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An extension theorem for $[n, k, d]_q$ codes with $\gcd(d, q) = 2$

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Abstract

As a continuation of Maruta [*Finite Fields Appl.* 10 (2004), 674–685], we investigate the extendability of $[n, k, d]_q$ codes with $d \equiv -2 \pmod q$ whose weights are congruent to 0, -1 or $-2 \pmod q$ for even $q \geq 4$. We show that such codes are extendable for all even $q \geq 8$, giving a new extension theorem for $[n, k, d]_q$ codes with $gcd(d, q) = 2$. We also consider the extendability of such codes for $q = 4$.

1 Introduction

Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. The *weight* of a vector $\boldsymbol{a} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{a})$, is the number of nonzero coordinate positions in $\boldsymbol{a}$. A k-dimensional subspace of $\mathbb{F}_q^n$ is called a linear code over $\mathbb{F}_q$ of length n with dimension k, or an $[n, k]_q$ *code*. An $[n, k, d]_q$ *code* is an $[n, k]_q$ code with minimum weight d. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. We only consider *non-degenerate* codes having no coordinate which is identically zero. The code obtained by deleting the same coordinate from each codeword of an $[n, k, d]_q$ code $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n + 1, k, d + 1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$.

It is well-known that every binary linear code with odd minimum distance is extendable ([1]). Hill and Lizak [3] generalized this fact to linear codes over $\mathbb{F}_q$ by showing that every $[n, k, d]_q$ code with $gcd(d, q) = 1$ whose weights (i's such that $A_i > 0$) are congruent to 0 or $d \pmod q$ is extendable, see also [2]. Maruta [12] gave another extension theorem as follows.

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Theorem 1.1 ([12]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with odd $q \geq 5$, $d \equiv -2 \pmod q$, whose weights are congruent to 0, -1 or $-2 \pmod q$, $k \geq 3$. Then $\mathcal{C}$ is extendable.*

See [9], [10], [11], [15] for other results on the extendability of linear codes over $\mathbb{F}_q$. We note that all of known extension theorems in these papers require the condition $gcd(d, q) = 1$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with q even, $d \equiv -2 \pmod q$, whose weights are congruent to 0, -1 or $-2 \pmod q$, $k \geq 3$. Based on the weight distribution of the code, we define the *diversity* of $\mathcal{C}$ as the pair (Φ_0, Φ_1) with

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i>0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \equiv -1 \pmod q} A_i.$$

We assume $q = 2^h$, $h \geq 2$. Our goal is to prove the following new extension theorems:

Theorem 1.2. *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q = 2^h$ and $h \geq 3$, $d \equiv -2 \pmod q$, whose weights are congruent to 0, -1 or $-2 \pmod q$, $k \geq 3$. Then $\mathcal{C}$ is extendable.*

Theorem 1.3. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with diversity (Φ_0, Φ_1), $k \geq 3$, $d \equiv 2 \pmod 4$ such that $A_i = 0$ for all $i \equiv 1 \pmod 4$. Then*

(1) *$\mathcal{C}$ is extendable if there is a codeword $c \in \mathcal{C}$ with $wt(c) \equiv 3 \pmod 4$, i.e., $\Phi_1 > 0$.*

(2) *$\mathcal{C}$ is extendable if $\Phi_1 = 0$ and $\Phi_0 \in \{\theta_{k-2}, (\theta_{k-1} + \theta_{k-2} + 4^{k-2})/2\}$, where $\theta_j = (4^{j+1} - 1)/3$.*

See also Theorems 5.1, 5.3, 5.5 in Section 5 for the case $q = 4$ with $k = 3, 4$. From Theorems 1.1 and 1.2, we get the following.

Theorem 1.4. *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q \geq 5$, $d \equiv -2 \pmod q$, whose weights are congruent to 0, -1 or $-2 \pmod q$, $k \geq 3$. Then $\mathcal{C}$ is extendable.*

Applications. (1) Let $\mathcal{C}_1$ be a $[q^2 - 1, 4, q^2 - q - 2]_q$ code with $q \geq 5$. Let c be a codeword of $\mathcal{C}_1$ with weight $q^2 - q + e$. For $1 \leq e \leq q - 3$, the residual code of $\mathcal{C}_1$ with respect to c is a $[q - 1 - e, 3, q - 2 - e]_q$ code, which does not exist (see Theorem 2.7.1 of [7] for the residual code). Thus we have $A_i = 0$ for all $i \notin \{q^2 - q - 2, q^2 - q - 1, q^2 - q, q^2 - 2, q^2 - 1\}$. Applying Theorem 1.4, $\mathcal{C}_1$ is extendable. Actually, the extension of $\mathcal{C}_1$ is also extendable. It is known that the weight distribution of $\mathcal{C}_1$ is given by

$$(a_0, a_1, a_{q-1}, a_q, a_{q+1}) = (2, q^2 - 1, q + 1, 2(q^2 - 1), q^3 - 2q^2 + 1),$$

where $a_i = A_{q^2-1-i}/(q - 1)$. Hence the diversity of $\mathcal{C}_1$ is $(\theta_1, 2q^2)$. Considering the columns of a generator matrix of $\mathcal{C}_1$ as a $(q^2 - 1)$-set in $\mathrm{PG}(3, q)$ (see Section 2), we get a $(q^2 - 1)$-*cap* in $\mathrm{PG}(3, q)$, that is, a set of $q^2 - 1$ points no three of which are

collinear. Hence the above result is equivalent to saying that, for $q \geq 5$, a (q^2-1)-cap in PG$(3, q)$ is incomplete and extends to a (q^2+1)-cap (see Chapter 18 of [5]).
(2) Let $\mathcal{C}_2$ be a $[q, 3, q-2]_q$ code with $q \geq 5$. Since $\mathcal{C}_2$ is MDS, the weight distribution is uniquely determined, satisfying the condition of Theorem 1.4. Hence $\mathcal{C}_2$ is extendable. It is also known that the extension of $\mathcal{C}_2$ is also extendable for even q but not for odd q.
(3) Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$, where ω and $\bar{\omega}$ are the roots of $x^2 + x + 1 \in \mathbb{F}_2[x]$ and let $\mathcal{C}_3$ be the $[14, 3, 10]_4$ code with generator matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & \omega & \omega & \omega & \omega & \bar{\omega} & \bar{\omega} & \bar{\omega} & \bar{\omega} \\ \omega & \bar{\omega} & \omega & \bar{\omega} & \omega & \bar{\omega} & 0 & 1 & \omega & \bar{\omega} & 0 & 1 & \omega & \bar{\omega} \end{bmatrix}.$$

Then, $\mathcal{C}_3$ has weight distribution $(A_0, A_{10}, A_{12}) = (1, 42, 21)$ with diversity $(7, 0)$. Since $\Phi_0 + A_d/3 = 7 + 14 = 21$, $\mathcal{C}_3$ is not extendable by Theorem 5.3 in Section 5, although $\mathcal{C}_3$ satisfies $d \equiv -2 \pmod 4$ and the weights of codewords are congruent to 0 or $-2 \pmod 4$ as required in Theorem 1.2. Thus, the case $q = 4$ is exceptional. Similarly, the case $q = 3$ is also exceptional, see [13].
(4) Extension theorems are often used for optimal linear codes problem, especially to prove the nonexistence of linear codes with certain parameters. For example, the nonexistence of $[328, 4, 286]_8$, $[474, 4, 414]_8$, $[803, 4, 702]_8$ and $[858, 4, 750]_8$ codes attaining the Griesmer bound can be proved applying Theorem 1.2, see [8].

2 Geometric preliminaries

We denote by PG(r, q) the projective geometry of dimension r over $\mathbb{F}_q$. A *j-flat* is a projective subspace of dimension j in PG(r, q). 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes*, respectively. The number of points in a j-flat is $|\mathrm{PG}(j, q)| = \theta_j = (q^{j+1} - 1)/(q - 1)$, where $|T|$ denotes the number of elements in the set T. We refer to [4], [5] and [6] for geometric terminologies. We investigate linear codes over $\mathbb{F}_q$ through the projective geometry.

We assume that $k \geq 3$, see [10] for $k = 1, 2$. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix $G = [g_{ij}] = [g_1, \cdots, g_k]^{\mathrm{T}}$. Put $\Sigma =$PG$(k-1, q)$, the projective space of dimension $k - 1$ over $\mathbb{F}_q$. We consider the mapping w_G from Σ to $\{i \mid A_i > 0\}$, the set of non zero weights of $\mathcal{C}$. For $P = \mathbf{P}(p_1, \ldots, p_k) \in \Sigma$ we define *the weight of P with respect to G*, denoted by $w_G(P)$, as

$$w_G(P) = |\{j \mid \sum_{i=1}^{k} g_{ij} p_i \neq 0\}| = wt(\sum_{i=1}^{k} p_i g_i).$$

Let $F_d = \{P \in \Sigma \mid w_G(P) = d\}$. Recall that a hyperplane H of Σ is defined by a non-zero vector $h = (h_0, \ldots, h_{k-1}) \in \mathbb{F}_q^k$ as $H = \{\mathbf{P}(p_0, \ldots, p_{k-1}) \in \Sigma \mid h_0 p_0 + \cdots + h_{k-1} p_{k-1} = 0\}$. The vector h is called the *defining vector of H*.

Lemma 2.1 ([14]). *$\mathcal{C}$ is extendable if and only if there exists a hyperplane H of Σ such that $F_d \cap H = \emptyset$. Moreover, the extended matrix of G by adding the defining vector of H as a column generates an extension of $\mathcal{C}$.*

Proof For an $[n,k,d]_q$ code $\mathcal{C}$ with a generator matrix G, there exists a vector $h = (h_0,\dots,h_{k-1}) \in \mathbb{F}_q^k$ such that $[G, h^{\mathrm{T}}]$ generates an $[n+1,k,d+1]_q$ code if and only if $\sum_{i=0}^{k-1} h_i p_i \neq 0$ holds for all $P = \mathbf{P}(p_0,\dots,p_{k-1}) \in F_d$. Equivalently, there exists a hyperplane H with defining vector h such that $F_d \cap H = \emptyset$. □

Now, let

$$\begin{aligned} F_0 &= \{P \in \Sigma \mid w_G(P) \equiv 0 \pmod q\}, \\ F_1 &= \{P \in \Sigma \mid w_G(P) \not\equiv 0, d \pmod q\}, \\ F_2 &= \{P \in \Sigma \mid w_G(P) \equiv d \pmod q\}, \quad F = \Sigma \setminus F_2. \end{aligned}$$

Since $F_d \subset F_2$, we obtain the following lemma by Lemma 2.1.

Lemma 2.2. *$\mathcal{C}$ is extendable if there exists a hyperplane H of Σ such that $H \subset F$.*

Lemma 2.3 ([2])**.** *For two linearly independent vectors $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{F}_q^n$, it holds that*

$$\sum_{\lambda \in \mathbb{F}_q} wt(\boldsymbol{a}_1 + \lambda \boldsymbol{a}_2) + wt(\boldsymbol{a}_2) \equiv 0 \pmod q.$$

As a consequence of Lemma 2.3, we get the following.

Lemma 2.4. *For a line $L = \{P_0, P_1, \cdots, P_q\}$ in Σ, it holds that*

$$\sum_{i=0}^{q} w_G(P_i) \equiv 0 \pmod q.$$

A t-flat Π of Σ with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an $(i,j)_t$ *flat*. An $(i,j)_1$ flat is called an (i,j)-*line*. An (i,j)-*plane*, an (i,j)-*solid* and so on are defined similarly. We denote by $\mathcal{F}_j$ the set of j-flats of Σ. Let Λ_t be the set of all possible (i,j) for which an $(i,j)_t$ flat exists in Σ.

Now, let $\mathcal{C}$ be an $[n,k,d]_q$ code with diversity (Φ_0, Φ_1), $q = 2^h$, $h \geq 2$, $k \geq 3$, $d \equiv -2 \pmod q$ such that $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod q$. By Lemma 2.4, for any (i,j)-line l in Σ, we have

$$\sum_{P \in l} w_G(P) = 0i + (-1)j + (-2)(\theta_1 - (i+j)) \equiv 0 \pmod q.$$

So, $2i + j \equiv 2 \pmod q$. This yields the following.

Lemma 2.5.

$$\begin{aligned} \Lambda_1 &= \{(i,j) \mid 2i + j \equiv 2 \pmod q,\ 0 \leq i, j \leq \theta_1,\ i + j \leq \theta_1\} \\ &= \{(\frac{q}{2} + 1 - s, 2s) \mid 0 \leq s \leq \frac{q}{2}\} \cup \{(1,0), (0,2), (\theta_1, 0)\}. \end{aligned}$$

For a $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ flat Π_t, let $c_{i,j}^{(t)}$ be the number of $(i,j)_{t-1}$ flats in Π_t. Note that $\Phi_u = \varphi_u^{(k-1)}$ for $u = 1, 2$. The list of $c_{i,j}^{(t)}$'s is called the *spectrum* of Π_t. Then we have the following equations by usual counting arguments:

$$\sum_{(i,j)\in\Lambda_{t-1}} c_{i,j}^{(t)} = \theta_t, \tag{2.1}$$

$$\sum_{(i,j)\in\Lambda_{t-1}} i c_{i,j}^{(t)} = \theta_{t-1}\varphi_0^{(t)}, \tag{2.2}$$

$$\sum_{(i,j)\in\Lambda_{t-1}} j c_{i,j}^{(t)} = \theta_{t-1}\varphi_1^{(t)}, \tag{2.3}$$

$$\sum_{(i,j)\in\Lambda_{t-1}} \binom{i}{2} c_{i,j}^{(t)} = \theta_{t-2}\binom{\varphi_0^{(t)}}{2}, \tag{2.4}$$

$$\sum_{(i,j)\in\Lambda_{t-1}} \binom{j}{2} c_{i,j}^{(t)} = \theta_{t-2}\binom{\varphi_1^{(t)}}{2}, \tag{2.5}$$

$$\sum_{(i,j)\in\Lambda_{t-1}} \binom{i+j}{2} c_{i,j}^{(t)} = \theta_{t-2}\binom{\varphi_0^{(t)}+\varphi_1^{(t)}}{2}. \tag{2.6}$$

From (2.2), (2.3) and (2.6), we get

$$\sum_{(i,j)\in\Lambda_{t-1}} ij c_{i,j}^{(t)} = \binom{\varphi_0^{(t)}+\varphi_1^{(t)}}{2} - \binom{\varphi_0^{(t)}}{2} - \binom{\varphi_1^{(t)}}{2} = \theta_{t-2}\varphi_0^{(t)}\varphi_1^{(t)}. \tag{2.7}$$

Lemma 2.6. $\varphi_0^{(t)} \ge \theta_{t-2}$ *for* $t \ge 2$.

Proof We proceed by induction on t. For $t = 2$, if $\varphi_0^{(2)} = 0$, then every line in Π_2 is a $(0,2)$-line. We count the value of $\varphi_1^{(2)}$ in two ways: considering all lines passing through a fixed point of $F_d \cap \Pi_2$ we have $\varphi_1^{(2)} = \theta_1 + 1$, while the lines through a fixed point of $F_1 \cap \Pi_2$ yields $\varphi_1^{(2)} = 2\theta_1$, a contradiction. Hence $\varphi_0^{(2)} \ge 1$.

Assume our assertion for $t-1$, $t \ge 3$. Then, by the induction hypothesis and from the equations (2.1) and (2.2), we get

$$0 \le \sum_{(i,j)\in\Lambda_{t-1}} (i-\theta_{t-3}) c_{i,j}^{(t)} = \theta_{t-1}\varphi_0^{(t)} - \theta_{t-3}\theta_t,$$

which yields ${\varphi_0}^{(t)} \ge \theta_t\theta_{t-3}/\theta_{t-1} = \theta_{t-2} - 1 + \theta_{t-3}/\theta_{t-1} > \theta_{t-2} - 1$, so that ${\varphi_0}^{(t)} \ge \theta_{t-2}$. □

To prove Theorems 1.2 and 1.3, by Lemma 2.2 it suffices to prove the following three theorems.

Theorem 2.7. *Assume* $q = 2^h$, $h \geq 2$. *Let* Π_t *be a* $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ *flat,* $2 \leq t \leq k-1$. *Then,* Π_t *contains a* $(\theta_{t-2}, q^{t-1})_{t-1}$ *flat if* $\varphi_1^{(t)} > 0$.

Theorem 2.8. *Let* Π_t *be a* $(\varphi_0^{(t)}, 0)_t$ *flat,* $2 \leq t \leq k-1$. *Then,* Π_t *contains a* $(\theta_{t-1}, 0)_{t-1}$ *flat if* $q = 2^h, h \geq 3$.

Theorem 2.9. *Assume* $q = 4$. *Let* Π_t *be a* $(\varphi_0^{(t)}, 0)_t$ *flat,* $2 \leq t \leq k-1$. *Then,* Π_t *contains a* $(\theta_{t-1}, 0)_{t-1}$ *flat if* $\varphi_0^{(t)} = \theta_{t-1}$ *or* $(\theta_t + \theta_{t-1} + 4^{t-1})/2$.

For $t = k-1$, a $(\theta_{t-2}, q^{t-1})_{t-1}$ flat and a $(\theta_{t-1}, 0)_{t-1}$ flat are hyperplanes of Σ which are contained in F. Hence Theorem 1.2 follows from Theorems 2.7 and 2.8, and (1) and (2) of Theorem 1.3 follow from Theorems 2.7 and 2.9, respectively.

3 Proof of Theorem 2.7

Assume $q = 2^h$, $h \geq 2$. Let Π_t be a $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ flat, $2 \leq t \leq k-1$, with spectrum $c_{i,j}^{(t)}$'s. We assume that $\varphi_1^{(t)} > 0$ throughout this section. We first determine all possible $(\varphi_0^{(2)}, \varphi_1^{(2)}) \in \Lambda_2$ and the corresponding spectra.

Lemma 3.1. *When* $\varphi_1^{(2)} > 0$, *the following three equalities hold:*

$$\theta_1(2\varphi_0^{(2)} + \varphi_1^{(2)}) - (q+2)\theta_2 = -qc_{1,0}^{(2)} - qc_{0,2}^{(2)} + qc_{\theta_1,0}^{(2)} \tag{3.1}$$

$$\varphi_0^{(2)}(2\varphi_0^{(2)} + \varphi_1^{(2)} - \theta_2 - 1) = -qc_{1,0}^{(2)} + \theta_1 qc_{\theta_1,0}^{(2)} \tag{3.2}$$

$$\varphi_1^{(2)}(2\varphi_0^{(2)} + \varphi_1^{(2)} - \theta_2 - \theta_1) = -2qc_{0,2}^{(2)} \tag{3.3}$$

Proof Calculating $\sum_{(i,j)\in\Lambda_1}(2i + j - q - 2)c_{i,j}^{(2)}$ by way of (2.1)-(2.3), we get

$$2\theta_1\varphi_0^{(2)} + \theta_1\varphi_1^{(2)} - (q+2)\theta_2 = -qc_{1,0}^{(2)} - qc_{0,2}^{(2)} + qc_{\theta_1,0}^{(2)}.$$

Similarly, we can get (3.2) and (3.3) by calculating $\sum_{(i,j)\in\Lambda_1} i(2i+j-q-2)c_{i,j}^{(2)}$ and $\sum_{(i,j)\in\Lambda_1} j(2i+j-q-2)c_{i,j}^{(2)}$, respectively. □

A point P of $\Pi_t \cap F_0$ is *singular* if every line through P meets $\Pi_t \cap F_0$ in exactly one point or $q+1$ points. The set $\Pi_t \cap F_0$ is called *singular* or *non-singular* according as it has singular points or not [6].

An s-flat S is called an *axis* of Π_t *of type* (a, b) if every hyperplane of Π_t not containing S has diversity (a, b) and if there is no hyperplane of Π_t through S whose diversity is (a, b). Then the spectrum of $\Pi_t \cap F_0$ has $c_{a,b}^{(t)} = \theta_t - \theta_{t-1-s}$ and the axis is unique if it exists. The axis helps characterize the geometrical structure of Π_t. Π_t has an axis if $\Pi_t \cap F_0$ is singular.

Lemma 3.2. *Let* Π_2 *be a* $(\varphi_0^{(2)}, \varphi_1^{(2)})$*-plane with* $\varphi_1^{(2)} > 0$. *If* $c_{0,2}^{(2)} = 0$ *in* Π_2, *then*

$$(\varphi_0^{(2)}, \varphi_1^{(2)}; c_{1,0}^{(2)}, c_{r+1,q-2r}^{(2)}, c_{1,q}^{(2)}, c_{\theta_1,0}^{(2)}) = ((r+1)q+1, q^2-2rq; r, q^2, q-2r, r+1)$$

for some r *with* $0 \le r \le \frac{q}{2}$ *and there is a point of* F_0 *which is the axis of* Π_2 *of type* $(r+1, q-2r)$.

Proof From (3.3), (3.1) and (3.2), we get $2\varphi_0^{(2)} + \varphi_1^{(2)} = \theta_2 + \theta_1$, $c_{\theta_1,0}^{(2)} = c_{1,0}^{(2)} + 1$ and $\theta_1 c_{\theta_1,0}^{(2)} = \varphi_0^{(2)} + c_{1,0}^{(2)}$. We assume that $c_{1,0}^{(2)} = r$. Then we get $\varphi_0^{(2)} = qr + \theta_1$ and $\varphi_1^{(2)} = q^2 - 2rq$. If $r = 0$, we obtain $(\varphi_0^{(2)}, \varphi_1^{(2)}) = (\theta_1, q^2), c_{\theta_1,0}^{(2)} = 1$ and $c_{1,q}^{(2)} = q^2 + q$. If $r \ge 1$, we have $(\varphi_0^{(2)}, \varphi_1^{(2)}) = ((r+1)q+1, q^2-2rq), c_{\theta_1,0}^{(2)} = r+1$ and $c_{1,0}^{(2)} = r$. It follows from $\varphi_0^{(2)} + \varphi_1^{(2)} + rq = \theta_2$ that $\varphi_2^{(2)} = rq$. Let l_1, l_2 be $(\theta_1, 0)$-lines and put $P = l_1 \cap l_2$. Then r $(1,0)$-lines must be passing through P. Since $\varphi_2^{(2)} = rq$, other lines through P have no points of F_2, which must be $(\theta_1, 0)$-lines or $(1,q)$-lines from Λ_1. Hence, when $(\varphi_0^{(2)}, \varphi_1^{(2)}) = ((r+1)q+1, q^2-2rq)$, we obtain $c_{1,q}^{(2)} = q - 2r$, $c_{r+1,q-2r}^{(2)} = q^2$, and P is the axis of Π_2 of type $(r+1, q-2r)$. □

An n-set K in $\mathrm{PG}(2,q)$ is called an *n-arc* if every line of $\mathrm{PG}(2,q)$ meets K in at most two points. In the dual space of $\mathrm{PG}(2,q)$, the set of lines K is called an *n-arc of lines*. When q is even, it is known that $n \le q+2$ and that every q-arc is contained in a unique $(q+2)$-arc, see [4].

Lemma 3.3. *Let* Π_2 *be a* $(\varphi_0^{(2)}, \varphi_1^{(2)})$*-plane with* $\varphi_1^{(2)} > 0$. *If* $c_{0,2}^{(2)} > 0$ *and* $c_{1,0}^{(2)} = 0$ *in* Π_2, *then*

$$(\varphi_0^{(2)}, \varphi_1^{(2)}; c_{0,2}^{(2)}, c_{\frac{q}{2}+1,0}^{(2)}, c_{\frac{q}{2},2}^{(2)}, c_{1,q}^{(2)}) = (\frac{q^2-q+2}{2}, 2q; q, q-1, q^2-q, 2)$$

and the $(0,2)$*-lines and* $(1,q)$*-lines form a* $(q+2)$*-arc of lines.*

Proof We have $c_{\theta_1,0}^{(2)} = 0$ from $c_{0,2}^{(2)} > 0$. From (3.2), (3.3) and (3.1), we get $2\varphi_0^{(2)} + \varphi_1^{(2)} = \theta_2 + 1$, $c_{0,2}^{(2)} = \frac{\varphi_1^{(2)}}{2}$ and $c_{0,2}^{(2)} = q$. So, $(\varphi_0^{(2)}, \varphi_1^{(2)}, \varphi_2^{(2)}) = (\frac{q^2-q+2}{2}, 2q, \frac{q^2-q}{2})$. Let $l_1, l_2, \cdots, l_q$ be the $(0,2)$-lines. Then, we have $|l_1 \cup l_2 \cup \cdots \cup l_q| \ge \theta_1 q - \binom{q}{2} = \frac{q^2+3q}{2}$, where the equality holds when $l_1, \cdots, l_q$ form an arc of lines, that is, no three of $l_1, ..., l_{q+2}$ are concurrent. Since $|(l_1 \cup l_2 \cup \cdots \cup l_q) \cap F_1| \le 2q = \varphi_1$, we have

$$\begin{aligned} |(l_1 \cup l_2 \cup \cdots \cup l_q) \cap F_2| &= |l_1 \cup l_2 \cup \cdots \cup l_q| - |(l_1 \cup l_2 \cup \cdots \cup l_q) \cap F_1| \\ &\ge \frac{q^2+3q}{2} - 2q = \varphi_2. \end{aligned}$$

Hence it holds that $|(l_1 \cup l_2 \cup \cdots \cup l_q) \cap F_2| = \varphi_2$ and $|(l_1 \cup l_2 \cup \cdots \cup l_q) \cap F_1| = \varphi_1$. Thus, $l_1, \cdots, l_q$ form an arc of lines. Since there is a unique $(q+2)$-arc containing a given q-arc by Corollary 10.19 in [4], let l_{q+1}, l_{q+2} be the two lines so that $l_1, \cdots, l_q, l_{q+1}, l_{q+2}$ form a $(q+2)$-arc of lines. It follows from $(l_1 \cup l_2 \cup \cdots \cup l_q)^c = F_0 \cap \Pi_2$ that $l_{q+1} \cap l_{q+2} \in F_0$. And the points of $l_{q+1} \cup l_{q+2}$ other than $l_{q+1} \cap l_{q+2}$ are points

Table 1: Types of planes with ${\varphi_1}^{(2)} > 0$ where $1 \le r \le \frac{q}{2} - 2$ and $c_r^{(2)} = c_{r+1,q-2r}^{(2)}$

Type	${\varphi_0}^{(2)}$	${\varphi_1}^{(2)}$	$c_{1,0}^{(2)}$	$c_{0,2}^{(2)}$	$c_r^{(2)}$	$c_{\frac{q}{2},2}^{(2)}$	$c_{\frac{q}{2}+1,0}^{(2)}$	$c_{1,q}^{(2)}$	$c_{\theta_1,0}^{(2)}$
(a-1)	θ_1	q^2	0	0	0	0	0	$\theta_2 - 1$	1
(a-2)	$(r+1)q+1$	$q^2 - 2rq$	r	0	q^2	0	0	$q - 2r$	$r+1$
(a-3)	$(\frac{q}{2})q+1$	$2q$	$\frac{q}{2} - 1$	0	0	q^2	0	2	$\frac{q}{2}$
(a-4)	$\frac{q^2-q+2}{2}$	$2q$	0	q	0	$q^2 - q$	$q - 1$	2	0
(a-5)	1	$2q$	$q - 1$	q^2	0	0	0	2	0

of F_1 meeting $l_1, \cdots, l_q$ since $c_{1,0}^{(2)} = 0$. Hence, l_{q+1} and l_{q+2} are $(1, q)$-lines. Thus, we obtain the spectrum as claimed and that the $(0, 2)$-lines and $(1, q)$-lines form a $(q + 2)$-arc of lines. □

Lemma 3.4. *Let Π_2 be a $(\varphi_0^{(2)}, \varphi_1^{(2)})$-plane with $\varphi_1^{(2)} > 0$. If $c_{0,2}^{(2)} > 0$ and $c_{1,0}^{(2)} > 0$ in Π_2, then $(\varphi_0^{(2)}, \varphi_1^{(2)}; c_{1,0}^{(2)}, c_{0,2}^{(2)}, c_{1,q}^{(2)}) = (1, 2q; q-1, q^2, 2)$ and there is a point of F_0 which is the axis of Π_2 of type $(0, 2)$.*

Proof We have $c_{\theta_1,0}^{(2)} = 0$ from $c_{0,2}^{(2)} > 0$. Calculating $(2\varphi_0^{(2)} + \varphi_1^{(2)})\theta_1 - 2\theta_2 = \sum_{(i,j)\in\Lambda_1}(2i + j - 2)c_{i,j}^{(2)} = q\sum_{s=1}^{\frac{q}{2}} c_{\frac{q}{2}+1-s,2s}^{(2)} \ge 0$, we obtain $2\varphi_0^{(2)} + \varphi_1^{(2)} \ge 2q + \frac{2}{q+1}$. So, $2\varphi_0^{(2)} + \varphi_1^{(2)} \ge 2q + 1$. For any point R of $F_2 \cap \Pi_2$, considering the numbers of (i, j)-lines through R in Π_2, it follows from $2i + j \equiv 2 \pmod q$ for $(i, j) \in \Lambda_1$ that $2\varphi_0^{(2)} + \varphi_1^{(2)} \equiv 2 \pmod q$. Thus, $2\varphi_0^{(2)} + \varphi_1^{(2)} \ge 2\theta_1$. Let $2\varphi_0^{(2)} + \varphi_1^{(2)} = 2\theta_1 + xq$ for $x \in \mathbb{N} \cup \{0\}$. Then, we have $c_{1,0}^{(2)} = \varphi_0^{(2)}(q - x - 1)$ from (3.2) and $x \le q - 2$ from $c_{1,0}^{(2)} > 0$. Calculating $-2 \times (3.1) + 2 \times (3.2) + (3.3)$, we obtain

$$(2\varphi_0^{(2)} + \varphi_1^{(2)})^2 - (q^2 + 3q + 4)(2\varphi_0^{(2)} + \varphi_1^{(2)}) + 2q^3 + 6q^2 + 6q + 4 - q\varphi_1^{(2)} = 0 \quad (3.4)$$

from $c_{\theta_1,0}^{(2)} = 0$. Substituting $2\varphi_0^{(2)} + \varphi_1^{(2)} = 2\theta_1 + xq$ to (3.4), we get

$$\varphi_1^{(2)} = 2q + xq(x - q + 1) \quad (3.5)$$

which implies $x = 0$, for the right hand side of (3.5) is at most 0 for $1 \le x \le q - 2$, a contradiction. Since $(\varphi_0^{(2)}, \varphi_1^{(2)}) = (1, 2q)$, we get $c_{1,0}^{(2)} = q - 1, c_{0,2}^{(2)} = q^2$ from (3.2) and (3.3). Let P be the unique point of F_0. Then, all $(1, 0)$-lines are passing through P. The remaining two lines through P contain $2q$ points of F_1, so $c_{1,q}^{(2)} = 2$. Hence, P is the axis of Π_2 of type $(0, 2)$. □

From Theorems 3.2, 3.3 and 3.4, we obtain Table 1.
We denote by $\langle \chi_1, \chi_2, \cdots \rangle$ the smallest flat containing subsets $\chi_1, \chi_2, \cdots$ of Σ.

Lemma 3.5. *Let Π_2 be a $(\varphi_0^{(2)}, \varphi_1^{(2)})$-plane with $\varphi_1^{(2)} > 0$.*

(1) *If* $(\varphi_0^{(2)}, \varphi_1^{(2)}) \neq (\theta_1, q^2)$*, then there is a point* $P \in F_0$ *such that* $\langle P, Q\rangle$ *is a* $(1,q)$*-line for any* $Q \in F_1$*.*
(2) *If there exist two* $(1,q)$*-lines meeting in a point of* $F_1 \cap \Pi_2$*, then* Π_2 *is a* (θ_1, q^2)*-plane.*

Proof We consider the geometric structure of the planes of Table 1. In the cases (a-2) and (a-3), the plane has a point of F_0 which is the axis, so that any two $(1,q)$-lines meet in a point of F_0. And the two $(1,q)$-lines also meet in a point of F_0 in the cases (a-4) and (a-5). Hence, (a-1) is the only case that there exist two $(1,q)$-lines meeting in a point of $F_1 \cap \Pi_2$. □

Lemma 3.6 ([10])**.** *Let* π *be a proper subset of* Σ*. Then* π *is a hyperplane of* Σ *if and only if every line in* Σ *meets* π *in one point or in* $q+1$ *points.*

Lemma 3.7. *Let* Π_t *be a* $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ *flat with* $\varphi_1^{(t)} > 0$ *and* $\varphi_2^{(t)} = 0$*. Then* $(\varphi_0^{(t)}, \varphi_1^{(t)}; c^{(t)}_{\theta_{t-1},0}, c^{(t)}_{\theta_{t-2},q^{t-1}}) = (\theta_{t-1}, q^t; 1, \theta_t - 1)$*.*

Proof It follows from the condition $\varphi_2^{(t)} = 0$ and Λ_1 that Π_t has only $(\theta_1, 0)$-lines or $(1,q)$-lines. From $\varphi_1^{(t)} > 0$, $\Pi_t \cap F_0$ is a proper subset of Π_t. By Lemma 3.6, $\Pi_t \cap F_0$ is a hyperplane of Π_t. Hence our assertion follows. □

We prove the following lemma in the proof of Lemma 3.9 while the assertion for $t = 2$ follows from Lemma 3.5 (2).

Lemma 3.8. *Let* Π_t *be a* $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ *flat with* $t \geq 2$ *containing two* $(\theta_{t-2}, q^{t-1})_{t-1}$ *flats* π_1 *and* π_2*. If* $\pi_1 \cap \pi_2$ *is a* $(\theta_{t-3}, q^{t-2})_{t-2}$ *flat, then* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, q^t)$*.*

Lemma 3.9. *Let* Π_t *be a* $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ *flat with* $\varphi_1^{(t)} > 0$ *and* $\varphi_2^{(t)} > 0$*. Then,*
(1) Π_t *contains a* $(\theta_{t-3}, q^{t-2})_{t-2}$ *flat.*
(2) *For a given* $(\theta_{t-3}, q^{t-2})_{t-2}$ *flat* Δ_1*, there exists a* $(\theta_{t-2}, q^{t-1})_{t-1}$ *flat* π_0 *in* Π_t *containing* Δ_1*.*
(3) Π_t *contains a* $(\theta_{t-2}, 0)_{t-2}$ *flat* S *which is the axis of* Π_t*.*
(4) *Let* Q *be a point of* F_1*. Then,* Q *is contained in a* $(\theta_{t-2}, q^{t-1})_{t-1}$ *flat through* S*.*

Proof We proceed by induction on t. Assume $t = 3$. (1) is obvious, for Π_3 contains a $(1,q)$-line from Table 1. Let l_1 be a $(1,q)$-line in Π_3 and Q_1 be a point of $F_1 \cap l_1$. Since $\varphi_2^{(t)} > 0$, let R be a point of F_2 and put $l = \langle Q_1, R\rangle$. Now, take a plane δ of Π_3 through l which does not contain l_1. By Lemma 3.5 (1), we can take a $(1,q)$-line l_2 through Q_1 in δ. Then, $\delta_0 = \langle l_1, l_2\rangle$ is a (θ_1, q^2)-plane by Lemma 3.5 (2). Hence, (2) holds. From Lemma 3.7, δ_0 contains a $(\theta_1, 0)$-line, say L. We prove that any point Q of F_1 is contained in a (θ_1, q^2)-plane through L. Assume that $Q \notin \delta_0$, since it is obvious when $Q \in \delta_0$. Let δ_1 be a plane which contains $R \in F_2$ and Q, not containing L. Take a point $P_1 = L \cap \delta_1$. Then, $\delta_1 \cap \delta_0$ is a $(1,q)$-line, and P_1 is on the $(1,q)$-lines of δ_1 by Lemma 3.5 (1). So, the line $l' = \langle Q, P_1\rangle$ is a $(1,q)$-line. Take the plane $\delta' = \langle L, l' \rangle$. Let P_2 be a point of L which is not P_1 and let δ_2 be a plane through the line $\langle P_2, R \rangle$, not containing L. Then, $\delta_0 \cap \delta_2$ is a $(1,q)$-line. Hence P_2 is on the $(1,q)$-lines of δ_2 by Lemma 3.5 (1). From $l' \cap \delta_2 \in F_1$, $\delta_2 \cap \delta'$ is a $(1,q)$-line. It follows from Lemma 3.5 (2) that δ' is a (θ_1, q^2)-plane. Thus, every point of F_1 is on a (θ_1, q^2)-plane through L, and other planes through L are $(\theta_1, 0)$-planes. This implies that L is the axis of π. Hence (3) and (4) hold. It also follows that the solid containing two (θ_1, q^2)-planes through a $(1,q)$-line is a (θ_2, q^3)-solid. Thus, Lemma 3.8 holds for $t = 3$.

Next, assume (1)-(4) and Lemma 3.8 for $t-1$, $t \geq 4$. Let Π_t be a $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ flat with $\varphi_1^{(t)} > 0$ and $\varphi_2^{(t)} > 0$. Take a hyperplane $\hat{\pi}$ of Π_t containing a point of F_1 and a point of F_2. Then, by the induction hypothesis for (2), $\hat{\pi}$ contains a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat. Hence (1) holds. Let Δ_1 be a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat in Π_t and δ be a $(\theta_{t-4}, q^{t-3})_{t-3}$ flat in Δ_1. Let R be a point of F_2 and put $\Delta = \langle \delta, R \rangle$. Now, take a $(t-1)$-flat π of Π_t through Δ which does not contain Δ_1. By the induction hypothesis for (3) and (4), we can take a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat Δ_2 through δ in π. Then, $\pi_0 = \langle \Delta_1, \Delta_2 \rangle$ is a $(\theta_{t-2}, q^{t-1})_{t-1}$ flat by the induction hypothesis for Lemma 3.8. Hence (2) holds. From Lemma 3.7, π_0 contains a $(\theta_{t-2}, 0)_{t-2}$ flat, say S. We prove that any point $Q \in F_1$ is contained in a $(\theta_{t-2}, q^{t-1})_{t-1}$ flat through S. Assume that $Q \notin \pi_0$, since it is obvious when $Q \in \pi_0$. Let π_1 be a $(t-1)$-flat which contains $R \in F_2$ and Q, not containing S. Take a $(\theta_{t-3}, 0)_{t-3}$ flat $\delta_1 = S \cap \pi_1$. Then, $\pi_0 \cap \pi_1$ is a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat, and δ_1 is on the $(\theta_{t-3}, q^{t-2})_{t-2}$ flats of π_1 by the induction hypothesis for (3) and (4). So, $\Delta' = \langle Q, \delta_1 \rangle$ is a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat. Take the $(t-1)$ flat $\pi' = \langle S, \Delta' \rangle$. Let δ_2 be a $(\theta_{t-3}, 0)_{t-3}$ flat of S which is not δ_1 and let π_2 be a $(t-1)$-flat through δ_2, not containing S. Then, $\pi_0 \cap \pi_2$ is a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat. Hence δ_2 is on the $(\theta_{t-3}, q^{t-2})_{t-2}$ flat by the induction hypothesis for (3). Since $\Delta' \cap \pi_2$ is a $(\theta_{t-4}, q^{t-3})_{t-3}$ flat, $\pi_2 \cap \pi'$ is a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat. It follows from the induction hypothesis for Lemma 3.8 that π' is a $(\theta_{t-2}, q^{t-1})_{t-1}$ flat. Thus, every point of F_1 is on a $(\theta_{t-2}, q^{t-1})_{t-1}$ flat through S, and other $(t-1)$-flats through S are $(\theta_{t-2}, 0)_{t-1}$ flats. This implies that S is the axis of Π_t. Hence (3) and (4) hold. It also follows that the t flat containing two $(\theta_{t-2}, q^{t-1})_{t-1}$ flats through a $(\theta_{t-3}, q^{t-2})_{t-2}$ flat is a $(\theta_{t-1}, q^t)_t$ flat. Thus, we also complete the proof of Lemma 3.8. □

As a consequence of Lemma 3.9, we get the following.

Lemma 3.10. *Let* Π_t *be a* $(\varphi_0^{(t)}, \varphi_1^{(t)})_t$ *flat with* $\varphi_1^{(t)} > 0$ *and* $\varphi_2^{(t)} > 0$. *Then,* Π_t *contains a* $(\theta_{t-2}, q^{t-1})_{t-1}$ *flat.*

Hence Theorem 2.7 follows from Lemmas 3.10 and 3.7.

4 Proof of Theorem 2.8

We assume that $\varphi_1^{(t)} = 0$ and $q = 2^h$, $h \geq 3$ throughout this section.

We proceed by induction on t. For $t = 2$, we get Table 2 from Theorem 19.4.4 in [5] with $n = \frac{q}{2} + 1$, since $|F_0 \cap l| = 1, \frac{q}{2} + 1$ or θ_1 for any line l (we use the symbols in [5] as Type in Tables 2-6). Any plane has a $(\theta_1, 0)$-line as claimed.

Table 2: Types of planes with ${\varphi_1}^{(2)}=0$, $q=2^h$, $h\geq 3$

Type	${\varphi_0}^{(2)}$	${\varphi_1}^{(2)}$	$c^{(2)}_{1,0}$	$c^{(2)}_{\frac{q}{2}+1,0}$	$c^{(2)}_{\theta_1,0}$
(III)	$\frac{q^2+q+2}{2}$	0	θ_1	q^2-1	1
(IV)	$\frac{q^2+3q+2}{2}$	0	0	q^2-1	$q+2$
(V)	$\frac{q^2+2q+2}{2}$	0	$\frac{q}{2}$	q^2	$\frac{q}{2}+1$
(VI)	θ_1	0	θ_2-1	0	1
(VII)	θ_2	0	0	0	θ_2

Table 3: Types of solids with ${\varphi_1}^{(3)}=0$, $q=2^h$, $h\geq 3$

Type	${\varphi_0}^{(3)}$	${\varphi_1}^{(3)}$	$c^{(3)}_{\frac{q^2+q+2}{2},0}$	$c^{(3)}_{\frac{q^2+3q+2}{2},0}$	$c^{(3)}_{\frac{q^2+2q+2}{2},0}$	$c^{(3)}_{\theta_1,0}$	$c^{(3)}_{\theta_2,0}$
(III)	$\frac{q^3+q^2+2q+2}{2}$	0	$\theta_3-\theta_2$	0	q^2-1	θ_1	1
(IV)	$\frac{q^3+3q^2+2q+2}{2}$	0	0	$\theta_3-\theta_2$	q^2-1	0	$q+2$
(V)	$\frac{q^3+2q^2+2q+2}{2}$	0	0	0	$\theta_3-\theta_1$	$\frac{q}{2}$	$\frac{q}{2}+1$
(VI)	θ_2	0	0	0	0	θ_3-1	1
(VII)	θ_3	0	0	0	0	0	θ_3
$(\mathscr{R}_3)$	$\frac{q^3+2q^2+2q+2}{2}$	0	$\frac{\theta_3-\theta_2}{2}$	$\frac{\theta_3-\theta_2}{2}$	θ_2-1	0	1

For $t=3$, we also get Table 3 for possible solids from Theorems 19.4.8 and 19.4.9 in [5]. Thus, any solid has a $(\theta_2,0)$-plane.

Assume the assertion of Theorem 2.8 for $t-1$, $t\geq 4$. We first assume that Π_t does not contain a $(\frac{q^2+3q+2}{2},0)$-plane. If $\Pi_t\cap F_0$ is non-singular, then we have $\frac{q}{2}+1=n=\sqrt{q}+1$ by Lemma 23.5.15 in [6], a contradiction. Hence $\Pi_t\cap F_0$ is singular. Let $P\in F_0\cap\Pi_t$ be a singular point and let π be a $(t-1)$-flat in Π_t not containing P. By the induction hypothesis, π contains a $(\theta_{t-2},0)_{t-2}$ flat δ. It follows from the singularity that $\langle\delta,P\rangle$ is a $(\theta_{t-1},0)_{t-1}$ flat. Thus Π_t contains a $(\theta_{t-1},0)_{t-1}$ flat. Next, assume that Π_t contains a $(\frac{q^2+3q+2}{2},0)$-plane. Then, Π_t has a $(\theta_{t-1},0)_{t-1}$ flat from Theorem 23.6.1 in [6]. This completes the proof of Theorem 2.8.

5 Proof of Theorem 2.9

We assume that $q=4$ throughout this section. Let $\mathcal{C}$ be an $[n,k,d]_4$ code with diversity (Φ_0,Φ_1), $k\geq 3$, $d\equiv -2$ satisfying $A_i=0$ for all $i\equiv 1 \pmod 4$. If $\Phi_1>0$, then $\mathcal{C}$ is extendable by Theorem 2.7. So, we only consider the case when $\Phi_1=0$. Let Π_t be a $(\varphi_0^{(t)},0)_t$ flat in $\Sigma=\mathrm{PG}(k-1,4)$.

For $t=2$, we can obtain Table 4 for possible planes from Theorem 19.4.4 in [5]. Hence, when $k=3$, $\mathcal{C}$ is extendable if $\Phi_0\notin\{7,9\}$, for $c^{(2)}_{\theta_1,0}>0$.

Theorem 5.1. *Let $\mathcal{C}$ be an $[n,3,d]_4$ code with diversity $(\Phi_0,0)$, $d\equiv 2 \pmod 4$ such that $A_i=0$ for all $i\equiv 1 \pmod 4$. Then $\mathcal{C}$ is extendable if $\Phi_0\notin\{7,9\}$.*

Table 4: Types of planes with ${\varphi_1}^{(2)} = 0, q = 4$

Type	${\varphi_0}^{(2)}$	${\varphi_1}^{(2)}$	$c^{(2)}_{1,0}$	$c^{(2)}_{3,0}$	$c^{(2)}_{\theta_1,0}$
(I)	9	0	9	12	0
(II)	7	0	14	7	0
(III)	11	0	5	15	1
(IV)	15	0	0	15	6
(V)	13	0	2	16	3
(VI)	5	0	20	0	1
(VII)	21	0	0	0	21

The next lemma follows from the spectra of (II) and (I) in Table 4.

Lemma 5.2. *Let* Π_2 *be a* $(\varphi_0^{(2)}, 0)$*-plane in* Σ.
(1) $F_0 \cap \Pi_2$ *forms a Fano plane if* $\varphi_0^{(2)} = 7$.
(2) $F_0 \cap \Pi_2$ *forms a Hermitian curve if* $\varphi_0^{(2)} = 9$.

In the cases of Lemma 5.2, every line in Π_2 meets F_d in at least one point if $|(\Pi_2 \cap F_2) \setminus F_d| \leq 1$. Since $|(\Pi_2 \cap F_2) \setminus F_d| = \theta_2 - \varphi_0^{(2)} - |\Pi_2 \cap F_d|$, the following holds.

Theorem 5.3. *Let* $\mathcal{C}$ *be an* $[n, 3, d]_4$ *code with diversity* $(\Phi_0, 0)$, $\Phi_0 \in \{7, 9\}$, $d \equiv 2 \pmod 4$ *such that* $A_i = 0$ *for all* $i \equiv 1 \pmod 4$. *Then* $\mathcal{C}$ *is not extendable if* $\Phi_0 + A_d/3 \geq 20$.

For $t = 3$, we can obtain Table 5 for possible solids from Theorems 19.4.8, 19.4.9 and 19.5.13 in [5].

Lemma 5.4. *Let* Π_3 *be a* $(\varphi_0^{(3)}, 0)$*-solid. Then,* $\Pi_3 \cap F_0$ *contains a plane if* $\varphi_0^{(3)} \in \{21, 53, 61, 85\}$.

Proof From Table 5, every $(\varphi_0^{(3)}, 0)$-solid with $\varphi_0^{(3)} \in \{21, 53, 61, 85\}$ contains a $(21, 0)$-plane. □

As a consequence of Lemma 5.4, we get the following. Note that $\Phi_0 \neq \theta_3$ since $F_d \neq \emptyset$.

Theorem 5.5. *Let* $\mathcal{C}$ *be an* $[n, 4, d]_4$ *code with diversity* $(\Phi_0, 0)$, $d \equiv 2 \pmod 4$ *such that* $A_i = 0$ *for all* $i \equiv 1 \pmod 4$. *Then* $\mathcal{C}$ *is extendable if* $\Phi_0 \in \{21, 53, 61\}$.

Lemma 5.6. *Let* Π_t *be a* $(\theta_{t-1}, 0)_t$ *flat. Then,* $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-1},0}) = (\theta_t - 1, 1)$.

Table 5: Types of planes with ${\varphi_1}^{(2)} = 0, q = 4$

Type	${\varphi_0}^{(3)}$	${\varphi_1}^{(3)}$	$c^{(3)}_{9,0}$	$c^{(3)}_{7,0}$	$c^{(3)}_{11,0}$	$c^{(3)}_{15,0}$	$c^{(3)}_{13,0}$	$c^{(3)}_{5,0}$	$c^{(3)}_{21,0}$
(I)	37	0	64	0	0	0	12	9	0
(II)	29	0	0	64	0	0	7	14	0
(III)	45	0	0	0	64	0	15	5	1
(IV)	61	0	0	0	0	64	15	0	6
(V)	53	0	0	0	0	0	80	2	3
(VI)	21	0	0	0	0	0	0	84	1
(VII)	85	0	0	0	0	0	0	0	85
($\mathscr{U}_3$)	45	0	40	0	0	0	45	0	0
($\mathscr{R}_3$)	53	0	0	0	32	32	20	0	1
($\mathscr{K}^*$)	37	0	16	32	32	0	4	1	0
($\mathscr{S}_{IV}$)	33	0	15	45	18	1	0	6	0
($\mathscr{S}_{III}$)	41	0	15	15	46	3	5	1	0
($\mathscr{S}_{II}$)	49	0	7	1	42	21	14	0	0
($\mathscr{T}$)	45	0	8	8	48	8	13	0	0

Proof Suppose Π_t contains a $(3,0)$-line l. Then every plane in Π_t through l contains at least seven points of F_0 from Table 4. Counting the number of points of F_0 in the planes through l, we have $\varphi_0^{(t)} \geq (7-3)\theta_{t-2} + 3 = \theta_{t-1} + 2$, a contradiction. Hence, Π_t has no $(3,0)$-line, and possible lines are $(1,0)$-lines or $(5,0)$-lines. Since $|\Pi_t \cap F_0| = \theta_{t-1}$, $\Pi_t \cap F_0$ is a hyperplane of Π_t by Lemma 3.6, and we get the desired spectrum. □

Now, we set $\nu_t = \frac{\theta_t + \theta_{t-1}}{2}$, $\eta_t = \frac{\theta_t + \theta_{t-1} + 4^{t-1}}{2}$ for $t \geq 3$.

Lemma 5.7. *Let Π_t be an $(\eta_t, 0)_t$ flat. Then, Π_t has no $(i,0)$-plane for $i \in \{7,9,11\}$.*

Proof In case $t = 3$, Π_t has no $(i,0)$-plane for $i \in \{7,9,11\}$ since $(\varphi_0^{(3)}, 0) = (61,0), (c^{(3)}_{13,0}, c^{(3)}_{15,0}, c^{(3)}_{21,0}) = (15,64,6)$.
In case $t \geq 4$, suppose Π_t contains a $(7,0)$-plane δ. Then, every solid in Π_t through δ contains at most 49 points of F_0 from Table 5. Counting the number of points of F_0 in the solids through δ, we have

$$\varphi_0^{(t)} \leq (49-7)\theta_{t-3} + 7 = \frac{\theta_t + \theta_{t-1} + \theta_{t-2} - 13}{2} < \eta_t,$$

a contradiction. Hence, Π_t has no $(7,0)$-plane. Similarly, it can be checked that Π_t has no $(9,0)$-plane and no $(11,0)$-plane by counting arguments. Thus, there is no $(i,0)$-plane for $i \notin \{5,13,15,21\}$. □

Lemma 5.8. *Let Π_t be a $(\varphi_0^{(t)}, 0)_t$ flat. Then, $\varphi_0^{(t)} \in \{\eta_t, \nu_t, \theta_{t-1}, \theta_t\}$ if Π_t has no $(i,0)$-plane with $i \in \{7,9,11\}$.*

Table 6: Possible types of solids in Π_t of Lemma 5.8

Type	${\varphi_0}^{(3)}$	${\varphi_1}^{(3)}$	$c_{5,0}^{(3)}$	$c_{13,0}^{(3)}$	$c_{15,0}^{(3)}$	$c_{21,0}^{(3)}$
(VI)	21	0	84	0	0	1
(V)	53	0	2	80	0	3
(IV)	61	0	0	15	64	6
(VII)	85	0	0	0	0	85

Proof A possible plane in Π_t is a $(5,0)$-plane, a $(13,0)$-plane, a $(15,0)$-plane or a $(21,0)$-plane. Then, we have Table 6 as possible types of solids in Π_t from Table 5.
(1) Suppose Π_t contains a $(15,0)$-plane δ_1. Then, every solid in Π_t through δ_1 is a $(61,0)$-solid from Table 6. Counting the number of points of F_0 in the solids through δ_1, we have $\varphi_0^{(t)} = (61-15)\theta_{t-3} + 15 = \eta_t$.
(2) Suppose Π_t has no $(15,0)$-plane and contains a $(13,0)$-plane δ_2. Then, every solid in Π_t through δ_2 is a $(53,0)$-solid from Table 6. Counting the number of points of F_0 in the solids through δ_2, we have $\varphi_0^{(t)} = (53-13)\theta_{t-3} + 13 = \nu_t$.
(3) Suppose Π_t has no $(15,0)$-plane and no $(13,0)$-plane and that Π_t contains a $(5,0)$-plane δ_3. Then, every solid in Π_t through δ_3 is a $(21,0)$-solid from Table 6. Counting the number of points of F_0 in the solids through δ_3, we have $\varphi_0^{(t)} = (21-5)\theta_{t-3} + 5 = \theta_{t-1}$.
(4) Suppose Π_t has none of a $(15,0)$-plane, a $(13,0)$-plane and a $(5,0)$-plane. Then, every solid in Π_t is a $(85,0)$-solid from Table 6, and we have $\varphi_0^{(t)} = \theta_t$. □

Lemma 5.9. *Let Π_t be an $(\eta_t, 0)_t$ flat. Then, the spectrum of Π_t is*

$$(c_{\nu_{t-1},0}^{(t)}, c_{\eta_{t-1},0}^{(t)}, c_{\theta_{t-1},0}^{(t)}) = (15, \theta_t - \theta_2, 6).$$

Proof We proceed by induction on t. For $t = 3$, the result follows from Table 5. Assume this for $t-1$, $t \geq 4$. From Lemma 5.8, a possible $(t-1)$-flat in Π_t is a $(\theta_{t-2}, 0)_{t-1}$ flat, an $(\eta_{t-1}, 0)_{t-1}$ flat, a $(\nu_{t-1}, 0)_{t-1}$ flat or a $(\theta_{t-1}, 0)_{t-1}$ flat. And Π_t has an $(\eta_{t-1}, 0)_{t-1}$ flat since a solid through a $(15,0)$-plane is only a $(61,0)$-solid. Then, by the induction hypothesis, Π_t has $(\eta_{t-2}, 0)_{t-2}$ flats, $(\nu_{t-2}, 0)_{t-2}$ flats and $(\theta_{t-2}, 0)_{t-2}$ flats. In Π_t, there are exactly four $(\eta_{t-1}, 0)_{t-1}$ flats and a $(\nu_{t-1}, 0)_{t-1}$ flat through a $(\nu_{t-2}, 0)_{t-2}$ flat, there are exactly five $(\eta_{t-1}, 0)_{t-1}$ flats through an $(\eta_{t-2}, 0)_{t-2}$ flat and there are either

(a) four $(\eta_{t-1}, 0)_{t-1}$ flats and a $(\theta_{t-1}, 0)_{t-1}$ flat or

(b) three $(\nu_{t-1}, 0)_{t-1}$ flats and two $(\theta_{t-1}, 0)_{t-1}$ flats

through a $(\theta_{t-2}, 0)_{t-2}$ flat. From the induction hypothesis, the spectrum of an $(\eta_{t-1}, 0)_{t-1}$ flat is $(c_{\nu_{t-2},0}^{(t-1)}, c_{\eta_{t-2},0}^{(t-1)}, c_{\theta_{t-2},0}^{(t-1)}) = (15, \theta_{t-1} - \theta_2, 6)$. Hence, we have $c_{\nu_{t-1},0}^{(t)} = 15 \times 1 = 15$, $c_{\eta_{t-1},0}^{(t)} = 15 \times (4-1) + (5-1) \times (\theta_{t-1} - \theta_2) + 6 \times (4-1) + 1 = \theta_t - \theta_2$ and $c_{\theta_{t-1},0}^{(t)} = 6 \times 1 = 6$. □

Now, Theorem 2.9 follows from Lemmas 5.6 and 5.9.

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