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Note on cubic symmetric graphs of order $2p^n$

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Abstract

Let p be a prime and n a positive integer. In [*J. Austral. Math. Soc.* 81 (2006), 153–164], Feng and Kwak showed that if $p > 5$ then every connected cubic symmetric graph of order $2p^n$ is a Cayley graph. Clearly, this is not true for $p = 5$ because the Petersen graph is non-Cayley. But they conjectured that this is true for $p = 3$. This conjecture is confirmed in this paper. Also, for the case when $p = 2$, we prove a slightly more general result, that is, every connected cubic vertex-transitive graph of order a power of 2 is a Cayley graph.

1 Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, $u \sim v$ means that u is adjacent to v and denote by $\{u, v\}$ the edge incident to u and v in X. A graph X is said to be *vertex-transitive*, and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.

It is well known that a vertex-transitive graph is a Cayley graph if and only if its automorphism group contains a subgroup acting regularly on its vertex set (see, for example, [19, Lemma 4]). There are vertex-transitive graphs which are not Cayley graphs and the smallest one is the well-known Petersen graph. Such a graph will be called a *vertex-transitive non-Cayley graph*, or a *VNC-graph* for short. Many publications have been put into investigation of VNC-graphs from different perspectives. For example, in [10], Marušič asked for a determination of the set *NC* of non-Cayley numbers, that is, those numbers for which there exists a VNC-graph of order n, and to settle this question, a lot of VNC-graphs were constructed in [5, 7, 11, 15, 12, 13, 14, 16, 18, 20]. In [3], Feng considered the question of determining the smallest valency for VNC-graphs of a given order and it was solved for the graphs of odd prime power order. Recently, Zhou et al. considered the problem of classifying cubic VNC-graphs with specific orders. Let p and q be two primes. Note that from [2, 9] all cubic VNC-graphs of order $2p$ are known. In [23, 24, 25], Zhou et al. classified all cubic VNC-graphs of order $2pq$.

In [4, Corollary 3.4], Feng and Kwak proved that if $p > 5$ then every connected cubic symmetric graph of order $2p^n$ is a Cayley graph. Clearly, this is not true for $p = 5$ because the Petersen graph is a VNC-graph. However, they conjectured this holds for $p = 3$.

FK-Conjecture: *Every connected cubic symmetric graph of order $2 \cdot 3^n$ is a Cayley graph for each positive integer n.*

In this paper, this conjecture is confirmed. Moreover, the case of $p = 2$ is also investigated, and for this case we have a slightly more general result, that is, every connected cubic vertex-transitive graph of order a power of 2 is a Cayley graph.

2 Preliminaries

In this section, we introduce some notation and definitions as well as some preliminary results which will be used later in the paper. For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. For a regular graph X, use $d(X)$ to represent the valency of X, and for any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.1 [6, Chapter I, Theorem 4.5] *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. The following proposition gives a characterization for Cayley graphs in terms of their automorphism groups.

Proposition 2.2 [19, Lemma 4] *A graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X.*

Let X be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(X)$ be vertex-transitive on X. For a normal subgroup N of G, the *quotient graph* X_N of X relative to the orbit set of N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between vertices lying in these two orbits. If X is a symmetric cubic graph and G acts arc-transitively on X, then by [8, Theorem 9], we have

Proposition 2.3 *If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$ and X_N is a cubic symmetric graph with G/N as an arc-transitive group of automorphisms.*

3 Main results

First, we prove the FK-Conjecture is true by the following theorem.

Theorem 3.1 *Let X be a connected cubic symmetric graph of order $2 \cdot 3^n$ with n a positive integer. Then X is a Cayley graph.*

Proof. Let G be an arc-transitive automorphism group of X with smallest order. Let G_v be the stabilizer of a vertex $v \in V(X)$ in G. By Tutte [21], $|G_v| \mid 48$ and hence $|G| = 2^{\ell+1} \cdot 3^{n+1}$ for some $0 \leq \ell \leq 4$. To complete the proof, it suffices to prove that G has a subgroup acting regularly on $V(X)$.

Use induction on n. If $n = 1$ then X is the complete bipartite graph $K_{3,3}$. By MAGMA [1], $|G| = 18$ and G has a subgroup acting regularly on $V(K_{3,3})$, as required. In what follows, assume $n > 1$. Note that from Burnside's $p^a q^b$-Theorem we know that every group of order $p^a q^b$ is solvable for any primes p, q and integers a, b. Thus, G is solvable. Let M be a minimal normal subgroup of G. Then M is an elementary abelian 2- or 3-group. If M is a 2-group, then M has 3^n orbits on $V(X)$. By Proposition 2.3, the quotient graph X_M of X relative to the orbit set of M is a cubic symmetric graph of order 3^n. This is impossible. Thus, M is an elementary abelian 3-group.

Let M have exactly two orbits on $V(X)$, say Δ and Δ'. Then X is a bipartite graph with two parts Δ and Δ'. Suppose that M is not semiregular. Then the stabilizer M_u of $u \in \Delta$ in M is non-trivial. Since $|G_u| \mid 48$, one has $M_u \cong \mathbb{Z}_3$. Set $M_u = \langle \alpha \rangle$. Let x, y, z be three neighbors of u. Clearly, $x, y, z \in \Delta'$ and α cyclicly permutates x, y and z. Without loss of generality, assume that $x^\alpha = y, y^\alpha = z$ and $z^\alpha = x$. Let v, w be two neighbors of x different from u. Then $v, w \in \Delta$. Note that M is abelian. The transitivity of M on Δ implies that M_u fixes all vertices in Δ. Hence, $\{v, y\} = \{v, x\}^\alpha \in E(X)$, and $\{v, z\} = \{v, x\}^{\alpha^2} \in E(X)$. Similarly, $\{w, y\}$ and $\{w, z\}$ are also in $E(X)$. It follows that the subgraph of X induced by $\{u, v, w, x, y, z\}$ is the complete bipartite graph $K_{3,3}$. By the connectivity of X, we have $X \cong K_{3,3}$, contrary to the fact that $n > 1$. Thus, M is semiregular. Let $\Delta = \{L(h) \mid h \in M\}$ and $\Delta' = \{R(h) \mid h \in M\}$. One may assume that the actions of M on Δ and Δ' are just by right multiplication, that is, $L(h)^g = L(hg)$ and $R(h)^g = R(hg)$ for any $h, g \in M$. By the arc-transitivity of G on X, $X[\Delta]$ has valency 0. Let the neighbors of $R(1)$ be $L(g_1)$, $L(g_2)$ and $L(g_3)$ where $g_1, g_2, g_3 \in M$. Note that M is abelian. Then for any $g \in M$, the neighbors of $R(g)$ are $L(gg_1)$, $L(gg_2)$ and $L(gg_3)$, and furthermore, the neighbors of $L(g)$ are $R(gg_1^{-1})$, $R(gg_2^{-1})$ and $R(gg_3^{-1})$. Since M is abelian, it is easy to see that the map α defined by $L(g) \mapsto R(g^{-1}), R(g) \mapsto L(g^{-1}), \forall g \in M$, is an automorphism of X of order 2. Clearly, $\langle M, \alpha \rangle$ is transitive on $V(X)$. Since $M \trianglelefteq G$, one has $|\langle M, \alpha \rangle| = 2|M|$. It follows that $\langle M, \alpha \rangle$ acts regularly on $V(X)$. By Proposition 2.2, X is a Cayley graph.

Let M have more than two orbits in $V(X)$. Consider the quotient graph X_M of X relative to the orbit set of M. By Proposition 2.3, M is semiregular on $V(X)$, and X_M is a cubic graph with G/M as an arc-transitive group of automorphisms. It follows that $M \cong \mathbb{Z}_3^t$ for some $1 \leq t < n$. Then $|X_M| = 2 \cdot 3^{n-t} < |X|$, and furthermore, G/M is also an arc-transitive automorphism group of X_M with smallest order. By induction on n, G/M has a subgroup, say H/M, acting regularly on $V(X_M)$. Then H is regular on $V(X)$, and by Proposition 2.2, X is a Cayley graph. □

Below, we investigate the Cayley property of connected vertex-transitive graphs of order a power of 2.

Theorem 3.2 *Let X be a connected cubic vertex-transitive graph of order 2^n with an integer $n \geq 2$. Then X is a Cayley graph.*

Proof. Let G be a vertex-transitive automorphism group of X with smallest order. By [22, Theorem 3.4], G is a 2-group. By elementary group theory, the center $Z(G)$ of G is nontrivial. Take a subgroup N of order 2 in $Z(G)$. Then $N \trianglelefteq G$. Consider the quotient graph X_N of X relative to the orbit set of N, and let K be the kernel of G acting on $V(X_N)$. Then X_N has order 2^{n-1} and valency 1, 2 or 3.

Let $d(X_N) = 1$. By the connectivity of X, one has $|X_N| = 2$, implying that $|X| = 4$. Thus, X is isomorphic to the complete graph K_4 of order 4 which is a Cayley graph.

Let $d(X_N) = 2$. Let $V(X_N) = \{B_i \mid i \in \mathbb{Z}_{2^{n-1}}\}$ with $B_i \sim B_{i+1}$. Since $d(X) = 3$ and X is connected, $d(X[B_0]) = 0$ or 1. Assume $d(X[B_0]) = 1$. Then the stabilizer K_u of $u \in V(X)$ in K fixes the neighborhood of u in X pointwise because K fixes each orbit of N. By the connectivity of X, one has $K_u = 1$ and hence $K = N \cong \mathbb{Z}_2$. Since $X_N \cong C_{2^{n-1}}$, the vertex-transitivity of G/K on X_N implies that G/K has a subgroup, say H/K, acting regularly on $V(X_N)$. Then H is regular on $V(X)$. By Proposition 2.2, X is a Cayley graph. Assume $d(X[B_0]) = 0$. Since $d(X) = 3$, one may let $X[B_0 \cup B_1] \cong 2K_2$ and $X[B_0 \cup B_{2p-1}] \cong C_4$. Let $B_i = \{x_i, y_i\}$ for each $i \in \mathbb{Z}_{2^{n-1}}$. By the transitivity of A on $V(X)$, we may assume that $x_i \sim x_{i+1}, y_i \sim y_{i+1}, x_{2i} \sim y_{2i+1}$ and $y_{2i} \sim x_{2i+1}$ for each $i \in \mathbb{Z}_{2^{n-1}}$. Let $\alpha : x_i \mapsto x_{i+2}, y_i \mapsto y_{i+2}$ ($i \in \mathbb{Z}_{2^{n-1}}$), $\beta : x_i \mapsto y_i, y_i \mapsto x_i$ ($i \in \mathbb{Z}_{2^{n-1}}$), and $\gamma : x_i \mapsto x_{1-i}, y_i \mapsto y_{1-i}$ ($i \in \mathbb{Z}_{2^{n-1}}$) be three permutations on $V(X)$. It is easy to check that α, β and γ are automorphisms of X. Furthermore, $\langle \alpha, \beta, \gamma \rangle = \langle \alpha, \gamma \rangle \times \langle \beta \rangle \cong D_{2^{n-1}} \times \mathbb{Z}_2$ is regular on $V(X)$. Again, by Proposition 2.2, X is a Cayley graph.

Let $d(X_N) = 3$. By induction on n, X_N is a Cayley graph. Clearly, G/K is a vertex-transitive automorphism group of X_N with smallest order. Then $|G/K| = 2^{n-1}$. Also, since $d(X_N) = 3$, the stabilizer K_u of $u \in V(X)$ in K fixes the neighborhood of u in X pointwise because K fixes each orbit of N. By the connectivity of X, one has $K_u = 1$ and hence $K = N \cong \mathbb{Z}_2$. It follows that $|G| = 2^n$, and hence G is regular on $V(X)$. By Proposition 2.2, X is a Cayley graph. □

If X is a connected vertex-transitive graph of order 2^n and valency at least 4 with an integer $n \geq 2$, then X is not necessarily a Cayley graph.

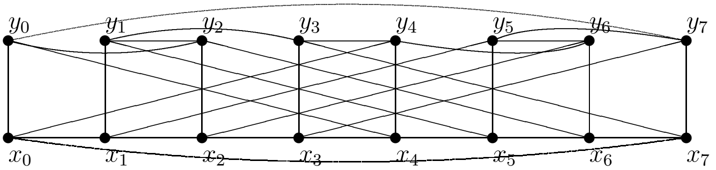

Figure 1: The graph X_{16}

Example 3.3 *Let X_{16} be a graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_8\}$ and edge set $\{\{x_i, x_{i+1}\}, \{x_i, y_i\}, \{x_i, y_{i+4}\} \mid i \in \mathbb{Z}_8\} \cup \{\{y_0, y_2\}, \{y_2, y_1\}, \{y_1, y_3\}, \{y_3, y_4\}, \{y_4, y_6\}, \{y_6, y_5\}, \{y_5, y_7\}, \{y_0, y_7\}\}$ (see Fig. (1)). By MAGMA [1] and [17], up to isomorphism, X_{16} is the unique connected tetravalent VNC-graph of order 2^4.*

The following corollary immediately follows from Theorems 3.1, 3.2 and [4, Corollary 3.4].

Corollary 3.4 *Let $p \neq 5$ be a prime and n be a positive integer. Then every connected cubic symmetric graph of order $2p^n$ is a Cayley graph.*

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