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Symmetry and unimodality of independence polynomials of path-like graphs

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Abstract

We prove the symmetry and unimodality of the independence polynomials of various graphs constructed by means of a recursive "path-like" construction. Our results provide a substantial generalization of the work of Zhu [*Australas. J. Combin.* 38 (2007), 27–33] and others.

1 Introduction

The study of independence polynomials mirrors the much older and more well-established study of chromatic polynomials. Both polynomials are known to yield a good deal of information about the graph from which they are derived, and so structural aspects of these polynomials (degree, behavior of coefficients, location of roots, *etc.*) are active subjects of research. Levit and Mandrescu [7] offer a relatively recent and robust overview of the general research on independence polynomials, and Rosenfeld [14] describes the state of affairs regarding the coefficients, roots, and other analytical properties of these polynomials.

Our goal in this paper is to show that the graphs obtained by certain inductive path-like constructions possess independence polynomials with particularly nice structure. We note at the outset that we use the term "path-like" loosely here; the graphs to which our results apply are described carefully below.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Recall that an *independent set* S in G is a set of pairwise non-adjacent vertices. The *independence number* of G, $\alpha(G)$, is the cardinality of a largest independent set in G. The *independence polynomial* of G, denoted $I(G; x)$, is defined by

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + \cdots + s_{\alpha(G)} x^{\alpha(G)},$$

where s_k is the number of independent sets with cardinality k.

We say a polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ is *unimodal* if the sequence of its coefficients satisfies $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some j, $0 \leq j \leq n$. We say that p is *symmetric* if $a_i = a_{n-i}$ for $0 \leq i \leq n$. For notational convenience, we abbreviate "symmetric unimodal" to "SU."

The binomial coefficients are probably the best-known example of a symmetric unimodal sequence.

We will say that a graph G is SU if the sequence of the coefficients of its independence polynomial is SU. We would like to develop an efficient means of constructing a large number of SU graphs. We will construct many families of such graphs in a "path-like" fashion, modifying a K_t-path (defined below) by attaching copies of a fixed graph G at each vertex of the path. Such a construction yields independence polynomials that form an easily analyzable sequence. Exploiting such a sequence allowed Zhu [16] to provide a partial answer to a question posed in [7] by showing that a caterpillar in which every "spine" vertex has precisely 2 legs is SU. Our work here substantially generalizes this construction by describing a means of generating infinitely many infinite families of SU graphs.

There are a number of other results concerning the coefficients of independence polynomials, many of which consider graphs formed by applying some sort of operation (such as those found in [12]) to simpler graphs. In [14], for instance, Rosenfeld examines the independence polynomials of graphs formed by taking various rooted products of simpler graphs. (See [2] for the definition of the rooted product of two graphs.) In particular, he shows that the property of having all real roots is preserved under forming rooted products. In [3], [4], and [5] Gutman analyzes independence polynomials (and the related matching polynomials) of graphs, often in terms of their subgraphs. He thereby obtains a number of new results regarding the coefficients, roots, and even derivatives of these polynomials. Stevanović [15] generalizes on Gutman's work and indicates several explicit means of constructing graphs with symmetric independence polynomials.

Still other results are known concerning independence polynomials of graphs satisfying certain properties. For example, Levit and Mandrescu develop a number of results concerning the independence polynomials of well-covered and very well-covered graphs in [8], [9], [10], and [11]. We note that because a typical graph resulting from our construction here is not well-covered as defined in [7], none of the latter authors' results applying to well-covered graphs will apply to the graphs resulting from our construction. In [13] Mandrescu develops means of building graphs, all of the roots of whose independence polynomials are real. It is not hard to show that this condition is sufficient to guarantee not only unimodality but also the stronger condition of log-concavity (see Section 5).

In the following section we define our construction and state our paper's main result. Section 3 will outline the main tools we will use in our proof, and in Section 4 we will prove the main theorem. Section 5 will offer a roadmap for related constructions and future work.

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2 Definitions and a statement of the main result

Before describing our main theorem we require a handful of structural definitions.

Let $G = (V, E)$ be a graph and let $v \in V$. For any two graphs, G_1 and G_2, we let $G_1 \cup G_2$ be the disjoint union of the two graphs. For $U \subseteq V$ we denote by $G - U$ the subgraph of G induced by the vertex set $V \setminus U$; if $U = \{v\}$ is a single vertex, we abuse notation and write $G - v$. Recall that for $v \in V$, $N[v] = \{u \in V \mid u = v \text{ or } \{u, v\} \in E\}$ denotes the closed neighborhood of v. We will often be concerned with $G - N[v]$.

Let $k \geq 1$. The *K_t-path of length k*, denoted $P(t, k)$, is the graph (V, E) in which

$$V = \{v_1, v_2, \ldots, v_{t+k-1}\}$$

and

$$E = \Big\{ \{v_i, v_{i+j}\} \mid 1 \leq i \leq t + k - 2,\ 1 \leq j \leq \min\{t - 1, t + k - i - 1\} \Big\}.$$

Such a graph consists of k copies of K_t, each glued to the previous one by identifying certain prescribed subgraphs isomorphic to K_{t-1}. In Figure 1 we see the graph $P(4, 7)$, with vertices labeled as in the definition.

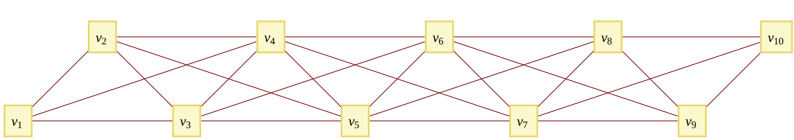

Figure 1: The graph $P(4, 7)$

Let $d \geq 0$ be an integer. We define the *d-augmented* K_t path $P(t, k, d)$, by introducing new vertices $\{u_{i,1}, u_{i,2}, \ldots, u_{i,d}\}_{i=0}^{t+k-2}$ and edges

$$\{\{v_i, u_{i,j}\}, \{v_{i+1}, u_{i,j}\} \mid j = 1, \ldots, d\}_{i=1}^{t+k-2} \cup \{\{v_1, u_{0,j}\} \mid j = 1, \ldots, d\}.$$

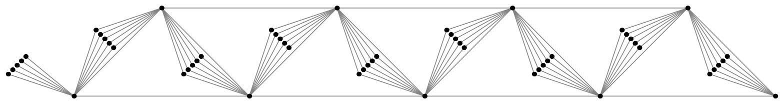

Figure 2: The graph $P(3,5,4)$

Roughly, every pair of consecutive vertices in $P(t,k)$ is made adjacent to a new set of d vertices; when $d=0$ we obtain an unmodified K_t-path. The graph $P(3,5,4)$ is shown in Figure 2.

Now suppose that $G=(V,E)$ is a graph and let $U \subseteq V$ be a subset of its vertices. Let $v \notin V$ and define the *cone of G on U with vertex v*, denoted $G^*(U,v)$ (or $G^*(U)$, if v is understood), by

$$V(G^*(U,v)) = V \cup \{v\} \text{ and } E(G^*(U,v)) = E \cup \Big\{\{u,v\} \mid u \in U\Big\}.$$

If $U=V$ we call $G^*(U,v)$ the *full cone* of G and denote it simply by $G^*(v)$ (or G^*, if v is understood). Given G and U as above and a vertex v in a graph Γ, we will often say that we *U-cone G off of v* in forming a cone of G with vertex v. Given G and U as above and a graph Γ, we denote by $\Gamma\nabla(G,U)$ the graph obtained by U-coning G off of every vertex in Γ. Figure 3 shows the graph $P(2,5)\nabla K_4$, obtained by full-coning a copy of K_4 off of each vertex of an ordinary path of length 5.

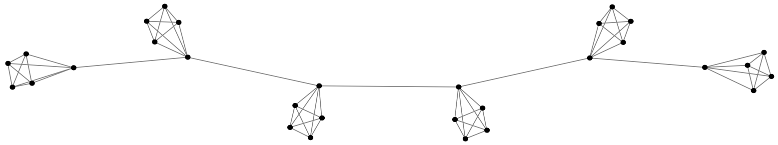

Figure 3: The graph $P(2,5)\nabla K_4$

We can now state our main theorem:

Theorem 2.1 *Let $t \geq 2$, $k \geq 1$, and $d \geq 0$ be integers, and let $G=(V,E)$ be a graph with $U \subseteq V$ a distinguished subset of vertices. Suppose that each of the graphs G, $G-U$, and $G^*(U,v)$ is SU, and that* $\deg(I(G;x)) = \deg(I(G^*(U,v);x)) = \deg(I(G-U;x))+2$. *Then the graph $P(t,k,d)\nabla(G,U)$ is SU.*

As a special case, consider $t=2$, $d=0$, $G=N_2$ (the null graph on 2 vertices), and $U=V$. In this case $G^*(U,v)=G^*$ is the path P_3 and $G-U$ is empty, so we have $I(G;x)=x^2+2x+1$, $I(G^*;x)=x^2+3x+1$, and $I(G-U;x)=1$. Because all of these polynomials are obviously SU and clearly satisfy the degree condition demanded in Theorem 2.1, we may apply the theorem and thus reestablish the primary result of [16], concerning caterpillars with exactly 2 legs on each nonperipheral vertex.

3 Useful lemmas and propositions

Following are the primary tools we will use in our proof, often without explicit mention.

Proposition 3.1 *Let G_1 and G_2 be disjoint graphs. Then*

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x).$$

Proposition 3.2 *Let $G = (V, E)$ be a graph and let $v \in V$. Then*

$$I(G; x) = I(G - v; x) + x \cdot I(G - N[v]; x).$$

For more details concerning these results, see [7]. We also have, from [1], the following lemma:

Lemma 3.3 *If $p(x)$ and $q(x)$ are symmetric unimodal polynomials with nonnegative coefficients, then so is $p(x)q(x)$. (In particular, any nonnegative power of a SU polynomial is SU.)*

Finally, we will frequently require the following lemma, whose proof is very elementary:

Lemma 3.4 *Let $p(x)$ and $q(x)$ be polynomials of degree r and $r-1$ respectively, for some $r \geq 2$, and let $p(0) \neq 0$ and $q(0) = 0$. If $p(x)$ and $q(x)$ are SU, then so is $p(x) + q(x)$.*

Proof: First assume that r is even. Let $p(x) = \sum_{i=0}^{2k} p_i x^i$ and $q(x) = \sum_{i=1}^{2k-1} q_i x^i$. We know that $p(x)$ will have $2k+1$ terms and $q(x)$ will have $2k-1$ terms. Since they both have an odd number of terms and are SU, their coefficient sequences will each have a single peak at the index k. Thus

$$p(x) + q(x) = p_0 + (p_1 + q_1)x + \cdots + (p_k + q_k)x^k + \cdots + (p_1 + q_1)x^{2k-1} + p_0 x^{2k},$$

which is clearly SU, its coefficients peaking at the index k.

If r is odd, we may argue in an entirely analogous fashion, preserving the peak at the indices $\frac{r-1}{2}$ and $\frac{r+1}{2}$. □

These tools in hand, we now turn to our main task.

4 A proof of the main theorem

Throughout this section we let $t \geq 2$, $k \geq 1$, and $d \geq 0$ be fixed natural numbers, and we refer to the vertices of $P(t, k, d)\nabla(G, U)$ using the notation from Section 2. Given a graph G and subset U of its vertices, we will let $b(x) = I(G; x)$, $c(x) = I(G - U, x)$, and $f(x) = I(G^*(U, v); x)$. For a positive integer r and graph Γ we let $r\Gamma$ represent r disjoint copies of Γ.

Our first lemma concerns the structure of $P(t, k, d)\nabla(G, U)$ upon removal of the first path vertex or its neighborhood.

Lemma 4.1 *Let $t, k, d, G,$ and U be as above, and let v_1 be the first vertex of the K_t-path in $P(t, k, d)$.*

(i) If $k \geq 2$ then $(P(t, k, d)\nabla(G, U)) - v_1 \cong G \cup dG^*(U) \cup (P(t, k-1, d)\nabla(G, U))$.

(ii) If $k = 1$ then $(P(t, 1, d)\nabla(G, U)) - v_1 \cong G \cup dG^*(U) \cup (P(t-1, 1, d)\nabla(G, U))$.

(iii) If $k \geq t + 1$ then

$$(P(t, k, d)\nabla(G, U)) - N[v_1] \cong \\ (G - U) \cup (2d + t - 1)G \cup d(t - 2)G^*(U) \cup (P(t, k - t, d)\nabla(G, U)).$$

(iv) If $1 \leq k \leq t$ then

$$(P(t, k, d)\nabla(G, U)) - N[v_1] \cong \\ (G - U) \cup (2d + t - 1)G \cup d(t - 2)G^*(U) \cup (P(k - 1, 1, d)\nabla(G, U)).$$

Proof: We prove (i) and (iii); the proofs of the other parts are entirely analogous.

For (i), note that on removing v_1 we obtain the isolated copies of $G^*(U, u_{0,j})$ ($j = 1, \ldots, d$) and a lone isolated copy of G that arises upon removing v_1 from $G^*(U, v_1)$. Each vertex v_i ($i = 2, \ldots, t + k - 1$) now takes on the role v'_{i-1} in the remaining "primed" copy of $P(t, k - 1, d)\nabla(G, U)$, and each $u_{i,j}$ ($i = 2, \ldots, t + k - 2$ and $j = 1, \ldots, d$) now takes on the role $u'_{i-1,j}$ in the remaining $P(t, k - 1, d)\nabla(G, U)$.

For (iii), note that on removing $N[v_1]$ we obtain (1) a lone copy of $G - U$ that arises upon removing U from the isolated G in $(P(t, k, d)\nabla(G, U)) - v_1$, (2) isolated copies of G for each

$$v \in \{u_{0,1}, \ldots, u_{0,d}, u_{1,1}, \ldots, u_{1,d}, v_2, \ldots, v_t\},$$

and (3) isolated copies of $G^*(U, v)$ for

$$v \in \{u_{i,j} \mid i = 2, \ldots, t - 1, j = 1, \ldots, d\}.$$

Moreover, each vertex v_i ($i = t + 1, \ldots, t + k - 1$) now takes on the role v'_{i-t} in the remaining "primed" copy of $P(t, k-t, d)\nabla(G, U)$, and each $u_{i,j}$ ($i = t+1, \ldots, t+k-2$ and $j = 1, \ldots, d$) now takes on the role $u'_{i-t,j}$ in the remaining $P(t, k - t, d)\nabla(G, U)$. □

We are now nearly able to prove Theorem 2.1; we require one more intermediate result concerning short d-augmented K_t-paths:

Lemma 4.2 *Let $G = (V, E)$ be a graph such that for some $U \subseteq V$ each of the polynomials $b(x)$, $c(x)$, and $f(x)$ (as above) are SU and that* $\deg(f) = \deg(b) = \deg(c) + 2$. *Then every graph of the form $P(t, 1, d)\nabla(G, U)$ is SU, for $t \geq 1$.*

Proof: Let $f_t(x) = I(P(t,1,d)\nabla(G,U);x)$ for $t \geq 1$. We argue by induction on t, including the additional inductive hypothesis that $\deg(f_t) = t(d+1)\deg(b)$. The base case is not difficult; applying Proposition 3.2 to the single vertex in K_1 itself we obtain

$$f_1(x) = I(P(1,1,d)\nabla(G,U);x) = b(x)\left(f(x)\right)^d + xc(x)\left(b(x)\right)^d.$$

Since our hypotheses imply $\deg(bf^d) = \deg(xcb^d) + 1$, we see that $\deg(f_1) = (d+1)\deg(b)$, as desired, and that Lemma 3.4 applies, showing that f_1 is SU.

Now suppose we have verified the lemma for all values up to and including some fixed $t-1$, and consider $P(t,1,d)\nabla(G,U)$. Applying Lemma 4.1 and Proposition 3.2 to the first vertex of $P(t,1,d)\nabla(G,U)$ in the K_t-path itself (that is, v_1), we obtain

$$f_t(x) = b(x)\left(f(x)\right)^d f_{t-1}(x) + xc(x)\left(b(x)\right)^{2d+t-1}\left(f(x)\right)^{d(t-2)}.$$

By inductive hypothesis the first term has degree

$$(d+1)\deg(b) + (t-1)(d+1)\deg(b) = t(d+1)\deg(b),$$

and the second term has degree

$$\begin{aligned} &(2d+t-1)\deg(b) + d(t-2)\deg(f) + \deg(c) + 1 \\ =\ & (t+td-1)\deg(b) + \deg(c) + 1 \\ =\ & t(d+1)\deg(b) - 1. \end{aligned}$$

Therefore $\deg(f_t) = t(d+1)\deg(b)$, as needed, and Lemma 3.4 applies, showing that f_t is SU. □

We can now prove the main theorem.

Proof:[Proof of Theorem 2.1] For each $k \geq 1$ let $p_k(x) = I(P(t,k,d)\nabla(G,U);x)$. Also, define $p_{1-t}(x) = 1$ and $p_k(x) = I(P(t+k-1,1,d)\nabla(G,U);x)$ for each $k \in \{2-t, 3-t, \ldots, 0\}$. (For example, $p_{2-t}(x) = I(P(1,1,d)\nabla(G,U);x)$.) By Lemma 4.2, p_k is SU for $k \in \{1-t, \ldots, 0\}$.

For $k \geq 1$ we prove $p_k(x)$ is SU by induction on k, with the additional inductive hypothesis that $\deg(p_k) = (t+k-1)(d+1)\deg(b)$. The base case ($k=1$) is supplied by Lemma 4.2. Suppose now that we have proven the proposition true for all values up to and including some fixed $k-1$. Applying Lemma 4.1 and Proposition 3.2 to the first vertex of $P(t,k,d)\nabla(G,U)$ in the K_t path itself, we obtain

$$p_k(x) = bf^d p_{k-1}(x) + xcb^{2d+t-1}f^{d(t-2)}p_{k-t}(x).$$

Note that this formula is valid even when $k-t \leq 0$, thanks to our definitions in the previous paragraph.

By inductive hypothesis

$$\begin{aligned} \deg(bf^d p_{k-1}) &= (d+1)\deg(b) + (t+k-2)(d+1)\deg(b) \\ &= (t+k-1)(d+1)\deg(b) \end{aligned}$$

and

$$\begin{aligned}&\deg(xcb^{2d+t-1}f^{d(t-2)}p_{k-t})\\ &= \deg(c)+(td+t-1)\deg(b)+(k-1)(d+1)\deg(b)+1\\ &= ((t+k-1)(d+1)-1)\deg(b)+\deg(c)+1\\ &= (t+k-1)(d+1)\deg(b)-1.\end{aligned}$$

Therefore $\deg(p_k)=(t+k-1)(d+1)\deg(b)$ as desired, and Lemma 3.4 applies, so that $p_k(x)$ is SU. □

Examples.

(i) Suppose that we demand $U=V(G)$ in our cone construction. In forming the cone G^* we introduce a new vertex which is adjacent to every vertex already present in G, and therefore the only change to the independence polynomial $b(x)=I(G;x)$ is the addition of a single new term x: $f(x)=b(x)+x$. If both b and f are to be SU, it must thus be that $\deg(b)=\deg(f)=2$. It is not hard to show that the only SU graphs with independence number 2 are of the form $G=K_s-e$, $s\geq 2$. Therefore if we insist on using full cones, the only graphs $P(t,k,d)\nabla(G,V)$ to which Theorem 2.1 applies are those in which $G=K_s-e$.

(ii) As a more nontrivial example, let us take G *itself* to be one of the 2-regular caterpillars considered by Zhu, that is, $G=P(2,k,0)\nabla N_2$ for some $k\geq 1$. Then $\deg(b(x))=2(k+1)$ and $b(x)$ is SU. If we let U consist of a single nonperipheral vertex and its two leaves, $G-U$ is clearly $P(2,k-1,0)\nabla N_2$ and so $\deg(c(x))=2k=\deg(b)-2$, and

$$f(x)=I(G^*(U,v);x)=I(P(2,k,0)\nabla N_2;x)+xI(P(2,k-1,0)\nabla N_2;x),$$

to which Lemma 3.4 easily applies, showing that $f(x)$ is SU and of degree $2(k+1)=\deg(b)$. We may therefore apply Theorem 2.1 with the caterpillar graphs as the cone graphs, yielding yet new examples of SU graphs.

5 Related constructions and open problems

Arguments analogous to those of the previous section allow us to analyze another recursively defined sequence of graphs not captured by the K_t-path construction.

For a given $k\geq 1$ we define the graph C_k to be the length=k *ladder* with pendant vertex v, consisting of vertices $V=\{u_1,v_1,\ldots,u_k,v_k,v\}$ and edges

$$E=\{\{u_i,v_i\}|\ i=1,\ldots,k\}\cup\{\{u_i,u_{i+1}\},\{v_i,v_{i+1}\}\mid i=1,\ldots,k-1\}\cup\{\{v,v_1\}\}.$$

An argument similar to the proof of Lemma 4.1 shows that

$$(C_k\nabla(G,U))-v_1\cong G\cup G^*(U)\cup C_{k-1}\nabla(G,U)$$

and

$$(C_k\nabla(G,U))-N[v_1]\cong(G-U)\cup 3G\cup(C_{k-2}\nabla(G,U)).$$

Proposition 5.1 *Let $k \geq 1$ be an integer, and let $G = (V, E)$ be a graph with $U \subseteq V$ a distinguished subset of vertices. Suppose that each of the graphs G, $G - U$, and $G^*(U, v)$ is SU, and that (with the notation of the previous section)* $\deg(b) = \deg(f) = \deg(c) + 2$. *Then the graph $C_k \nabla(G, U)$ is SU.*

Proof: Let $q_k(x) = I(C_k \nabla(G, U); x)$ for $k \geq 1$. As before our proof is an induction on $k \geq 1$, with the additional inductive hypothesis that $\deg(q_k) = (2k+3)\deg(b)$. If we let $q_{-1}(x) = f(x)$ and $q_0(x) = b(x)\left(f(x)\right)^2 + xc(x)\left(b(x)\right)^2$, then the following recursive formula holds for all $k \geq 1$, applying Proposition 3.2 to the vertex v_1:

$$q_k(x) = b(x)f(x)q_{k-1}(x) + xc(x)\left(b(x)\right)^3 q_{k-2}(x).$$

By inductive hypothesis,

$$\deg(bfq_{k-1}) = (2k+1)\deg(b) + 2\deg(b) = (2k+3)\deg(b)$$

and

$$\deg(xcb^3 q_{k-2}) = (2k-1)\deg(b) + 4\deg(b) - 1 = (2k+3)\deg(b) - 1,$$

so q_k has the desired degree and Lemma 3.4 applies to show that q_k is SU. □

The presence of the family $C_k \nabla(G, U)$ leads us to ask the following

Question 5.2 *Can we obtain an explicit description of the most general family of graphs to which Theorem 2.1 may be applied in order to show symmetry and unimodality?*

We direct the reader to [15], in which Stevanović introduces a number of other means of constructing graphs with symmetric independence polynomials.

Although we have concerned ourselves here with symmetry and unimodality, there are other interesting conditions one can place on real-valued sequences. For instance, a sequence $\{a_i\}_{i \geq 0}$ is said to be *logarithmically concave* (or *log-concave*) if for all $i \geq 1$ we have $a_i^2 \geq a_{i-1}a_{i+1}$. If the coefficient sequence of a polynomial p is log-concave, we may say that p itself is log-concave. It is not hard to show that log-concavity implies unimodality. Moreover, log-concave polynomials enjoy some of the same "closure" properties as do SU polynomials. Consider the following analogue of Lemma 3.3 (see [6] for more details):

Lemma 5.3 *Let $p(x)$ and $q(x)$ be polynomials with non-negative coefficients.*

(i) If p is log-concave and q is unimodal, then pq is unimodal.

(ii) If both p and q are log-concave, then so is pq.

However, there is no obvious analogue to Lemma 3.4, so proving that sums of log-concave polynomials are log-concave is a more challenging enterprise. Given the presence of the sum on the right hand side of all of our recurrence relations above, it is this obstacle that would have be surmounted in order to obtain recursive constructions of graphs with log-concave independence polynomials. Nevertheless, having used *Mathematica* to explicitly compute the independence polynomials of a very large number of the graphs $P(t, k, d)\nabla(G, U)$, we are led to posit the following

Conjecture 5.4 *Let $t \geq 2$, $k \geq 1$, and $d \geq 0$ be integers, and let $G = (V, E)$ be a graph with $U \subseteq V$ a distinguished subset of vertices. Suppose that each of the graphs G, $G - U$, and $G^*(U, v)$ is SU, and that* $\deg(I(G; x)) = \deg(I(G^*(U, v); x)) = \deg(I(G - U; x)) + 2$. *Then the graph $P(t, k, d)\nabla(G, U)$ is log-concave.*

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