edge xy asymmetrically oriented from y to x. Let $T' = T - \{y\}$. Then $n(T') = n(T) - 1$ and by Part 1 of Observation 6, $\beta_1(T) = \beta_1(T')$. By Part 1 of Observation 3, S contains y, and since x is a support, without loss of generality $S' = S - \{y\}$ is a dominating set of T' (x is dominated by a leaf of $L_x - \{y\}$ or $x \in S$). So, $\gamma(T) - 1 \leq \gamma(T') \leq |S'| = |S - \{y\}| = \gamma(T) - 1$. Thus $\gamma(T') = \gamma(T) - 1 = n(T) - \beta_1(T) - 1 = n(T') - \beta_1(T')$. By induction on T', we have $T' \in \mathcal{F}$, implying that $T \in \mathcal{F}$ because T is obtained by using Operation $\mathcal{O}_1$.

From now on we may assume that $|L_x| = 1$. Let $L_x = \{y\}$. Let z be the parent of x

in the rooted tree, since $\operatorname{diam}(T) \geq 4$, z exists.

Case 2. The edge xy is asymmetrically oriented from y to x, that is; (x, y) is not present. Let $T' = T - \{x, y\}$. Then $n(T') = n(T) - 2$ and by Part 3 of Observation 6, $\beta_1(T) = \beta_1(T') + 1$. Also, by Part 1 of Observation 3, S contains y. Without loss of generality, we suppose that $x \notin S$ (otherwise replace x by z). So $S' = S - \{y\}$ is dominating set of T'. Thus, $\gamma(T) - 1 \leq \gamma(T') \leq |S'| = |S - \{y\}| = \gamma(T) - 1$ which implies that $\gamma(T') = \gamma(T) - 1 = n(T) - \beta_1(T) - 1 = n(T) - \beta_1(T') - 2 = n(T') - \beta_1(T')$. By induction on T', we have $T' \in \mathcal{F}$, implying that $T \in \mathcal{F}$ because T is obtained by using Operation $\mathcal{O}_2$.

Case 3. The edge xy is oriented from x to y, possibly the arc (x, y) is symmetrical. Let us examine the following subcases:

Case 3.1. Assume z is a support vertex in T. Let $T' = T - \{x, y\}$. Then $n(T') = n(T) - 2$ and by Part 3 of Observation 6, $\beta_1(T) = \beta_1(T') + 1$. Without loss of generality, we suppose that $x \in S$ and $y \notin S$ (otherwise replace y by x) and since z is a support vertex of T', either z is contained in S or z is dominated by one vertex of L_z, so $S' = S - \{x\}$ is dominating set of T'. Thus, $\gamma(T) - 1 \leq \gamma(T') \leq |S'| = |S - \{y\}| = \gamma(T) - 1$ which implies that $\gamma(T') = \gamma(T) - 1 = n(T) - \beta_1(T) - 1 = n(T) - \beta_1(T') - 2 = n(T') - \beta_1(T')$. By induction on T', we have $T' \in \mathcal{F}$, implying that $T \in \mathcal{F}$ because T is obtained by using Operation $\mathcal{O}_3$.

Case 3.2. Assume z is not a support vertex in T. We can suppose that every child x of z in the rooted tree is a weak support with $L_x = \{y\}$ in the underlying tree and is a predecessor of y (otherwise we can apply **Case 2**). So, let $\overrightarrow{P_2^2(x_i, y_i)}$; $i = 1, \ldots, k$ $(k \geq 1)$ be oriented chains where every x_i is joined to the vertex z in T.

- If the edge $x_i z$ is asymmetrically oriented from z to x_i for $i = 1, \ldots, k$, then consider $T' = T - \bigcup_{i=1}^{k} \{x_i, y_i\}$. Since T has a diameter at least four, T' is nontrivial oriented tree and z is a pendant vertex in T'. Since $n(T') = n(T) - 2k$ and by Part 3 of Observation 6, $\beta_1(T) = \beta_1(T') + k$ and it is a routine matter to check $\gamma(T') = \gamma(T) - k$. Hence $\gamma(T') = \gamma(T) - k = n(T) - \beta_1(T) - k = n(T) - \beta_1(T') - 2k = n(T') - \beta_1(T')$. Applying the inductive hypothesis to T', we have $T' \in \mathcal{F}$. Since T is obtained from T' by using Operation $\mathcal{O}_4$, $T \in \mathcal{F}$.

- If there exists an edge $x_i z$ which is oriented from x_i to z (possibly symmetrically), then since $\operatorname{diam}(T) \geq 4$, let w be the parent of z in the rooted tree. Let $T' = T - (\bigcup_{i=1}^{k} \{x_i, y_i\} \cup \{z\})$, $n(T') = n(T) - 2k - 1$. Also, since T has a diameter at least four, T' is a nontrivial oriented tree. It is a routine matter to check $\gamma(T') = \gamma(T) - k$. If for every maximum matching M of T', w is incident with one edge of M, then $\beta_1(T) = \beta_1(T') + k$. So, $\gamma(T) = \gamma(T') + k \leq n(T') - \beta_1(T') + k = n(T') - \beta_1(T) + 2k = n(T) - 1 - \beta_1(T) < n(T) - \beta_1(T)$, a contradiction. Thus, there exists a maximum matching M with w as a $\overline{M}$-vertex. Hence, $\beta_1(T) = \beta_1(T') + k + 1$ and $\gamma(T') = \gamma(T) - k = n(T) - \beta_1(T) - k = n(T) - \beta_1(T') - 2k - 1 = n(T') - \beta_1(T')$. Applying the inductive hypothesis to T', we have $T' \in \mathcal{F}$. Since T is obtained from T' by using Operation $\mathcal{O}_5$, $T \in \mathcal{F}$. This achieves the proof. ■

4 Characterization of digraphs achieving the lower bound

Theorem 10 *Let D be an oriented graph. Then $\gamma(D) = s(D)$ if and only if the oriented graph D satisfies :*

1. *For every vertex z of $V(D) - (S(D) \cup L(D))$, $I(z) \cap S(D) \neq \varnothing$.*
2. *For every vertex $x \in S(D)$ with $|L_x| \geq 2$, $O(x) \cap L_x = L_x$.*
3. *Let $L' = \{y \in L \ / \ I(y) \cap S(D) = \varnothing\}$, for every $z \in V(D) - (S(D) \cup L(D))$, $(I(z) \setminus O(L')) \cap S(D) \neq \varnothing$.*

Proof. ($\Longleftarrow$) Suppose that one of the conditions is not satisfied. Then in all cases, $\gamma(D) > s(D)$, a contradiction.

($\Longrightarrow$) By Theorem 4, $\gamma(D) \geq s(D)$. We construct the dominating set S' as follow, set every support vertex with at least two leaves in S'. If x is a support vertex with one leaf and $O(x) \cap L_x = \varnothing$, then set the leaf in S', if not set x in S'. By construction, $|S'| = s(D)$ and S' dominates all vertices of $S(D) \cup L(D)$. Suppose there exists a vertex z of $V(D) - (S(D) \cup L(D))$ which is not dominated by S'. By Part 1 of Theorem 10, $I(z) \cap S(D) \neq \varnothing$. Let $S'' = I(z) \cap S(D)$, by construction of S' the leaves attached to support vertices of S'' are in S'. Therefore, for every vertex x of S'' $O(x) \cap L_x = \varnothing$, a contradiction with Part 3 of Theorem 10. So S' is a dominating set. $|S'| = s(D) \geq \gamma(D)$, which implies that $\gamma(D) = s(D)$. ∎

Theorem 11 *Let D be an oriented graph. Then $\gamma(D) = s(D) = n(D) - \beta_1(D)$ if and only if the underlying graph G of D is a corona.*

To prove Theorem 11, we use the following result due to Xu [8].

Theorem 12 *[8] Let G be a graph. Then $\beta(G) + \beta_1(G) \leq n(G)$.*

Proof of Theorem 11. ($\Longrightarrow$) If the underlying graph of D is a corona, then G has a perfect matching, $\beta_1(D) = \dfrac{n(D)}{2} = s(D)$. By Theorem 4, $\dfrac{n(D)}{2} = s(D) \leq \gamma(D) \leq n(D) - \beta_1(D) = n(D) - \dfrac{n(D)}{2} = \dfrac{n(D)}{2}$. Thus $\gamma(D) = \dfrac{n(D)}{2} = s(D)$.

($\Longleftarrow$) By Theorem 12, $s(D) = n(D) - \beta_1(D) \geq \beta(D)$ and $s(D) \leq l(D) \leq \beta(D)$. So, $s(D) = l(D) = \beta(D)$ which implies that $V(D) - (S(D) \cup L(D)) = \varnothing$. It follows that the underlying G of D is a corona. This completes the proof ∎

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(Received 4 Mar 2010; revised 31 May 2010)