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Ultimately bipartite subtraction games

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Abstract

We introduce the notion of ultimately bipartite impartial games. These are games that are ultimately periodic in the simplest possible manner. We examine ultimately bipartite subtraction games and demonstrate a curious feature: for 'large' games it is clear who has the winning position and how the game should be strategically played, but during play, as the game eventually becomes small, it is no longer so easy to know what the strategic moves are. We give examples which indicate that ultimately bipartite subtraction games are quite common.

1 Bipartite games

Throughout this paper, G denotes a finite two-player impartial combinatorial game; as usual, the players play alternately and the loser is the first player who has no possible move [3]. Before we can define ultimately bipartite games, we first treat the less interesting, bipartite games. A well known example of this kind of game is *Brussels sprouts*. Here, one starts with an agreed number of crosses on the page. A move is played by connecting two arms of (possibly equal) crosses by a simple curve (avoiding all other crosses and previously drawn curves) and introducing a new cross by marking a bar across the middle of the curve. Figure 1 shows the game with one initial cross; it is a first player win. Invented by John H. Conway, Brussels Sprouts is something of a mathematical joke. A simple Euler characteristic calculation shows that for Brussels Sprouts played with n initial crosses, the first player wins if and only if n is odd, regardless of how the game is played [8, 1, 6].

The fundamental property of Brussels sprouts is that the underlying game graph is bipartite. Recall that for a given impartial game, with a given starting configuration, the *game graph* is the directed graph whose vertices are the possible positions that can be potentially reached in the game, and there is a directed edge between two vertices if there is a legal move between the corresponding positions; see [2, 7]. We make a further proviso: if there is more than one terminal position, we coalesce these into a single position. In this way, the game graph has a unique source (the

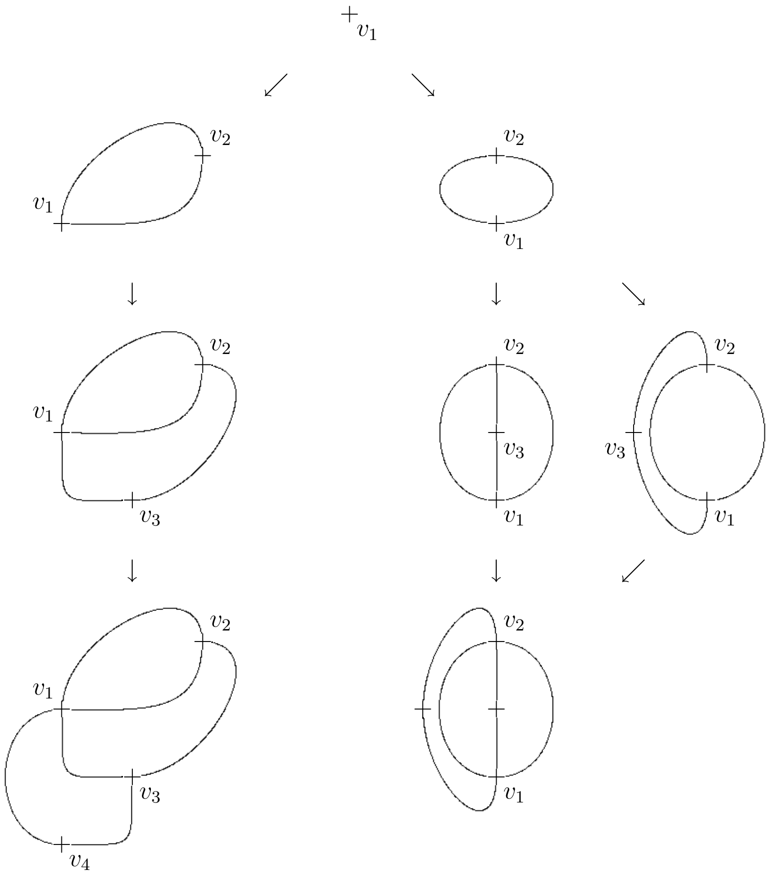

Figure 1: A game of Brussel sprouts with one initial cross

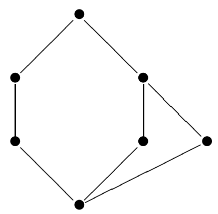

Figure 2: The game graph for Brussels sprouts with one initial cross

starting position) and unique sink (the terminal position). The game graph for Brussels sprouts with one initial cross is shown in Figure 2; here the edges are directed downwards. Recall that a graph is bipartite if the vertex set can be written as a disjoint union of two subsets (the "parts") such that every edge connects vertices in different parts. We say that an impartial game is bipartite if its game graph is bipartite.

Bipartite games, such as Brussels sprouts, are trivial from the perspective of the players. To see this, recall that a position is a *winning position* if from that position there is a winning strategy for the player whose turn it is to play. Positions that are not winning are *losing*. (Losing positions are also known as $\mathcal{P}$ positions, and winning positions are called $\mathcal{N}$ positions). The game is winning if its starting position is winning. Recall that a position is winning if and only if its nim value (also known as the Sprague-Grundy value) is positive [3]. Recall that the nim value is defined recursively as follows: the terminal positions have nim value 0, and the nim value of a position p is the smallest nonnegative integer m such that there is no move from p to a position with nim value m, but for all $0 \leq i < m$ there is a move from p to a position with nim value i.

Proposition 1. *A game G is bipartite if and only if all its winning positions have nim value 1.*

We will require the following elementary, general fact.

Lemma 1. *Suppose that the vertex set of the game graph of a game G is the disjoint union of two sets A, B such that*

(a) for every vertex p in A, all the moves from p terminate in B,

(b) for every vertex p in B, there is a move from p that terminates in A.

Then A is the set of losing positions, and B is the set of winning positions.

Note that if G is bipartite with respective parts A, B with the sink in A, then the sets A, B satisfy the hypotheses of Lemma 1 and so we have immediately:

Lemma 2. *If a game G is bipartite, then the two parts of G are the losing positions and the winning positions respectively.*

Proof of Proposition 1. If G is bipartite, Lemma 2 implies that the winning positions form one of the parts, so there are no moves in G from a winning position to a winning position. The winning positions therefore all have nim value 1. Conversely, if the winning positions all have nim value 1, then the losing and winning positions respectively form sets A, B with the required property. □

We can be more precise about the winning positions of bipartite games. We will use the following definition.

Definition 1. In a game G, the *height* $h(p)$ of a position p is the length of the longest directed path in the game graph from p to the sink.

Proposition 2. *In a bipartite game G, a position p is winning if and only if p has odd height.*

Proof. Consider a directed path γ of maximal length, h say, from p to the sink of G. By Lemma 2, as G is bipartite, γ alternately visits winning and losing positions. So, as the sink is losing, the position p is winning if and only if h is odd. □

Examples of bipartite games are provided by certain subtraction games. Let S be a finite set; $S = \{s_1, s_2, s_3, \dots, s_m\}$. Recall that the *subtraction game* with subtraction set S is the game $\mathcal{S}(n, S)$, commencing with n counters in which, on each move, one may remove k counters, where k is any element of S. Without loss of generality, we assume that the members of S are relatively prime. Indeed, if $\gcd(S) = g$, then playing the game $\mathcal{S}(n, S)$ is no different to playing the game $\mathcal{S}(\lfloor n/g \rfloor, S/g)$, where $\lfloor . \rfloor$ denotes the integer part; see [3, p. 98].

Theorem 1. *For a finite subtraction set S with $\gcd(S) = 1$, the subtraction game $\mathcal{S}(n, S)$ is bipartite for all n if and only if $1 \in S$ and the elements of S are all odd. Moreover, in this case, $\mathcal{S}(n, S)$ is winning if and only if n is odd.*

Proof. Let $s = \min(S)$. First suppose that $1 \in S$ and the elements of S are all odd. Since $1 \in S$, there is only one terminal position, namely the pile with 0 counters. Let A (resp. B) denote the set of positions having an even (resp. odd) number of counters. Each move starting in A ends in B, and visa-versa. So $\mathcal{S}(n, S)$ is bipartite.

Conversely, suppose that $\mathcal{S}(n, S)$ is bipartite for all n and let $s = \min(S)$. The piles with $0, 1, 2, \dots, s - 1$ counters are all terminal positions, and so in the game graph, they are coalesced into a single sink. Thus, as $\mathcal{S}(n, S)$ is bipartite, we have by induction that the pile with m counters is a losing position if and only if $\lfloor \frac{m}{s} \rfloor$ is even. Now let $t \in S$ and consider the piles with $t + i$ counters for $0 \le i < s$; these piles are obviously winning positions, and so $\lfloor \frac{t+i}{s} \rfloor$ is odd for all $0 \le i < s$. It follows that t is an odd multiple of s. Hence, as $\gcd(S) = 1$, we have $s = 1$, and the elements of S are all odd.

Finally, if $\mathcal{S}(n, S)$ is bipartite, we have $1 \in S$ and so the starting position in $\mathcal{S}(n, S)$ has height n. Thus, by Proposition 2, $\mathcal{S}(n, S)$ is winning if and only if n is odd. □

2 Ultimately Bipartite games

Ultimate periodicity is a ubiquitous concept in combinatorial game theory, so much so that authors commonly refer to it as periodicity, and refer to periodicity as "pure periodicity". The concept doesn't apply to a single game, but rather to a family of games G_n depending on some integer n. Such a family is *ultimately periodic* if for some integer N, the nim values $Nim(G_n)$ are periodic for $n \geq N$; that is, there exists p, the *period*, such that $Nim(G_{n+p}) = Nim(G_n)$ for all $n \geq N$; see [4, p. 529]. Usually one is interested in a family that is effectively a single game, but with different starting positions specified by n.

Definition 2. We say that a family of games G_n is *ultimately bipartite* if G_n is ultimately periodic with period 2 with, for sufficiently large n, alternating nim values $0, 1, 0, 1, 0, 1, \dots$.

Turning to subtraction games, one might be forgiven for thinking that most subtraction games are purely periodic. As remarked in [3, p. 86], the only one subtraction game in the tables [3, p. 84–85] that isn't purely periodic is $S = \{2, 5, 7\}$. It is known that all subtraction games are ultimately periodic but in general, little is known about their period [3]. In the following result we give examples that indicate that ultimately bipartite, non purely periodic, subtraction games are not uncommon. There are no ultimately bipartite subtraction games having a subtraction set with just 2 elements, apart from the purely periodic examples we considered in Theorem 1 (see [4], especially the bottom of p. 529 and the top of page p. 530). So the first interesting case has subtraction sets with 3 elements. We give three infinite families of ultimately bipartite subtraction games, each family commencing with the subtraction set $\{3, 5, 9\}$.

Theorem 2. *The subtraction game $\mathcal{S}(n, S_k)$ is ultimately bipartite for each member of the following three infinite families of subtractions sets:*

(a) $S_k = \{3, 5, 9, \dots, 2^k + 1\}$*, for* $k \geq 3$*,*

(b) $S_k = \{3, 5, 2^k + 1\}$*, for* $k \geq 3$*,*

(c) $S_k = \{k, k + 2, 2k + 3\}$*, for odd* $k \geq 3$*.*

Proof. For k fixed, and for each of the three given families of subtractions sets $S_k = \{s_1, s_2, \dots, s_k\}$, we will give a set A for which we claim that $\mathcal{S}(n, S_k)$ is losing if and only if n belongs to A. This claim follows from Lemma 1, provided we show that:

(i) if $n \in A$, then $n - s \notin A$, for all $s \in S_k$,

(ii) if $n \notin A$, then $n - s_i \in A$, for some $s \in S_k$.

For the family (a), let $S_k + 1$ denote the set obtained by adding 1 to each of the members of S_k, and let $\mathbb{E}$ denote the set of non-negative even integers. We set:

$$A = \{1\} \cup \mathbb{E} \backslash (S_k + 1).$$

That is, $A = \{0, 1, 2, 8, 12, 14, 16, 20, \dots\}$. For condition (i), note that if n is odd and $n \in A$, then $n = 1$, and so for all i, we have $n - s_i < 0$ and so $n - s_i \notin A$. For all i, if n is even, then $n - s_i$ is odd and so $n - s_i \in A$ only if $n - s_i = 1$, in which case $n = s_i + 1 \notin A$. Thus $n \in A$ gives $n - s_i \notin A$ for all i.

For condition (ii), note that if $n \notin A$ and n is even then $n = s_i + 1$ for some i, so $n - s_i = 1 \in A$. On the other hand, if $n \notin A$ and n is odd, then $n \geq 3$. If $n - 3 \notin A$, then $n = s_i + 4$ for some i. But then $n - 5 = s_i - 1$. Thus $n - 5 \in A$ unless $s_i - 1 = s_j + 1$ for some j. Obviously this can only happen for $s_i = 5, s_j = 3$, and in this case $n = 9$. But then $n - 9 = 0 \in A$.

We have thus shown that A is the set of n for which $\mathcal{S}(n, S_k)$ is losing. It remains to notice that for $n > 2^k + 2$, we have $n \in A$ if and only if n is even, so the game $\mathcal{S}(n, S_k)$ is ultimately bipartite. This completes the proof for family (a).

For families (b) and (c) the proof is similar, but the details are more subtle. We content ourselves in exhibiting the sets A in each case, and we leave the verification of conditions (i) and (ii) to the reader. For the family (b), we work with congruences modulo 8. Notice that $2^k \equiv 0$ for all $k \geq 3$. Set

$$\begin{aligned}
D_1 &= \{i : i \equiv 4,\ 0 < i < 2^k\} \\
D_2 &= \{i : i \equiv 6,\ 0 < i < 2^k\} \\
C &= \{i : i \equiv 2,\ 2^k < i < 2^{k+1}\} \\
B &= \{i : i \equiv 1,\ 0 < i < 2^k\} \\
A &= B \cup \mathbb{E} \backslash (C \cup D_1 \cup D_2).
\end{aligned}$$

Notice that for $n > 2^k + 2$, we have $n \in A$ if and only if n is even.

For the family (c), let $k = 2l + 1$ and set

$$\begin{aligned}
D_i &= \{j \in \mathbb{N} : j \text{ even}, k + 1 + i(3k + 5) \leq j \leq 2k + i(3k + 3)\} \\
C_i &= \{j \in \mathbb{N} : j \text{ odd}, 1 + i(3k + 5) \leq j \leq k - 2 + i(3k + 3)\} \\
D &= \bigcup_{i=0}^{l-1} (D_i \cup (D_{i+1} - (2k + 2))) \\
C &= \bigcup_{i=1}^{l-1} C_i \\
B &= \{j \in \mathbb{N} : 0 \leq j \leq k - 1\} \\
A &= B \cup C \cup \mathbb{E} \backslash D.
\end{aligned}$$

In this case, for $n \geq 3 \cdot \frac{k^2 - 1}{2}$, we have $n \in A$ if and only if n is even. $\square$

Remark 1. The games described in the above theorem are not bipartite in general. For example, the game $S(n, \{3, 5, 9\})$ has the nim value 2 for $n = 6, 7, 13$, and nim value 3 for $n = 9, 10$.

3 A curious feature of ultimately bipartite subtraction games

We conclude this paper by establishing an amusing feature of ultimately bipartite subtraction games that is not enjoyed by ultimately bipartite games in general. Notice that for the games in each of the families in the above theorem, for sufficiently large n, the game is winning if and only if n is odd. In fact, as we will show, there is no ultimately bipartite subtraction game such that, for sufficiently large n, the game is winning if and only if n is even. This is a rather curious situation: for n large, that is, for a large pile of counters, it is clear who has the winning position and furthermore, it doesn't matter how that player plays. However, as one plays the game and n eventually becomes small, it is no longer so easy to know what the strategic moves are. This is reminiscent of *nimania* and related games, where the first player can play arbitrarily for large n and still win, and it's only when the game becomes small that the first player has to pay attention to maintain a win [11, 9, 10]. This situation can be contrasted to the game *chomp*, where we know that a winning strategy for the first player exists, but at present, the winning first move is not known [12]. In chomp we know what happens for small cases, such as one or two rows, and there are partial results for three rows [5, 13, 14], but we do not know what a winning strategy is for more rows. In ultimately bipartite subtraction games we do not know what happens for small n but we do for large n.

Theorem 3. *If the subtraction game $\mathcal{S}(n, S)$ is ultimately bipartite, then for sufficiently large n, the game is winning if and only if n is odd.*

Proof. First notice that if $\mathcal{S}(n, S)$ is ultimately bipartite, then the elements of S are necessarily all odd. If $1 \in S$, the required result is given by Theorem 1. For $1 \notin S$, the result follows immediately from the following general fact:

Lemma 3. *Consider an arbitrary subtraction game $\mathcal{S}(n, S)$ for which the elements of the subtraction set S are all odd with $1 \notin S$. If n is odd and is a losing position, then $n - 1$ is also a losing position.*

Proof. This result is best understood by imagining that one is playing the game. Suppose that one finds oneself in a losing position n where n is odd. Then, regardless of how one plays, the other player can always force a win, provided they play intelligently. In this case, the game terminates after an even number of moves; so, as the elements of S are all odd, the game terminates at some odd position n_0. Note that $0 < n_0 < \min(S)$. Now consider the situation where one finds oneself initially in position $n - 1$. The other player can use the same strategy they used when the game started at n. The result is that, regardless of how one plays, after an even number of moves, one finds oneself in a position $n_0 - 1$, where $0 \leq n_0 - 1 < \min(S)$. So again one loses. □

This concludes the proof of Theorem 3. □

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