AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 145–156

# Cycle decompositions of $\lambda$-fold complete equipartite graphs

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**Abstract**

In this paper we give constructions for various new fixed length cycle decompositions of $\lambda$-fold complete equipartite graphs. In particular, we determine necessary and sufficient conditions for the existence of such a decomposition in the case where the cycle length is prime.

## 1 Introduction

Throughout this paper we deal frequently with so-called *multigraphs*; that is, graphs which may contain more than one edge joining the same pair of vertices. For our purposes we define a multigraph $G$ (hereby referred to as just a *graph*) to be a finite non-empty set of vertices $V(G)$ together with a multiset $E(G)$ of 2-element subsets of $V(G)$ called edges. Note that we do not allow loops on vertices in our graphs. We will often denote an edge $\{u, v\}$ by $uv$. The *multiplicity* of an edge $uv$ in $G$ is defined to be the number of occurences of $uv$ in $E(G)$.

A *decomposition* of a graph $G$ is a collection of subgraphs of $G$ whose edges partition $E(G)$. Suppose $H$ is a subgraph of $G$ and $\rho$ is a permutation on $V(G)$, then $\rho(H)$ is defined to be the graph with vertex set $\{\rho(v) \mid v \in V(H)\}$ and edge multiset $\{\rho(u)\rho(v) \mid uv \in E(H)\}$. (Note that $\rho(H)$ is isomorphic to $H$ but need not be a subgraph of $G$.) If $\rho$ has order $\alpha$, we say $H$ *decomposes* $G$ *under the permutation* $\rho$ if the graphs $\rho^i(H)$, $0 \leqslant i \leqslant \alpha - 1$, together decompose $G$. Similarly we say the collection of graphs $H_1, H_2, \ldots, H_t$ decompose $G$ under the permutation $\rho$, if the graphs $\rho^i(H_j)$, $0 \leqslant i \leqslant \alpha - 1$ and $1 \leqslant j \leqslant t$, together decompose $G$.

The lexicographic product $G * H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, and with precisely $\lambda$ edges joining $(g_1, h_1)$ to $(g_2, h_2)$ if and only if there are precisely $\lambda$ copies of the edge $g_1g_2$ in the multiset $E(G)$, or if $g_1 = g_2$ and there are precisely $\lambda$ copies of the edge $h_1h_2$ in the multiset $E(H)$. We note that if $G$ has a decomposition into graphs $G_1, G_2, \ldots, G_t$, then $G * H$ has a decomposition into graphs $G_1 * H, G_2 * H, \ldots, G_t * H$.

For any graph $G$ and any positive integer $\lambda$, we let $\lambda G$ denote the graph obtained from $G$ by increasing the multiplicity of each of the edges in $G$ by a factor of $\lambda$.

Using the above notation, we denote the $\lambda$-fold complete equipartite graph, having $n$ parts of size $m$, precisely $\lambda$ edges joining any two vertices in different partite sets and no edges joining any two vertices in the same partite set, by $\lambda K_n * \overline{K_m}$ (where $\overline{K_m}$ denotes the empty graph on $m$ vertices). We note that graphs of this type include the complete graphs $K_n$ (when $\lambda = m = 1$), complete multigraphs $\lambda K_n$ (when $m = 1$) and complete equipartite graphs $K_n * \overline{K_m}$ (when $\lambda = 1$).

In this paper we examine decompositions of $\lambda K_n * \overline{K_m}$ into subgraphs, each of which is a cycle of length $k$ (sometimes referred to as a $k$-cycle and denoted by $C_k$). Obvious necessary conditions for the existence of such a decomposition are that

(i) $nm \geqslant k$;

(ii) $\lambda(n-1)m$ is even; and

(iii) $\lambda n(n-1)m^2 \equiv 0 \pmod{2k}$.

For $n \geqslant 3$, these conditions have been proved sufficient by Hanani [9] in the case $k = 3$, and by Billington, Hoffman and Maenhaut [5] in the case $k = 5$. We note also that for $n \geqslant 3$ these conditions have been proved sufficient in the cases $(\lambda, m) = (1,1), (1,2), (2,1)$. In these cases the graphs being decomposed are, respectively, complete graphs (of necessarily odd order), complete graphs of even order minus a 1-factor and 2-fold complete multigraphs. These results are summarised in the following three theorems.

**Theorem 1.1** ([2],[14]) *Suppose $n$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the complete graph $K_n$ admits a decomposition into cycles of length $k$ if and only if $n \geqslant k$, $n$ is odd and $n(n-1) \equiv 0 \pmod{2k}$.*

**Theorem 1.2** ([2],[14]) *Suppose $n$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the complete equipartite graph $K_n * \overline{K_2}$ admits a decomposition into cycles of length $k$ if and only if $2n \geqslant k$ and $2n(n-1) \equiv 0 \pmod{k}$.*

**Theorem 1.3** ([3]) *Suppose $n$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the 2-fold complete multigraph $2K_n$ admits a decomposition into cycles of length $k$ if and only if $n \geqslant k$ and $n(n-1) \equiv 0 \pmod{k}$.*

The above results will be useful for us since it is clear that a decomposition of $\lambda K_n * \overline{K_m}$ into cycles of length $k$ exists whenever there exists a decomposition of either $\lambda K_n$ or $K_n * \overline{K}_m$ into cycles of length $k$ (see Section 2 for further explanation). However, not all decompositions of $\lambda$-fold complete equipartite graphs can be obtained in this way. For example, consider the values $(\lambda, n, m, k) = (2, 12, 7, 7)$. These values satisfy the necessary conditions (i), (ii) and (iii) above, and hence we "expect" that the graph $2K_{12} * \overline{K}_7$ admits a decomposition into cycles of length 7 (in fact we prove the existence of such a decomposition in this paper). However, neither $2K_{12}$ nor $K_{12} * \overline{K}_7$ admits a decomposition into 7-cycles since the number of edges in $2K_{12}$ is not a multiple of 7 and every vertex in $K_{12} * \overline{K}_7$ has odd degree.

In this paper we offer various new construction methods for obtaining $k$-cycle decompositions of $\lambda$-fold complete equipartite graphs. Of particular interest are those constructions which give decompositions whose existence is *not* implied by a known decomposition of either $\lambda K_n$ or $K_n * \overline{K}_m$ (as discussed above). We then use these constructions to prove necessary and sufficient conditions for decomposing $\lambda$-fold complete equipartite graphs into cycles of prime length. It is worth noting that Manikandan and Paulraja [13] have recently dealt with this problem in the (non-multigraph) case $\lambda = 1$; however, we do not rely on their result here.

In addition to the decompositions given in Theorems 1.1, 1.2 and 1.3, we will also make use of the following two results. The first is a result of Smith [17] involving cycle decompositions of complete multigraphs, the second is a result of Liu [10],[11] involving cycle decompositions of complete equipartite graphs. (Note that Liu's original result concerned *resolvable* decompositions. However, we omit this detail since we will not be concerned with that particular property here. This also allows us to remove the small number of "exceptions" that appear in Liu's original result which will be useful for our purposes.)

**Theorem 1.4** ([17]) *Suppose $\lambda$, $n$ and $k$ are positive integers with $n \geqslant k \geqslant 3$. Then the $\lambda$-fold complete multigraph $\lambda K_n$ admits a decomposition into cycles of length $k$ whenever $\lambda(n-1)$ is even and $\lambda \equiv 0 \pmod{k}$.*

**Theorem 1.5** ([10],[11]) *Suppose $n$, $m$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the complete equipartite graph $K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $(n-1)m$ is even and $nm \equiv 0 \pmod{k}$.*

## 2 Some useful decompositions

It is easy to see that $\lambda G$ admits a decomposition into $k$-cycles whenever $G$ admits a decomposition into $k$-cycles. It is also well-known (see for instance [7]) that $G * \overline{K}_m$ admits a decomposition into $k$-cycles, and a decomposition into $km$-cycles, whenever $G$ admits a decomposition into $k$-cycles. Combining these observations we have the following useful result. (Observe that, by the properties of the lexicographic product, $(G * \overline{K}_a) * \overline{K}_t \cong G * \overline{K}_{at}$ for all positive integers $a$ and $t$.)

**Lemma 2.1** *If $G$ admits a decomposition into cycles of length $s$, then $\mu G * \overline{K}_{at}$ admits a decomposition into cycles of length $st$ for all positive integers $\mu$, $a$ and $t$.*

The above lemma can be applied to Theorems 1.1, 1.2 and 1.3 to give the relevant "$\lambda$-fold equipartite graph" generalisations of these results.

**Theorem 2.2** *Suppose $\lambda$, $n$, $m$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $n \geqslant k$ and either*

*(i) $\lambda$, $n$ and $m$ are all odd, and $n(n-1) \equiv 0 \pmod{2k}$; or*

*(ii) at least one of $m$ and $\lambda$ is even, and $n(n-1) \equiv 0 \pmod{k}$.*

**Proof:** Suppose $\lambda$, $n$, $m$ and $k$ satisfy condition (i). Then $K_n$ admits a decomposition into cycles of length $k$ by Theorem 1.1 and the result follows by applying Lemma 2.1 with $(\mu, G, s, t, a) = (\lambda, K_n, k, 1, m)$.

Suppose then $\lambda$, $n$, $m$ and $k$ satisfy condition (ii). If $m$ is even then $K_n * \overline{K}_2$ admits a decomposition into cycles of length $k$ by Theorem 1.2 and the result follows by applying Lemma 2.1 with $(\mu, G, s, t, a) = (\lambda, K_n * \overline{K}_2, k, 1, m/2)$. Otherwise $\lambda$ is even, $2K_n$ admits a decomposition into cycles of length $k$ by Theorem 1.3 and the result follows by applying Lemma 2.1 with $(\mu, G, s, t, a) = (\lambda/2, 2K_n, k, 1, m)$. □

In this section we also make use of *pairwise balanced designs*. Formally, these are defined as follows.

**Definition 2.3** A *pairwise balanced design*, PBD$(v, K, \lambda)$, is a collection of subsets (or *blocks*) of a $v$-set $S$ such that the size of each block is in the set $K$ and every pair of distinct elements in $S$ occurs in exactly $\lambda$ blocks.

**Example 2.4** A PBD$(8, \{3, 5\}, 2)$. Blocks listed columnwise.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 2 | 3 | 4 | 2 | 3 | 4 | 6 | 7 | 5 | 3 | 4 | 2 |
| 2 | 5 | 5 | 6 | 7 | 5 | 6 | 7 | 4 | 2 | 3 | 7 | 5 | 6 |
| 3 | 6 | | | | | | | | | | | | |
| 4 | 7 | | | | | | | | | | | | |

We point out the well-known fact that a PBD$(v, K, \lambda)$ is equivalent to a decomposition of the $\lambda$-fold complete graph on $v$ vertices into a collection of complete graphs whose orders all lie in the set $K$. For example, the PBD$(8, \{3, 5\}, 2)$ given above induces a decomposition of $2K_8$ into two copies of $K_5$ and twelve copies of $K_3$. With this in mind we have the following useful result.

**Lemma 2.5** *Suppose $\lambda$ and $n$ are positive integers with $\lambda$ even and $n \geqslant 3$. Then there exist nonnegative integers $\alpha$ and $\beta$ such that the $\lambda$-fold complete multigraph $\lambda K_n$ admits a decomposition into $\alpha$ copies of $K_3$ and $\beta$ copies of $K_5$.*

**Proof:** By the remark above, and the fact that $\lambda$ is even, we need only show there exists a PBD$(n, \{3,5\}, 2)$ for each $n \geqslant 3$. This will induce a decomposition of $2K_n$ into copies of $K_3$ and $K_5$ and the result then follows (since $\lambda$ is even).

It is well-known that there exists a PBD$(n, \{3,4,5,6,8\}, 1)$ for each $n \geqslant 3$ (see for example [5]), so in fact we need only show there exists a PBD$(n, \{3,5\}, 2)$ for the values $n = 3,4,5,6,8$. This is immediate in the cases $n = 3,5$. Moreover, the case $n = 8$ is dealt with in Example 2.4. The remaining cases $n = 4,6$, follow from the existence of 3-cycle decompositions of $2K_4$ and $2K_6$ respectively (see Theorem 1.3). □

The next result is a generalisation of Theorem 1.5.

**Lemma 2.6** *Suppose $\lambda$, $n$, $m$ and $k$ are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $\lambda(n-1)m$ is even and $nm \equiv 0 \pmod{k}$.*

**Proof:** If either $n$ is odd or $m$ is even the result follows immediately by Theorem 1.5 and Lemma 2.1. Suppose then that $n$ is even, $m$ is odd and hence $\lambda$ is even. Let $s = \gcd(n,k)$ and $t = k/s$. Then since $nm \equiv 0 \pmod{k}$ we have $m = at$ for some positive integer $a$.

If $s \geqslant 3$ the graph $2K_n$ admits a decomposition into cycles of length $s$ by Theorem 1.3 and the result then follows by Lemma 2.1 with $(\mu, G) = (\lambda/2, 2K_n)$.

If $s = 2$ then $k = 2t$ and the result follows by first decomposing $\lambda K_n * \overline{K}_m$ into copies of the 2-fold complete bipartite graph $2K_2 * \overline{K}_{at}$, then decomposing each of these into the required $2t$-cycles (such a decomposition exists by [15]).

Finally if $s = 1$, then by Lemma 2.5, there exist nonnegative integers $\alpha$ and $\beta$ such that the graph $\lambda K_n$ admits a decomposition into $\alpha$ copies of $K_3$ and $\beta$ copies of $K_5$. Hence $\lambda K_n * \overline{K}_m$ admits a decomposition into $\alpha$ copies of $K_3 * \overline{K}_m$ and $\beta$ copies of $K_5 * \overline{K}_m$. Since $k = t$ and $m \equiv 0 \pmod{t}$, both $K_3 * \overline{K}_m$ and $K_5 * \overline{K}_m$ admit a decomposition into cycles of length $t$ (see [7] and [4] respectively) and the result follows. □

## 3 New decompositions

In this section we give several new constructions for $k$-cycle decompositions of $\lambda K_n * \overline{K}_m$ in cases where the multiplicity $\lambda$ is a multiple of the cycle length $k$. In particular, we consider those cases in which the cycle length $k$ is odd. We note that numerous decompositions of this type are obtained in [8]. In particular, decompositions are given for cases in which both $n$ and $m$ are even. These results are as follows.

**Theorem 3.1** ([8]) *Suppose $n$, $m$, $\lambda$ and $k$ are positive integers with $k \geqslant 3$, $n \geqslant 4$, $k$ odd and both $n$ and $m$ even. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $nm > k$ and $\lambda \equiv 0 \pmod{k}$.*

Thus here we will primarily be concerned with cases in which at least one of $n$ or $m$ is odd. Note also that if $m$ is odd but $n$ is even, then $\lambda$ must be even for such a decomposition to exist.

We make use of the following result which is a generalisation of Theorem 1.1 in [16]. The notation $\alpha$ MOLS($\ell$) means a set of $\alpha$ mutually orthogonal latin squares of order $\ell$; in the case $\alpha = 1$, it means any one latin square of order $\ell$.

**Theorem 3.2** *Let $G$ be a connected even graph on $k \geqslant 3$ edges with maximum degree $\Delta(G) = \Delta$ and vertex chromatic number $\chi(G) = \chi$. Then for all $\ell \geqslant \Delta/2$, the graph $G * \overline{K}_\ell$ admits a decomposition into cycles of length $k$ whenever there exist at least $(\chi - 2)$ MOLS($\ell$).*

**Proof:** Suppose $\ell \geqslant \Delta/2$ and there exist $(\chi - 2)$ MOLS($\ell$). Recall that $G * \overline{K}_\ell$ is the graph with vertex set $\{v_1, v_2, \ldots, v_\ell \mid v \in V(G)\}$ and precisely $\lambda$ edges joining $u_x$ to $v_y$, $1 \leqslant x, y \leqslant \ell$, if and only if there are precisely $\lambda$ edges joining $u$ and $v$ in $G$. We place a proper colouring on the vertices of $G$ using the colours $1, 2, \ldots, \chi$, and for each $c \in \{1, 2, \ldots, \chi\}$ we let $V_c$ be the set of all vertices in $G$ which receive colour $c$ in this colouring.

Let $T_k$ be a particular traversal of the edges of $G$ which forms a closed trail of length $k$ (such a traversal exists since $G$ is connected and even). Note that if a vertex has degree greater than 2 in $G$, then it will appear more than once in the trail $T_k$. Hence we may talk about the "first" occurrence of a specific vertex in $T_k$, and where appropriate, the "second" occurrence, and so on. Note that for our purposes we will consider both the start and end vertex of $T_k$ to be the first occurrence of that particular vertex. Hence if $d(v)$ denotes the degree of $v$ in $G$, then each vertex $v \in V(G)$ has precisely $d(v)/2 \leqslant \ell$ occurrences in the trail $T_k$.

Let $S_{\chi-1}$ be an $\ell \times \ell$ array in which each entry in row $i$ is $i$, $1 \leqslant i \leqslant \ell$, and let $S_\chi$ be the transpose of $S_{\chi-1}$. If $\chi > 2$, then let $S_1, S_2, \ldots, S_{\chi-2}$ be a set of $(\chi - 2)$ MOLS($\ell$) on the elements $\{1, 2, \ldots, \ell\}$. Then for each $c \in \{1, 2, \ldots, \chi\}$ we let $(i, j)_c$ be the entry in row $i$, column $j$ of the array $S_c$. Note that for all $x, y \in \{1, 2, \ldots, \ell\}$ and each pair of distinct arrays $S_{c_1}, S_{c_2} \in \{S_1, S_2, \ldots, S_\chi\}$ there exists a unique row $i$ and column $j$ such that $(i, j)_{c_1} = x$ and $(i, j)_{c_2} = y$. For each ordered pair $(i, j)$ with $1 \leqslant i, j \leqslant \ell$, we form a $k$-cycle in $G * \overline{K}_\ell$, from the closed trail $T_k$, as follows.

For each vertex $v \in V_1$ (that is, for each vertex $v$ coloured 1), we replace the first occurrence of $v$ in $T_k$ with $v_{(i,j)_1}$; we replace the second occurrence of $v$ with $v_{(i,j)_1+1}$; and so on. In general, for each $1 \leqslant m \leqslant d(v)/2$, we replace the $m$th occurrence of $v$ with $v_{(i,j)_1+m-1}$ (subscripts taken modulo $\ell$ from residues $1, 2, \ldots, \ell$). Then for each vertex $v \in V_2$, we replace, for each $m$ in the range $1 \leqslant m \leqslant d(v)/2$, the $m$th occurrence of $v$ in $T_k$ with $v_{(i,j)_2+m-1}$. In general, for each vertex $v \in V_c$, with

$1 \leqslant c \leqslant \chi$, we replace, for each $m$ in the range $1 \leqslant m \leqslant d(v)/2$, the $m$th occurrence of $v$ in $T_k$ with $v_{(i,j)_c+m-1}$.

Observe that since no vertex appears more than $\ell$ times in $T_k$, the above construction does in fact form a cycle in each case. This gives us $\ell^2$ cycles in total. We now show that these cycles do indeed decompose $G * \overline{K}_\ell$.

Consider an arbitrary edge $u_x v_y$, with multiplicity $\lambda$ in $G * \overline{K}_\ell$. The edge $uv$ must have multiplicity $\lambda$ in $G$, and hence $u$ and $v$ must occur consecutively $\lambda$ times in the trail $T_k$. Assume that this happens at precisely the $\alpha_t$th occurrence of the vertex $u$ and the $\beta_t$th occurrence of the vertex $v$, for each $t \in \{1, 2, \ldots, \lambda\}$. (Note that $(\alpha_t, \beta_t) = (\alpha_s, \beta_s)$ if and only if $t = s$.) We may assume also that $u \in V_{c_1}$ and $v \in V_{c_2}$, where $c_1 \neq c_2$, since $u$ and $v$ are adjacent in $G$. For each $t \in \{1, 2, \ldots, \lambda\}$ we then choose $i_t$ and $j_t$ (uniquely) so that cell $(i_t, j_t)$ contains the entry $x - \alpha_t + 1$ in the array $S_{c_1}$, and $y - \beta_t + 1$ in the array $S_{c_2}$ (where both $x - \alpha_t + 1$ and $y - \beta_t + 1$ are calculated modulo $\ell$ from the residues $\{1, 2, \ldots, \ell\}$). The $k$-cycle constructed for this ordered pair $(i_t, j_t)$ will then include the edge $u_x v_y$ as required. □

The following result will be used to deal with cases in which $n$ is odd.

**Lemma 3.3** *Suppose $n$ and $k$ are odd with $k > n \geqslant 3$. Then the graph $kC_n$ has a decomposition into connected even graphs, each with $k$ edges and maximum degree $\Delta = 2\lceil k/n \rceil$.*

**Proof:** Let the vertex set of $kC_n$ be $\mathbb{Z}_n$, with vertex $v$ adjacent to vertex $u$ if and only if $u$ and $v$ differ modulo $n$ by 1. We write $k = n + 2r$ where $r \geqslant 1$. Furthermore, let $\rho$ be the permutation of order $n$ on $V(kC_n)$ defined by $\rho = (\,0\,1\,\cdots\,(n-1)\,)$.

Let $T$ be the subgraph of $kC_n$ formed by adding two edges joining $2j$ and $2j+1$ (where $2j$ and $2j+1$ are each calculated modulo $n$ from the residues $\mathbb{Z}_n$), for each $j \in \{0, 1, \ldots, r-1\}$, to the cycle $(0, 1, 2, \ldots, n-1)$. Hence $T$ is connected, even, has $2r + n = k$ edges, and has maximum degree

$$\Delta = 2\lceil 2r/n \rceil + 2 = 2\lceil (k-n)/n \rceil + 2 = 2\lceil k/n \rceil.$$

Then $T$ decomposes $kC_n$ under the permutation $\rho$. □

Noting that each graph in the above decomposition of $kC_n$ is of course 3-colourable, we can apply Theorem 3.2 to Lemma 3.3 and obtain the following simple corollary.

**Corollary 3.4** *Suppose $n$, $m$, $\lambda$ and $k$ are positive integers with $k > n \geqslant 3$ and both $n$ and $k$ odd. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $nm \geqslant k$ and $\lambda \equiv 0 \pmod{k}$.*

**Proof:** Since $n \geqslant 3$ is odd, $K_n$ admits a decomposition into Hamilton cycles. Thus, since $\lambda \equiv 0 \pmod{k}$, the graph $\lambda K_n * \overline{K}_m$ admits a decomposition into $\lambda n(n-1)/2k$

copies of the graph $kC_n * \overline{K}_m$. Hence we need only show that $kC_n * \overline{K}_m$ admits a decomposition into cycles of length $k$.

By Lemma 3.3, $kC_n$ admits a decomposition into 3-colourable, connected even graphs, each with $k$ edges and maximum degree $\Delta = 2\lceil k/n \rceil$. Moreover, since $nm \geqslant k$ we have $m \geqslant \lceil k/n \rceil = \Delta/2$. Hence by applying Theorem 3.2 to each graph in this decomposition of $kC_n$ (note that there exists a latin square of each positive integer order) we have that $kC_n * \overline{K}_m$ admits a decomposition into cycles of length $k$. The result follows. □

The following result will be used to deal with cases in which $m$ is odd and both $\lambda$ and $n$ are even.

**Lemma 3.5** *Suppose $n$ and $k$ are integers with $n$ even, $k$ odd and $k > n \geqslant 4$. Then the graph $2kK_n$ has a decomposition into 4-colourable, connected even graphs, each with $k$ edges and maximum degree $\Delta = 2\lceil k/n \rceil$.*

**Proof:** Let the vertex set of $2kK_n$ be $\mathbb{Z}_{n-1} \cup \{\infty\}$. Furthermore, let $\rho$ be the permutation of order $n-1$ on $V(2kK_n)$ defined by $\rho = (\,0\,1\,\cdots\,(n-2)\,)(\infty)$. For each $1 \leqslant i \leqslant n$ we let $H_i = W$ where $W$ is the $n$-cycle

$$W = (0, 1, n-2, 2, n-3, \ldots, n/2-1, n/2, \infty).$$

(Note that this is the "starter" cycle from the well-known Walecki construction for decomposing $K_n - F$ into Hamilton cycles in the case $n$ even; see [12].) The cycle $W$ decomposes $2K_n$ under the permutation $\rho$ and hence the cycles $H_i$, $1 \leqslant i \leqslant n$, together decompose $2nK_n$ under the permutation $\rho$. We write $k = qn + \lambda$, where $0 \leqslant \lambda < n$ and $\lambda$ is odd, and then split the problem according to whether $\lambda = 1$ or $\lambda \geqslant 3$.

**Case I:** $\lambda \geqslant 3$
Since $n > \lambda \geqslant 3$ and $n$ is even, there exist $\lambda$-cycles $J_i$, $1 \leqslant i \leqslant n$, which together decompose $2\lambda K_n$ under the permutation $\rho$ (see Theorem 3.1 in [17]). Such a decomposition is commonly called a 1-rotational decomposition since exactly 1 point, namely $\infty$, is fixed by the permutation $\rho$. Hence the graphs $qH_i \cup J_i$, $1 \leqslant i \leqslant n$, together decompose $2(qn+\lambda)K_n$ under the permutation $\rho$ (note that $2(qn+\lambda)K_n = 2kK_n$). Moreover, each of the graphs $qH_i \cup J_i$ is clearly even and connected with maximum degree $2q+2 = 2(q+1) = 2\lceil k/n \rceil$. It remains to show that these graphs are 4-colourable. In fact we need only show that the graphs $H_i \cup J_i$, $1 \leqslant i \leqslant n$, are 4-colourable (since if $H_i \cup J_i$ is 4-colourable then clearly so is $qH_i \cup J_i$). Now each of the graphs $H_i \cup J_i$ has maximum degree 4 and is not a complete graph. Hence by the well-known Brooks' Theorem [6], the graphs $H_i \cup J_i$ are each 4-colourable. This completes Case I.

**Case II:** $\lambda = 1$
Since $\lambda = 1$ we have $k = qn + 1$. Our aim here is to construct a collection of 4-colourable, even connected graphs $G_i$, $1 \leqslant i \leqslant n$, each of which has $k$ edges and

maximum degree $\Delta = 2\lceil k/n \rceil$, which together decompose $2kK_n$ under the permutation $\rho$. Since $k = qn + 1 = (q-1)n + (n-1) + 2$, we will form each graph by combining an $(n-1)$-cycle with $q-1$ Hamilton cycles (or no Hamilton cycles in the case $q = 1$) and two extra edges. We deal first with the case in which $q > 1$.

Let $J_i$, $1 \leqslant i \leqslant n$, be a collection of $(n-1)$-cycles which together decompose $2(n-1)K_n$ under the permutation $\rho$ (see Case I above). Hence the graphs $(q-1)H_i \cup J_i$, $1 \leqslant i \leqslant n$, together decompose $2(k-2)K_n$ under the permutation $\rho$ (note that each of the graphs $(q-1)H_i \cup J_i$ has $(q-1)n + (n-1) = k-2$ edges). Moreover, each of the graphs $(q-1)H_i \cup J_i$, $1 \leqslant i \leqslant n$, is even and connected with maximum degree $2(q-1) + 2 = 2q = 2\lceil k/n \rceil - 2$. Also, by the remarks in Case I above, each of the graphs $(q-1)H_i \cup J_i$ is 4-colourable.

Recall that each $H_i$ is equal to the $n$-cycle $W = (0, 1, n-2, 2, n-3, \ldots, n/2 - 1, n/2, \infty)$. For each $1 \leqslant i \leqslant n$ we form the graph $G_i$ by adding two copies of some edge (selected from the cycle $W$) to the graph $(q-1)H_i \cup J_i$. More formally we do this as follows.

- Add two copies of the edge $\{0, 1\}$ to the graph $(q-1)H_1 \cup J_1$ to form $G_1$.
- Add two copies of the edge $\{1, n-2\}$ to the graph $(q-1)H_2 \cup J_2$ to form $G_2$.
- Add two copies of the edge $\{n-2, 2\}$ to the graph $(q-1)H_3 \cup J_3$ to form $G_3$;

$\vdots$

- Add two copies of the edge $\{n/2, \infty\}$ to the graph $(q-1)H_{n-1} \cup J_{n-1}$ to form $G_{n-1}$.
- Add two copies of the edge $\{\infty, 0\}$ to the graph $(q-1)H_n \cup J_n$ to form $G_n$.

Hence each graph $G_i$ has $k$ edges and is even and connected. Moreover, the maximum degree of $G_i$ is $2(q-1) + 2 + 2 = 2(q+1) = 2\lceil k/n \rceil$. It is also clear that since $(q-1)H_i \cup J_i$ is 4-colourable, so is $G_i$ (since we have not added any "new" edges). Finally, since the graphs $(q-1)H_i \cup J_i$, $1 \leqslant i \leqslant n$, together decompose $2(k-2)K_n$ under the permutation $\rho$, the graphs $G_i$, $1 \leqslant i \leqslant n$, together decompose $2kK_n$ under the permutation $\rho$ (since in total we have "added" exactly two copies of each edge in $W$ to the graphs $(q-1)H^i \cup J^i$). This completes the proof in the case that $q > 1$.

If $q = 1$ we do exactly as in the case $q > 1$ above, replacing $(q-1)H_i \cup J_i$ with $J_i$ throughout. Each of the graphs $G_i$ is clearly still 4-colourable, even and connected (since each $G_i$ still contains the underlying $(n-1)$-cycle $J_i$). Moreover each $G_i$ has maximum degree $\Delta = 2 + 2 = 2\lceil (n+1)/n \rceil = 2\lceil k/n \rceil$ as required. This completes the proof. □

We now have the following simple corollary.

**Corollary 3.6** *Suppose $n$, $m$, $\lambda$ and $k$ are positive integers with $k > n \geqslant 4$, $n$ even and both $m$ and $k$ odd. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $nm > k$ and $\lambda \equiv 0 \pmod{2k}$.*

**Proof:** By Lemma 3.5, the graph $2kK_n$ admits a decomposition into 4-colourable, connected even graphs, each with $k$ edges and maximum degree $\Delta = 2\lceil k/n \rceil$. Since $nm > k$ we have $m \geqslant \lceil k/n \rceil = \Delta/2$, and moreover, since $m$ is odd there exist 2 MOLS($m$) (see [1]). The result then follows by Theorem 3.2. □

Combining Corollaries 3.4 and 3.6 with Theorems 3.1 and 1.4 we have the following result.

**Theorem 3.7** *Suppose $\lambda$, $n$, $m$ and $k$ are positive integers with $n \geqslant 3$, $k$ odd and $k \geqslant 3$. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $k$ whenever $nm \geqslant k$, $\lambda(n-1)m$ is even and $\lambda \equiv 0 \pmod{k}$.*

**Proof:** We split the problem according to whether $n$ is odd or even.

**Case I:** $n$ is odd
If $n \geqslant k$ the graph $\lambda K_n$ admits a decomposition into cycles of length $k$ by Theorem 1.4 and the result follows by Lemma 2.1. If $n < k$, the result follows by Corollary 3.4.

**Case II:** $n$ is even
If $m$ is even, the result follows by Theorem 3.1. If $m$ is odd, then $\lambda$ is even and hence $\lambda \equiv 0 \pmod{2k}$. The result then follows by either Theorem 1.4 in the case that $n \geqslant k$, or Corollary 3.6 otherwise. □

## 4 Prime length cycles

Combining the results of the previous section, we now prove our main result.

**Theorem 4.1** *Suppose $\lambda$, $n$, $m$ and $p$ are positive integers with $n \geqslant 3$ and $p$ an odd prime. Then the $\lambda$-fold complete equipartite graph $\lambda K_n * \overline{K}_m$ admits a decomposition into cycles of length $p$ if and only if*

*(i)* $nm \geqslant p$;

*(ii)* $\lambda(n-1)m$ *is even; and*

*(iii)* $\dfrac{\lambda n(n-1)m^2}{2} \equiv 0 \pmod{p}$.

**Proof:** By condition (iii) and the fact that $p$ is prime, we must have one of $\lambda \equiv 0 \pmod p$, $n \equiv 0, 1 \pmod p$ or $m \equiv 0 \pmod p$.

If $\lambda \equiv 0 \pmod p$ the result follows by Theorem 3.7. If $n \equiv 0, 1 \pmod p$, then $n \geqslant p$ and the result follows by Theorem 2.2. Finally, if $m \equiv 0 \pmod p$ the result follows by Lemma 2.6. This completes the proof. □

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(Received 21 July 2009)