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A family of translation planes

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Abstract

An infinite family of non-Desarguesian translation planes of order q^4 with kernel $\mathrm{GF}(q^2)$ is constructed, for any odd prime power q. The collineation group of each plane has orbits of lengths 1, q^2, and $q^4 - q^2$ on the translation line. The method used for the construction is net replacement, starting from a Dickson-Knuth semifield plane.

The planes are constructed via spreads and the spreads via spread sets.

A **spread set** is a set $\mathbb{S}$ of q^2 matrices, each 2 by 2, with entries in $\mathrm{GF}(q)$ such that the difference of any two of them is non-singular. Given a spread set $\mathbb{S}$, a spread $\pi_{\mathbb{S}}$ of $\mathrm{PG}(3,q)$ arises. (The elements of $\pi_{\mathbb{S}}$ in $\mathrm{GF}(q)^4$ are the line $x_1 = x_2 = 0$ and the lines $\{(x_1, x_2, x_3, x_4) \mid (x_1, x_2)A = (x_3, x_4)\}$, for $A \in \mathbb{S}$.) Applying the Andre/Bruck-Bose construction to a spread of $\mathrm{PG}(3,q)$ gives a translation plane of order q^2 with kernel containing $\mathrm{GF}(q)$ [1].

Let $F = \mathrm{GF}(q^2) > \mathrm{GF}(q) = K$, q odd, with corresponding automorphism $x \mapsto x^q = \bar{x}$ defining the conjugate $\bar{x}$ of x, and the corresponding norm $x \mapsto x^\nu = x\bar{x}$, $x \in F$. Fix $\eta \in F, \eta \notin K$ such that $\bar{\eta} = -\eta$; in general if $\bar{\alpha} = -\alpha$, then α is **skew-symmetric**.

Remark 1 If $\alpha, \beta \in F$, with $\bar{\alpha} = -\alpha$, then $\alpha \pm \beta^\nu = 0$ implies $\alpha = \beta = 0$. (This follows from adding the equation to its conjugate.)

We start with two copies of a Dickson-Knuth semifield plane.

Lemma 1
The following sets of matrices are spread sets closed under addition.

$$\begin{aligned}
\Delta^+ &= \left\{ \begin{pmatrix} s & t \\ \bar{s} + \bar{t} & \eta\bar{s} + \bar{t} \end{pmatrix} \mid s, t \in F \right\}; \\
\Delta^- &= \left\{ \begin{pmatrix} s & t \\ \bar{s} - \bar{t} & \eta\bar{s} + \bar{t} \end{pmatrix} \mid s, t \in F \right\}.
\end{aligned}$$

Proof. Since Δ^+ and Δ^- are closed under addition, we need only check that their non-zero elements are nonsingular. Since $\bar{\eta} = -\eta$, the determinants of the generic elements of Δ^+ and Δ^- are the sum of a skew symmetric part $(\eta s\bar{s} + (s\bar{t} - \bar{s}t))$ and $\pm t^\nu$, which by the remark is non-zero if s and t are non-zero. □

Fix $c \in F^*$ to be a non-square, and let $\delta_{1,2}$ denote the entry at location $(1,2)$ of any matrix $M \in \Delta^+ \cup \Delta^-$. Then M is considered **square** or **non-square** according to whether $\delta_{1,2} = \theta^2$, $\theta \in F^*$ or $\delta_{1,2} = c\theta^2$, $\theta \in F^*$, respectively. Let $Sq(\Delta^\pm)$ and $NSq(\Delta^\pm)$ be respectively the partial spread sets of all square and non-square matrices in $\Delta^\pm$. We claim that the partial subspread associated with $NSq(\Delta^+) \subset \Delta^+$ and $NSq(\Delta^-) \subset \Delta^-$ are replacements of each other. This will follow by noting the following.

Lemma 2
If $A \in NSq(\Delta^+)$, $B \in Sq(\Delta^-)$, *then* $\det(B - A) = 0$.

Proof. The difference of distinct typical elements may be expressed (using the notation $\sigma = s - s'$) as

$$\begin{pmatrix} s & \theta^2 \\ \bar{s} - \bar{\theta}^2 & \eta\bar{s} + \bar{\theta}^2 \end{pmatrix} - \begin{pmatrix} s' & c\phi^2 \\ \bar{s} + \overline{c\phi^2} & \eta s + \overline{c\phi^2} \end{pmatrix} = \begin{pmatrix} \sigma & \theta^2 - c\phi^2 \\ \bar{\sigma} - \overline{(\theta^2 + c\phi^2)} & \eta\bar{\sigma} + \overline{(\theta^2 - c\phi^2)} \end{pmatrix}.$$

Equating the determinant to zero,

$$\eta\sigma\bar{\sigma} + (\sigma\overline{(\theta^2 - c\phi^2)} - \bar{\sigma}(\theta^2 - c\phi^2)) + \overline{(\theta^2 + c\phi^2)}((\theta^2 - c\phi^2)) = 0.$$

Now

$$\eta\sigma\bar{\sigma} + (\sigma\overline{(\theta^2 - c\phi^2)} - \bar{\sigma}(\theta^2 - c\phi^2))$$

is skew-symmetric. Expanding

$$\overline{(\theta^2 + c\phi^2)}((\theta^2 - c\phi^2))$$

gives

$$(\theta^2\bar{\theta}^2 - c\bar{c}\phi^2\bar{\phi}^2) - c\phi^2\bar{\theta}^2 - \theta^2\overline{c\phi^2},$$

and, again,

$$-c\phi^2\bar{\theta}^2 - \theta^2\overline{c\phi^2}$$

is skew-symmetric. Thus adding the determinant to its conjugate gives

$$(\theta^2\bar{\theta}^2 - c\bar{c}\phi^2\bar{\phi}^2) = 0.$$

Thus c^ν is a square in K, contrary to c being a non-square in F, as ν is surjective. □

Corollary 1
$\Delta := Sq(\Delta^+) \cup NSq(\Delta^-)$ *is a spread set, and the associated spread* π_Δ *is a replacement of the Knuth semifield spread* π_{Δ^+} *obtained by replacing the partial spread associated with* $NSq(\Delta^+)$ *by the partial spread associated with* $NSq(\Delta^-)$, *of the Knuth spread* Δ^-.

Let $m = (q^2 + 2q - 1)/2$. Then $t^m = -t^q$ for non-square t and similarly $t^m = t^q$ for square t. Then

$$\Delta = \left\{ S_{s,t} = \begin{pmatrix} s & t \\ s^q + t^m & \eta s^q + t^q \end{pmatrix} \mid s, t \in \mathrm{GF}(q^2) \right\}.$$

Let π be the plane (of order q^4) arising from the spread π_Δ via the Andre/Bruck-Bose construction.

Theorem 1
The plane π is non-Desarguesian, and the collineation group $\operatorname{Aut}\pi$ of π has orbits of lengths 1, q^2 and $q^4 - q^2$ on the translation line.

Proof. If S is any spread set then the additive group $\Sigma = \{A \in S : S + A \subset S\}$ corresponds to the y-axis elations. So a fixed matrix $A = S_{s_1,t_1}$ corresponds to a y-axis elation, if and only if

$$S + A \subset S,$$

if and only if $t^m - t_1^m = (t - t_1)^m$ for all t, which implies $t_1 = 0$. Thus the full elation group Σ with axis the y-axis has order q^2, from which it follows that π is neither Desarguesian nor a semifield. We claim that Σ consists of all elations in the translation complement C of π. If not, by the Hering-Ostrom theorem [1] Δ admits $SL(2, q^2)$, so, by the Schaeffer-Walker theorem [1] π is either a Hall plane (but this cannot have an elation group of order greater than 2) or a Hering plane (but this cannot have a kernel of square order). So Σ is a normal subgroup of C. Now π admits the homology group

$$H_Y = \{\mathrm{Diag}(x^2, \bar{x}^2, 1, 1) : x \in F\},$$

of order $(q^2 - 1)/2$, and the homology group

$$H_X = \{\mathrm{Diag}(1, 1, y^{2m}, y^2) : y \in F\},$$

of order $(q^2 - 1)/2$. Moreover, the orbits of $G = \langle \Sigma, H_X, H_Y \rangle$ on the translation line of π have lengths 1, q^2, $\dfrac{q^4 - q^2}{2}$, $\dfrac{q^4 - q^2}{2}$. Let O be the orbit of G of length q^2. There is an element δ of $\Gamma L(4, q^2)$ fixing Δ, O and the orbit of G of length 1, and interchanging the two remaining orbits of G, namely that induced by the map $A \mapsto X_1^{-1} \bar{A} X_2$, where $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $X_2 = \begin{pmatrix} 1 & 0 \\ \frac{\eta^2+1}{\eta} & \eta^2 \end{pmatrix}$. Now, by Andre's theorem [1], Σ is transitive on all the centers of homologies with axis X, so it follows that C (and hence $\operatorname{Aut}\pi$) stabilizes O. Hence $\operatorname{Aut}\pi$ has orbits of lengths 1, q^2 and $q^4 - q^2$ on the translation line. □

Note that since π is non-Desarguesian, it has kernel $\mathrm{GF}(q^2)$.

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References

[1] H. Lüneburg, *Translation planes*, Springer-Verlag, Berlin-New York, 1980.

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