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Bicritical total domination*

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Abstract

A graph G with no isolated vertex is total domination bicritical if the removal of any pair of vertices, whose removal does not produce an isolated vertex, decreases the total domination number. In this paper we study properties of total domination bicritical graphs, and give several characterizations.

1 Introduction

For notation and graph theory terminology in general we follow [8]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. We denote the degree of a vertex v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The *open neighborhood* of a vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The *boundary* of S, denoted $B(S)$, is $N(S) \setminus S$. The set S is a *dominating set* of G if $N[S] = V$, and a *total dominating set* of G (or just TDS) if $N(S) = V$. For sets $A, B \subseteq V$, we say that A *dominates* B if $B \subseteq N[A]$, while A *totally dominates* B if $B \subseteq N(A)$. The minimum cardinality of a dominating set of G is the *domination number*, denoted $\gamma(G)$, and the minimum cardinality of a TDS of G is the *total domination number*, denoted $\gamma_t(G)$. We call a TDS of cardinality $\gamma_t(G)$, a $\gamma_t(G)$-set. For references on total domination we refer [3, 4, 10].

An *end-vertex* in a graph G is a vertex of degree one and a *support vertex* is the vertex that is adjacent to an end-vertex.

In [6], Goddard et al. studied *total domination vertex critical graphs*. Let $S(G)$ be the set of all support vertices of G. A connected graph G is called *total domination vertex*

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critical, or just γ_t-vertex critical, if for any vertex v of $V(G)\backslash S(G)$, $\gamma_t(G-v) < \gamma_t(G)$. If G is a γ_t-critical graph and $\gamma_t(G) = k$, then G is called k-γ_t-critical.

We recall that the *corona* $\mathrm{cor}(H)$ of a graph H, is that graph obtained from H by adding a pendant edge to each vertex of H. The following is useful:

Theorem A (Goddard et al. [6]): *Let G be a connected graph of order at least 3, and with at least one end-vertex. Then G is k-γ_t-critical if and only if $G = \mathrm{cor}(H)$ for some connected graph H of order k, where $\delta(H) \geq 2$.*

We call a vertex v in a graph G a *total domination critical vertex*, or just a *γ_t-critical vertex*, if $\gamma_t(G-v) < \gamma_t(G)$. So a graph G is total domination vertex critical if each vertex of G is a total domination critical vertex.

In [2], Brigham et al. studied *bicritical domination* in graphs. A graph G is *domination bicritical*, or just γ-bicritical, if the removal of any pair of vertices decreases the domination number.

In this paper, we study bicritical total domination in graphs. We call a graph G with no isolated vertex *total domination bicritical*, or just γ_t-bicritical, if for any pair of vertices u, v such that $G - \{u, v\}$ contains no isolated vertex, $\gamma_t(G - \{u, v\}) < \gamma_t(G)$. We say that G is $k - \gamma_t$-bicritical if G is γ_t-bicritical and $\gamma_t(G) = k$. We characterize all γ_t-bicritical graphs with at least one end-vertex, and we obtain some results on γ_t-bicritical graphs. In addition we give some way of constructing γ_t-bicritical graphs from smaller ones.

2 Bicritical total domination

We begin with the following obvious observation.

Observation 1 *A disconnected graph G with no isolated vertex is γ_t-bicritical if and only if any component of G is γ_t-bicritical.*

Therefore we study only connected graphs.

Lemma 2 *If G is a γ_t-bicritical graph and $v, u \in V(G)$ such that $G - \{u, v\}$ has no isolated vertex, then $\gamma_t(G) - 2 \leq \gamma_t(G - \{u, v\}) \leq \gamma_t(G) - 1$.*

Proof. Let G be a γ_t-bicritical graph, and let $v, u \in V(G)$ such that $G - \{u, v\}$ has no isolated vertex. It is obvious that $\gamma_t(G - \{u, v\}) \leq \gamma_t(G) - 1$. Let S be a $\gamma_t(G - \{u, v\})$-set. Let $x \in N(u)$ and $y \in N(v)$. Then $S \cup \{x, y\}$ is a TDS for G. This implies that $\gamma_t(G - \{u, v\}) \geq \gamma_t(G) - 2$. ■

Lemma 3 *Let G be a γ_t-bicritical graph and $v, u \in V(G)$ be two vertices such that $G - \{u, v\}$ has no isolated vertex. If $\gamma_t(G - \{u, v\}) = \gamma_t(G) - 2$, then:*
(1)If u and v are non-adjacent then $d_G(u, v) \geq 3$.
(2) If u and v are adjacent then $N(u) \cap N(v) = \emptyset$.

Proof. Let G be a γ_t-bicritical graph and $v, u \in V(G)$ be two vertices such that $G - \{u, v\}$ has no isolated vertex. Assume that $\gamma_t(G - \{u, v\}) = \gamma_t(G) - 2$.
(1) Let u, v be non-adjacent. Assume that $d(u, v) \leq 2$. Then $d(u, v) = 2$. Let S be a $\gamma_t(G - \{u, v\})$-set, and let $x \in N(u) \cap N(y)$. Then $S \cup \{x\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Thus $d(u, v) \geq 3$.
The proof of (2) is similar. ■

Lemma 4 *If G is a γ_t-bicritical graph, then for any $v \in V(G) \setminus S(G)$, $\gamma_t(G - v) \leq \gamma_t(G)$.*

Proof. Let G be a γ_t-bicritical graph and let $v \in V(G) \setminus S(G)$. Suppose to the contrary that $\gamma_t(G - v) > \gamma_t(G)$. Let $y \in V(G) \setminus S(G)$ be a vertex of G different from v. We obtain $\gamma_t(G) < \gamma_t(G - v) \leq \gamma_t(G - \{v, y\}) + 1 \leq \gamma_t(G) - 1 + 1 = \gamma_t(G)$, a contradiction. ■

Lemma 4 leads to the following corollary which plays an important role in this paper.

Corollary 5 *If G is a γ_t-bicritical graph, then either G is γ_t-vertex critical, or $G - v$ is γ_t-vertex critical for any vertex v such that $\gamma_t(G - v) = \gamma_t(G)$.*

Corollary 6 *If G is a γ_t-bicritical graph and G is not γ_t-vertex critical, then there is a non-support vertex v in G such that $G - v$ is γ_t-vertex critical, and $\gamma_t(G - v) = \gamma_t(G)$.*

3 Graphs with end-vertices

In this section we characterize the γ_t-bicritical graphs with end-vertices. For this purpose we introduce some extremal families of graphs. Let $\mathcal{E}$ be the class of all graphs G such that $G \in \mathcal{E}$ if and only if one of the following holds:

- $G = \text{cor}(H)$ for some connected graph H with $\delta(H) \geq 2$,
- G is obtained from $\text{cor}(H)$ for some connected graph H with $\delta(H) \geq 2$, by adding a new vertex and joining it to at least one vertex of H,
- G is obtained from $\text{cor}(H)$ for some connected graph H with $\delta(H) \geq 2$, by adding a new vertex and joining it to exactly one end-vertex of $\text{cor}(H)$, and to its support vertex,
- G is obtained from $\text{cor}(H_1)$, $\text{cor}(H_2)$,..., $\text{cor}(H_m)$ for some integer $m \geq 2$ by adding a new vertex and joining it to at least one support vertex in $\text{cor}(H_i)$ for each $i = 1, 2, \ldots, m$, where H_i is a connected graph with minimum degree at least two for $i = 1, 2, \ldots, m$,

- G is obtained from $\text{cor}(H_1)$, $\text{cor}(H_2)$,..., $\text{cor}(H_m)$ for some integer $m \geq 2$ by adding a new vertex and joining it to a leaf and its support vertex in $\text{cor}(H_1)$ and then joining to at least one support vertex in $\text{cor}(H_i)$ for each $i = 2, \ldots, m$, where H_i is a connected graph with minimum degree at least two for $i = 1, 2, \ldots, m$.

Theorem 7 *Let G be a connected graph of order at least 3 with at least one end-vertex. Then G is γ_t-bicritical if and only if $G \in \mathcal{E}$.*

Proof. It is a routine matter to see that any graph in $\mathcal{E}$ is γ_t-bicritical. Suppose that G is a γ_t-bicritical graph. By Corollary 5 either G is γ_t-vertex critical, or $G - v$ is γ_t-vertex critical for any vertex v such that $\gamma_t(G - v) = \gamma_t(G)$. If G is γ_t-vertex critical, then by Theorem A, $G = \text{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$, and so $G \in \mathcal{E}$. Suppose that G is not γ_t-vertex critical. By Corollary 6 there is a non-support vertex y such that $G - y$ is γ_t-vertex critical, and $\gamma_t(G - y) = \gamma_t(G)$. We consider the following cases.

Case 1. $G - y$ is connected. Then G is obtained from $\text{cor}(H)$ for some connected graph H with $\delta(H) \geq 2$, by adding a new vertex y and joining y to some vertex of $\text{cor}(H)$. Let $|V(H)| = t$. So $\gamma_t(\text{cor}(H)) = t$, and $\gamma_t(G) = \gamma_t(G - y) = t$.

Fact 1. y is adjacent to some vertex in H.

Proof of Fact 1. Suppose to the contrary, that y is not adjacent to a vertex of H. If y is adjacent to only one end-vertex of $\text{cor}(H)$, then $\gamma_t(G) = t + 1$, which is a contradiction. If y is adjacent to exactly two end-vertices, then we let y_1 and y_2 be the two neighbors of y in G. Let $z_1 \in V(H)$ be adjacent to y_1 and $z_2 \in V(H)$ be adjacent to y_2. Then $\gamma_t(G - \{z_1, z_2\}) = t$, a contradiction. So y is adjacent to at least three end-vertices. Now $\gamma_t(G) < \gamma_t(G - y)$, a contradiction. We deduce that y is adjacent to some vertex in H.□

If y is not adjacent to an end-vertex in $\text{cor}(H)$, then $G \in \mathcal{E}$. Suppose that y is adjacent to an end vertex a in $\text{cor}(H)$. Let $b \in V(H)$ be adjacent to a.

Fact 2. $N_G(y) \cap V(H) = \{b\}$.

Proof of Fact 2. We first show y is not adjacent to an end-vertex of $\text{cor}(H)$ different from a. Suppose that y is adjacent to some end-vertex of $\text{cor}(H)$ different from a. If y is adjacent to exactly one end-vertex $c \neq a$ of $\text{cor}(H)$, then we let c_1 be the support vertex of $\text{cor}(H)$ adjacent to c. Then $\gamma_t(G - \{c_1, b\}) \geq \gamma_t(G)$, a contradiction. So y is adjacent to at least two end-vertices of $\text{cor}(H)$ different from a. Now $\gamma_t(G) \leq t - 1$, a contradiction. Thus y is not adjacent to an end-vertex of $\text{cor}(H)$ different from a. If y is not adjacent to b, then by Fact 1, y is adjacent to vertex $d \in V(H)$. Let d_1 be the leaf of $\text{cor}(H)$ which is adjacent to d, and let D be a $\gamma_t(G - \{b, d_1\})$-set. Since a is an end-vertex in $G - \{b, d_1\}$, we obtain $y \in D$. But any vertex in $V(H) \setminus \{b\}$ is a support vertex in $G - \{b, d_1\}$. This leads that $|D| \geq t$, a contradiction since G is γ_t-bicritical. We deduce that y is adjacent to b. If y is adjacent to a vertex in $V(H) \setminus \{b\}$, we get a similar contradiction. Hence $N_G(y) \cap V(H) = \{b\}$. Thus, $N_G(y) \cap V(\text{cor}(H)) = \{a, b\}$. □

Now Fact 2 implies that $G \in \mathcal{E}$.

Case 2. $G-y$ is disconnected. By Theorem A and Corollary 6, $G-y$ is disjoint union of the graphs $\mathrm{cor}(H_1)$, $\mathrm{cor}(H_2)$,... and $\mathrm{cor}(H_m)$ for some integer $m \geq 2$, where H_i is a connected graph with minimum degree at least two for $1 \leq i \leq m$. Let $\gamma_t(G) = k$ and $|V(H_i)| = t_i$ for $1 \leq i \leq m$. Then $|V(H_1)| + |V(H_2)| + \cdots + |V(H_m)| = k$. Clearly $\deg_G(y) \geq m$.

Claim 1. $N_G(y) \cap (V(H_1) \cup V(H_2) \cup \cdots \cup V(H_m)) \neq \emptyset$.

Proof of Claim 1. Suppose to the contrary, that $N_G(y) \cap (V(H_1) \cup V(H_2) \cup \cdots \cup V(H_m)) = \emptyset$. We show that $\deg_G(y) > m$. Suppose to the contrary that $\deg_G(y) = m$. If $m = 2$ then y is adjacent to an end-vertex y_1 of $\mathrm{cor}(H_1)$ and an end-vertex y_2 of $\mathrm{cor}(H_2)$. Let x_1 be the vertex of H_1 adjacent to y_1 and x_2 be the vertex of H_2 adjacent to y_2. If $|V(H_1)| = |V(H_2)| = 3$ then $\gamma_t(G) = 6$, and $\gamma_t(G - \{y_1, w\}) \geq 6$, where w is an end-vertex of $\mathrm{cor}(H_1)$ different from y_1. This is a contradiction. So we suppose that $|V(H_1)| \geq 4$. Let $x_3 \in V(H_1)$ be a vertex such that there is a path of length two between x_3 and x_1, and let y_3 be the leaf adjacent to x_3. It follows that $\gamma_t(G - \{x_1, y_3\}) \geq \gamma_t(G)$, since any $\gamma_t(G - \{x_1, y_3\})$-set contains y. This contradiction implies that $m \geq 3$. But then y, y_1 together with the support vertices of G form a TDS for G of cardinality at most $k-1$, and so $\gamma_t(G) \leq k-1$, a contradiction. We conclude that $\deg(y) > m$. Suppose that $|N_G(y) \cap V(\mathrm{cor}(H_1))| \geq |N_G(y) \cap V(\mathrm{cor}(H_i))|$ for $2 \leq i \leq m$. We show that y is adjacent to exactly two leaves of $\mathrm{cor}(H_1)$. Suppose that y is adjacent to at least three leaves y_1, y_2, y_3 of $\mathrm{cor}(H_1)$. Let $z_1 \in V(H_1)$ be adjacent to y_1 and $z_2 \in V(H_1)$ be adjacent to y_2. Also let y be adjacent to a leaf y_4 in $\mathrm{cor}(H_2)$, and $y_5 \in V(H_2)$ be adjacent to y_4. If $|V(H_1)| = 3$, then $\{y, y_1, z_1\} \cup (V(H_2) - \{y_5\}) \cup V(H_3) \cup \cdots \cup V(H_m)$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. If $|V(H_1)| \geq 4$, then $\{y, y_1\} \cup (V(H_1) - \{z_1, z_2\}) \cup (V(H_2) - \{y_5\}) \cup V(H_3) \cup \cdots \cup V(H_m)$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Thus, y is adjacent to exactly two leaves a, b in $\mathrm{cor}(H_1)$. Let a_1 be the support vertex in $\mathrm{cor}(H_1)$ which is adjacent to a, and b_1 be the support vertex in $\mathrm{cor}(H_1)$ which is adjacent to b. If y is adjacent to two leaves in $\mathrm{cor}(H_j)$, for some $j \neq 1$, then we let a_2, b_2 be two support vertices of $\mathrm{cor}(H_j)$ such that y is adjacent to the leaves adjacent to them. If $|V(H_1)| > 3$ and $|V(H_j)| = 3$, then we let $V(H_j) = \{a_2, b_2, \alpha\}$. Let b_3 be the leaf in $\mathrm{cor}(H_j)$ which is adjacent to b_2. Then $\{y, \alpha, b_2, b_3\} \cup (V(H_1) - \{a_1, b_1\}) \cup \bigcup_{i \notin \{1,j\}} V(H_i)$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Similarly $|V(H_1)| = 3$ and $|V(H_j)| > 3$ produce a contradiction. So $|V(H_1)| = |V(H_j)| = 3$. Then $\gamma_t(G - \{a_1, b_1\}) \geq k$, a contradiction, since any $\gamma_t(G - \{a_1, b_1\})$-set is a TDS for G. Thus $|V(H_1)| > 3$ and $|V(H_j)| > 3$. Now $((V(H_1) \cup V(H_2) \cup \cdots \cup V(H_m)) \cup \{y, a\}) \setminus \{a_1, b_1, a_2, b_2\}$ is a TDS for G of cardinality $t_1 + t_2 + \cdots + t_m - 2 < k$, a contradiction. So y is adjacent to exactly one leaf in $\mathrm{cor}(H_j)$ for $2 \leq j \leq m$. This time y together a and the support vertices of G form a TDS for G of cardinality less than k, and so $\gamma_t(G) < k$, a contradiction. Thus, $N_G(y) \cap (V(H_1) \cup V(H_2) \cup \cdots \cup V(H_m)) \neq \emptyset$. □

Let $z_2 \in V(H_1)$ be a vertex adjacent to y, and let z_1 be the leaf in $\mathrm{cor}(H_1)$ which is adjacent to z_2. If y is adjacent to no leaf in $\mathrm{cor}(H_1) \cup \mathrm{cor}(H_2) \cup \cdots \cup \mathrm{cor}(H_m)$, then G is obtained from $\mathrm{cor}(H_1)$, $\mathrm{cor}(H_2)$,..., $\mathrm{cor}(H_m)$ by joining y to at least one support vertex in $\text{cor}(H_i)$ for $i = 1, 2, .., m$ and so $G \in \mathcal{E}$.

Suppose that y is adjacent to some leaf of $\text{cor}(H_1) \cup \text{cor}(H_2) \cup \cdots \cup \text{cor}(H_m)$. We show that z_1 is the only leaf of $\text{cor}(H_1) \cup \text{cor}(H_2) \cup \cdots \cup \text{cor}(H_m)$ which is adjacent to y. Suppose that y is adjacent to some leaf in $\text{cor}(H_k)$ for some $2 \leq k \leq m$. We first show that y is adjacent to exactly one leaf in $\text{cor}(H_k)$. Suppose that y is adjacent to at least two leaves $\hat{x}_1, \hat{x}_2$ of $\text{cor}(H_k)$. Let $\hat{y_1}$ be the vertex of H_k which is adjacent to $\hat{x_1}$ and $\hat{y_2}$ be the vertex of H_k which is adjacent to $\hat{x_2}$. If $|V(H_k)| \geq 4$, then $(V(H_k) - \{\hat{y_1}, \hat{y_2}\}) \cup \{y\} \cup \bigcup_{i \neq k} V(H_i)$ form a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. So $|V(H_k)| = 3$. If y is adjacent to all leaves of $\text{cor}(H_j)$, then $\gamma_t(G - \{\hat{x_2}, \hat{y_1}\} \geq k$, a contradiction. So y is adjacent to two leaves of $\text{cor}(H_j)$. Then $\gamma_t(G - \{\hat{x_1}, \hat{x_2}\}) \geq k$, a contradiction. Thus, y is adjacent to exactly one leaf in $\text{cor}(H_k)$. As a result y is adjacent to at most one leaf in $\text{cor}(H_i)$ for each $i = 2, 3, .., m$. Now if y is adjacent to a leaf u in $\text{cor}(H_l)$ for some $l \neq 1$, then we let v be the support vertex of $\text{cor}(H_l)$ which is adjacent to u. Then $\gamma_t(G - \{v, z_1\}) \geq k$, a contradiction. We conclude that y is adjacent to no leaf in $\text{cor}(H_2) \cup \text{cor}(H_3) \cup \cdots \cup \text{cor}(H_m)$.

If y is adjacent to at least two leaves d_1, d_2 in $\text{cor}(H_1)$, then we let $\hat{d}_1, \hat{d}_2$ be the support vertices adjacent to d_1, d_2, respectively. Then $(V(H_1) - \{\hat{d}_1, \hat{d}_2\}) \cup \{y\} \cup V(H_2) \cup \cdots \cup V(H_m)$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. So y is adjacent to exactly one leaf w of $\text{cor}(H_1)$. If $w \neq z_1$, then we let w_1 be the support vertex in $\text{cor}(H_1)$ which is adjacent to w. Then $(V(H_1) \cup \cdots \cup V(H_m)) - \{w_1\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Thus $w = z_1$. Hence, $G \in \mathcal{E}$. ■

Corollary 8 *No tree is γ_t-bicritical.*

Now it is straightforward to check any graph in $\mathcal{E}$ and obtain the following.

Corollary 9 *If G is a γ_t-bicritical graph with at least one end-vertex, then diam(G) $\leq k$ if $k \in \{3, 4\}$ and diam$(G) \leq 2k - 2$ if $k \geq 5$.*

Henceforth, all graphs we handle have minimum degree at least two.

4 Graphs without end-vertices

In this section we study γ_t-bicritical graphs with no end-vertex. We begin with the following known results.

Observation 10 ([9]) *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

Proposition 11 ([6]) *A cycle C_n is γ_t-vertex critical if and only if $n \equiv 1, 2$ (mod 4).*

Proposition 12 *A cycle C_n is γ_t-bicritical if and only if $n = 5$.*

Proof. First notice that C_5 is γ_t-bicritical. Let C_n be a γ_t-bicritical cycle. It is obvious that $n \geq 5$. Assume that $n \geq 6$. Let x, y be two vertices with $d(x, y) = 3$. It follows that $\gamma_t(C_n - \{x, y\}) = \gamma_t(P_{n-4}) + \gamma_t(P_2) = \gamma_t(C_n)$, a contradiction. Thus, $n = 5$. ■

We remark that Propositions 11 and 12 imply that a γ_t-vertex critical graph is not γ_t-bicritical in general. We have also seen in the previous section that a γ_t-bicritical graph is not γ_t-vertex critical in general.

We call a graph G with no isolated vertex *strong total domination bicritical*, or just strong γ_t-bicritical if $\gamma_t(G - \{u, v\}) = \gamma_t(G) - 2$ for any two vertices u, v such that $G - \{u, v\}$ has no isolated vertex.

Theorem 13 *No graph with minimum degree at least two is strong γ_t-bicritical.*

Proof. Assume G is a strong γ_t-bicritical graph with minimum degree at least two. It is obvious that $\gamma_t(G) \geq 4$. We show that $\Delta(G) = 2$. Suppose to the contrary that $\Delta(G) \geq 3$. Let x be a vertex of maximum degree. Since G is not a complete graph, there is a vertex y at distance two from x. By Lemma 3, $G - \{x, y\}$ has an isolated vertex z. This means that $N(z) = \{x, y\}$ and $\deg(z) = 2$. Let $w \neq z$ be a vertex adjacent to x. If $G - \{y, w\}$ has some isolated vertex, then there is a vertex $w_1 \in N(y) \cap N(w)$ such that $\deg(w_1) = 2$. Since $\deg(x) \geq 3$, $G - \{w, z\}$ has no isolated vertex. Let S be a $\gamma_t(G - \{w, z\})$-set. Then $y \in S$, since y is a support vertex in $G - \{w, z\}$. Then $S \cup \{w_1\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. Thus $G - \{y, w\}$ has no isolated vertex. Let S_1 be a $\gamma_t(G - \{y, w\})$-set. Then $x \in S_1$, and $S_1 \cup \{z\}$ is a TDS for G of cardinality less than $\gamma_t(G)$, a contradiction. We deduce that $\Delta(G) = 2$. Since $\delta(G) \geq 2$, G is a cycle. By Proposition 12, $G = C_5$. But $\gamma_t(C_5) = 3$, a contradiction. ■

A graph G is *vertex diameter k-critical* if $diam(G) = k$ and $diam(G - v) > k$ for all $v \in V(G)$.

Theorem 14 ([7]) *For a graph G, $\gamma_t(G) = 2$ if and only if $\overline{G}$ has diameter greater than two.*

Theorem 15 ([6]) *A connected graph G is $3-\gamma_t$-vertex critical if and only if $\overline{G}$ is vertex diameter 2-critical or $G = \mathrm{cor}(K_3)$.*

Theorem 16 *If a connected graph G with minimum degree at least two is $3-\gamma_t$-bicritical, then either $\overline{G}$ is vertex diameter 2-critical or $\overline{G} - v$ is vertex diameter 2-critical, for some vertex v with $\gamma_t(G - v) = \gamma_t(G)$.*

Proof. Let G be a $3-\gamma_t$-bicritical graph with minimum degree at least two. If G is γ_t-vertex critical, then by Theorem 15, $\overline{G}$ is vertex diameter 2-critical. Suppose that G is not γ_t-vertex critical. By Corollary 6 there is a vertex v such that $G - v$ is γ_t-vertex critical, and $\gamma_t(G - v) = 3$. Theorem 15, implies that $\overline{G} - v$ is vertex diameter 2-critical. ■

Lemma 17 *If G is a γ_t-vertex critical graph of order n, then $n \leq (\gamma_t(G)-1)\Delta(G)+1$.*

Proof. Let G be a γ_t-vertex critical graph of order n and let $v \in V(G)$. Let S be a $\gamma_t(G-v)$-set. We know that $|S| = \gamma_t(G) - 1$. Any vertex of S dominates at most $\Delta(G)+1$ vertices of $G-v$. Thus $n \leq (\gamma_t(G)-1)\Delta(G)+1$. ■

Lemma 18 ([11]) *If G is a γ_t-vertex critical graph of order $(\gamma_t(G)-1)\Delta(G)+1$, then G is regular.*

Proposition 19 *If G is a γ_t-bicritical graph of order n, then $n \leq (\gamma_t(G)-1)\Delta(G) + 2$.*

Proposition 20 *If G is a regular γ_t-bicritical graph of order n, then $n \leq (\gamma_t(G)-1)\Delta(G)+1$.*

Proof. Let G be a regular γ_t-bicritical graph of order n. If G is γ_t-vertex critical, then the result follows by Lemma 17. Suppose that G is not γ_t-vertex critical. By Corollary 6 there is a vertex v such that $G-v$ is γ_t-vertex critical. Since G is regular, $G-v$ is not regular. This means that $|V(G-v)| \leq (\gamma_t(G)-1)\Delta(G)$. Thus, $n \leq (\gamma_t(G)-1)\Delta(G)+1$. ■

Proposition 21 *Let G be a $k-\gamma_t$-bicritical graph with minimum degree at least two. Then $diam(G) \leq 2k-3$.*

Proof. Let G be a $k-\gamma_t$-bicritical graph with minimum degree at least two. Let x, y be two diameterical vertices, and for $i = 0, 1, 2, \ldots, d = diam(G)$, let V_i be the set of all vertices of G at distance i from x. Let $u \in V_1$ and $v \in V_2$ be two vertices such that $G - \{u, v\}$ has no isolated vertex. Let S be a $\gamma_t(G - \{u, v\})$-set. Since S dominates x, $|S \cap (V_0 \cup V_1 \cup V_2)| \geq 2$. Also for any $i \geq 3$, $|S \cap (V_i \cup V_{i+1} \cup V_{i+2} \cup V_{i+3})| \geq 2$. Let $d = 2 + 4j + r$, where $0 \leq r \leq 3$. But $|S| \leq k-1$. If $r = 0$, then $k-1 \geq 2+2j$ which implies that $d \leq 2k-4$. If $r = 1$, then $k-1 \geq 2+2j+1$ which implies that $d \leq 2k-3$. The cases of $r = 2$ and $= 3$ are similar, and in all cases we obtain $d \leq 2k-3$. ■

5 Constructions

In this section, we give two ways of constructing a γ_t-bicritical graph from smaller γ_t-bicritical graphs.

5.1 Coalescence

Brigham et al. in [1] gave a construction that used to build a bicritical graph from two smaller ones, [2]. Suppose F and H are nonempty graphs. Let u and w be non-isolated vertices of F and H, respectively. The coalescence of F and H via u and w, denoted by $(FH)(u, w : v)$ is the graph obtained from F and H by identifying u and w in a vertex labelled v. The following has a straightforward proof and therefore we omit a proof.

Proposition 22 *Let F and H be two graphs that are both γ_t-vertex critical and γ_t-bicritical. Let G be the coalesence of F and H. If $\gamma_t(G) = \gamma_t(F) + \gamma_t(H) - 1$, then G is both γ_t-vertex critical and γ_t-bicritical.*

5.2 Mycielski's construction

For a graph G, Mycielski's construction produces a graph $M(G)$ with $V(M(G)) = V \cup U \cup \{w\}$ where $V = V(G) = \{v_1, \ldots, v_n\}$, $U = \{u_1, \ldots, u_n\}$ and $E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}, i = 1, \ldots, n\}$, [12]. We define the k-th Mycielski graph of G, recursively by $M^0(G) = G$ and $M^{k+1}(G) = M(M^k(G))$ for $k \geq 1$.

Theorem 23 ([5]) *For any graph G, $\gamma_t(M(G)) = 1 + \gamma_t(G)$.*

Theorem 24 *Let G be a graph that is both γ_t-vertex critical and γ_t-bicritical. For any positive integer k, $M^k(G)$ is both γ_t-vertex critical and γ_t-bicritical.*

Proof. Let G be a graph that is both γ_t-vertex critical and γ_t-bicritical. Let $V(M(G)) = V \cup U \cup \{w\}$, where $V = V(G) = \{v_1, \ldots, v_n\}$, $U = \{u_1, \ldots, u_n\}$ and $E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}, i = 1, \ldots, n\}$. We prove γ_t-bicriticality. The proof for γ_t-vertex criticality is similar. We prove that $M(G)$ is γ_t-bicritical. Let $x, y \in V(M(G))$. We consider the following cases.

Case 1. $\{x, y\} = \{w, u_i\}$ for some i. In this case any $\gamma_t(G)$-set is a TDS for $M(G) - \{x, y\}$, and the result follows.

Case 2. $\{x, y\} = \{w, v_i\}$ for some i. Let S_1 be a $\gamma_t(G - v_i) - set$. Since G is γ_t-vertex critical, $|S_1| = \gamma_t(G) - 1$, and $S_1 \cap N_G(v_i) = \emptyset$. Let $v_{i+1} \in N_G(v_i)$. Then $S_1 \cup \{v_{i+1}\}$ is a TDS for $M(G)$ of cardinality $\gamma_t(G) < \gamma_t(M(G))$.

Case 3. $\{x, y\} = \{u_i, u_j\}$ for some i, j. Let S_2 be a $\gamma_t(G)$-set. It is obvious that $N_G(v_i) \cap S_2 \neq \emptyset$. Let $v_k \in N_G(v_i) \cap S_2$. It follows that $(S_2 \setminus \{v_k\}) \cup \{u_k, w\}$ is a TDS for $M(G)$ of cardinality $\gamma_t(G) < \gamma_t(M(G))$.

Case 4. $\{x, y\} = \{v_i, v_j\}$ for some i, j. Let S_3 be a $\gamma_t(G - v_i)$-set. Since G is γ_t-vertex critical, $|S_3| = \gamma_t(G) - 1$. If v_i and v_j are adjacent, then $v_j \notin S_3$. Let $v_k \in S_3$. Then $(S_3 \setminus \{v_k\}) \cup \{u_k, w\}$ is a TDS for $M(G) - \{x, y\}$ of cardinality less than $\gamma_t(M(G))$. So suppose that v_i and v_j are not adjacent. If $G - \{v_i, v_j\}$ contains an isolated vertex, then v_i and v_j have some common neighbor of degree 2. Further, $v_j \in S_3$. Now $(S_3 \setminus \{v_j\}) \cup \{u_j, w\}$ is a TDS for $M(G) - \{x, y\}$ of cardinality less than $\gamma_t(M(G))$. It remains to assume $G - \{v_i, v_j\}$ contains no isolated vertex. Let S_4 be a $\gamma_t(G - \{v_i, v_j\})$-set. Since G is γ_t-bicritical, $|S_4| \leq \gamma_t(G) - 1$. Let $v_k \in S_4$. It follows that $(S_4 \setminus \{v_k, w\}) \cup \{u_k\}$ is a TDS for $M(G)$ of cardinality $\gamma_t(G) < \gamma_t(M(G))$. Now the result follows by induction. ■

Similarly, the following is verified.

Theorem 25 *If G is a γ_t-vertex critical graph, then for any positive integer k, $M^k(G)$ is γ_t-vertex critical.*

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