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Characterization of total restrained domination edge critical unicyclic graphs

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Abstract

A graph with no isolated vertices is edge critical with respect to total restrained domination if for any non-edge e of G, the total restrained domination number of $G+e$ is less than the total restrained domination number of G. We call these graphs γ_{tr}-edge critical. In this paper, we characterize all γ_{tr}-edge critical unicyclic graphs.

1 Introduction

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if each vertex not in S is dominated by some vertex of S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a *total dominating set* if each vertex x is dominated by some vertex y of S with $y \neq x$. The *total domination number* of G, denoted $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. For terminology and notation in general we follow [5].

A *leaf* in a graph G is a vertex of degree one, and a *remote vertex* is a vertex which is adjacent to a leaf. Let $R(G)$, or just R, denote the set of remote vertices of G.

Chen et al. [1] and Zelinka [6] introduced the study of *total restrained domination*, which was further studied by Cyman et al. [2] and Hattingh et al. [4]. $S \subseteq V(G)$ is a *total restrained dominating set*, denoted TRDS, if every vertex is adjacent to a vertex in S and every vertex in $V(G) \setminus S$ is also adjacent to a vertex in $V(G) \setminus S$. The

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total restrained domination number of G, denoted $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A TRDS of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set.

Gera et al. [3] introduced the study of total restrained domination edge critical graphs. A graph G is *total restrained domination edge critical*, or just γ_{tr}-edge critical, if for any $e \notin E(G)$, we have $\gamma_{tr}(G+e) < \gamma_{tr}(G)$.

In this paper, we continue the study of total restrained domination edge critical graphs, and characterize all γ_{tr}-edge critical unicyclic graphs.

2 Unicyclic graphs

We begin with some known results.

Theorem 1 *([1]) (1) For a path P_n on $n \geq 2$ vertices, $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$,*
(2) For a cycle C_n on $n \geq 3$ vertices, $\gamma_{tr}(C_n) = n - 2\lfloor \frac{n}{4} \rfloor$.

Proposition 2 *([3]) Assume G is a γ_{tr}-edge critical graph. If R is the set of remote vertices, then R induces the complete graph $K_{|R|}$.*

Proposition 3 *([3]) Assume G is a γ_{tr}-edge critical graph. Let $\{r_1, \ldots, r_\ell\}$ be the set of remote vertices of G, and let L_i be the set of leaves adjacent to r_i for $i = 1, \ldots, \ell$. If $\ell \geq 2$, then $|L_i| = 1$ for $i = 1, \ldots, \ell$.*

The following is easily verified.

Observation 4 *Let x, y be two non-adjacent vertices in a γ_{tr}-edge critical graph G, and let S be a $\gamma_{tr}(G+xy)$-set.*
(1) If $\{x, y\} \subseteq S$, then $N_G(x) \cap S = \emptyset$ or $N_G(y) \cap S = \emptyset$.
(2) If $\{x, y\} \cap S = \emptyset$, then $N_G(x) \subseteq S$, or $N_G(y) \subseteq S$.
(3) If $x \in S, y \notin S$, then $N_G(y) \cap S = \emptyset$.

Theorem 5 *For $n \geq 3$, the cycle C_n is γ_{tr}-edge critical if and only if $n = 5, 6$, or $n \equiv 3$ (mod 4).*

Proof. Let $G = C_n$ and let $V(G) = \{v_1, v_2, ..., v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, ..., n-1$, and v_n is also adjacent to v_1. We consider the following cases.

Case 1. $n \equiv 0$ (mod 4).
Let $n = 4k$ for some integer k. By Theorem 1, $\gamma_{tr}(G) = 2k$. If $n \in \{4, 8\}$, then $\gamma_{tr}(G) = \gamma_{tr}(G + v_1v_3)$, and so G is not γ_{tr}-edge critical. Suppose that $n \geq 12$. Let S be a $\gamma_{tr}(G+v_1v_6)$-set. It is obvious that $|S \cap \{v_2, v_3, v_4, v_5\}| \geq 2$. If $\{v_1, v_6\} \subseteq S$, then $|S| \geq \gamma_{tr}(C_{n-4}) + 2 = 2k$, where C_{n-4} is the cycle obtained from $G+v_1v_6$ by removing

v_2, v_3, v_4, v_5. So assume $v_6 \notin S$. We show that $|S| \geq 2k$. If $v_1 \in S$, then $v_4 \in S$. If $v_3 \in S$, then $\{v_2, v_3, v_4\} \subseteq S$, and we may assume that $v_7 \notin S$. Then we consider the path P_{n-5} obtained by removing $v_3, v_4, ..., v_7$. It follows that $|S| \geq 2+\gamma_{tr}(P_{n-5}) \geq 2k$. If $v_3 \notin S$, then $v_5 \in S$, and so $v_7 \notin S$. This time we consider the path P_{n-6} obtained by removing $v_2, v_3, ..., v_7$. It follows that $|S| \geq \gamma_{tr}(P_{n-6}) + 2 \geq 2k$. It remains to suppose that $v_1 \notin S$. If $\{v_7, v_n\} \subseteq S$, then we consider the path P_{n-6} obtained from G by removing $v_1, v_2, ..., v_6$. So $|S| \geq \gamma_{tr}(P_{n-6}) + 2 \geq 2k$. So without loss of generality assume that $v_7 \notin S$. With a similar argument we obtain $|S| \geq 2k$.

Case 2. $n \equiv 1 \pmod 4$.
If $n = 5$, then we can easily see that G is γ_{tr}-edge critical. So we let $n \geq 9$. Let $n = 4k+1$ for some integer k. By Theorem 1, $\gamma_{tr}(G) = 2k+1$. Let S be a $\gamma_{tr}(G+v_1v_5)$-set. We show that $|S| \geq 2k+1$. If $\{v_1, v_5\} \subseteq S$, then $|S| \geq 1+\gamma_{tr}(C_{n-3}) = 2k+1$, where C_{n-3} is the cycle obtained from $G+v_1v_5$ by removing v_2, v_3, v_4. So suppose that $v_5 \notin S$. If $v_1 \in S$, then $\{v_2, v_3\} \subseteq S$, and so $|S| \geq 1+\gamma_{tr}(P_{n-4}) = 2k+1$, where P_{n-4} is the path obtained from G by removing v_3, v_4, v_5, v_6. It remains to assume that $v_1 \notin S$. This time with a similar argument we obtain $|S| \geq 2k+1$.

Case 3. $n \equiv 2 \pmod 4$.
It is a routine matter to see that C_6 is γ_{tr}-edge critical. Also $\gamma_{tr}(C_{10}) = \gamma_{tr}(C_{10}+v_1v_6)$ which implies that C_{10} is not γ_{tr}-edge critical. So we let $n \geq 14$. Let $n = 4k+2$ for some integer k. By Theorem 1, $\gamma_{tr}(G) = 2k+2$. Let S be a $\gamma_{tr}(G+v_1v_6)$-set. We show that $|S| \geq 2k+2$. If $\{v_1, v_6\} \subseteq S$, then $|S| \geq \gamma_{tr}(C_{n-4})+2 = 2k+2$, where C_{n-4} is the cycle obtained from $G+v_1v_6$ by removing v_2, v_3, v_4, v_5. We may so assume that $v_6 \notin S$. Assume $v_1 \notin S$. We may assume that $|S \cap \{v_2, v_3, v_4, v_5\}| = 2$; other possiblities are similarly verified. Let P_{n-6} be the path obtained from G by removing $v_1, v_2, ..., v_6$. It follows that $|S| \geq 2+\gamma_{tr}(P_{n-6}) = 2k+2$. It remains to assume that $v_1 \in S$. This time a similar argument gives that $|S| \geq 2k+2$.

Case 4. $n \equiv 3 \pmod 4$.
By definition C_3 is γ_{tr}-edge critical. It is also straightforward to see that C_7 is γ_{tr}-edge critical. So suppose that $n \geq 11$. Let $n = 4k+3$ for some integer k. By Theorem 1, $\gamma_{tr}(G) = 2k+3$. Let

$$S = \{v_{4i+3}, v_{4i+4} : 0 \leq i \leq k-1\} \cup \{v_{4k+1}, v_{4k+2}, v_{4k+3}\}.$$

Then S is a $\gamma_{tr}(G)$-set. Notice that v_{4k+3} only dominates v_1. Since G is vertex transitive, we prove that for each $j = 2, 3, ..., diam(G) = 2k+1$, there is a vertex $x \in S$ such that $d(x, v_1) = j$. For any $i > 2k+2$, we rename v_i by $w_{(4k+3)-i+2}$. So $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, ..., v_{2k+2}\}$ and $V_2 = \{w_2, w_3, ..., w_{2k+2}\}$. Let $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. It follows that

$$\{d(v_1, v) : v \in S_1\} = \left\{4i+2, 4i+3 : 0 \leq i < \left\lceil \frac{k}{2} \right\rceil \right\},$$

while

$$\{d(v_1, v) : v \in S_2\} = \left\{1, 2, 3, 4i, 4i+1 : 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

Hence the result follows. ■

We are now ready to give the main result of this paper. Let $\mathcal{E}$ be the class of graphs G such that G belongs to $\mathcal{E}$ if and only if one of the following holds.
(1) G is obtained from $K_{1,n}$ for a positive integer $n \geq 5$ by joining two leaves,
(2) G is obtained from the corona of C_3 by removing at most one leaf,
(3) $G \in \{C_5, C_6, C_n : n \equiv 3 \pmod 4\}$.

Theorem 6 *A unicyclic graph G is γ_{tr}-edge critical if and only if $G \in \mathcal{E}$.*

Proof. It is a routine matter to see that any graph in $\mathcal{E}$ is a γ_{tr}-edge critical unicyclic graph. Let G be a γ_{tr}-edge critical unicyclic graph, and let C be the unique cycle in G. If $G = C$, then by Theorem 5, $G \in \mathcal{E}$. Suppose that $G \neq C$. Let $v_1 \in L(G)$ such that the distance between v_1 and C is maximum, and let $P : v_1v_2 \ldots v_k$ be the shortest path from v_1 to C. Note that $V(C) \cap V(P) = \{v_k\}$. Let $V(C) = \{u_1, u_2, ..., u_t\}$, where $u_1 = v_k$, and $E(C) = \{u_iu_{i+1} : 1 \leq i \leq t-1\} \cup \{u_1u_t\}$. So $N_G[v_k] \cap V(C) = \{u_1, u_2, u_t\}$. By Propositions 2 and 3, if $k > 3$, then $deg(v_i) = 2$ for $3 \leq i \leq k-1$. Moreover, if $k > 2$, then $deg(u_1) = 3$ and $deg(u_i) = 2$ for $2 \leq i \leq t$. We first show that $k \leq 6$. Suppose to the contrary that $k \geq 7$. Let S_1 be a $\gamma_{tr}(G + v_2v_7)$-set. If $v_7 \in S_1$, then $N_G(v_7) \cap S_1 = \emptyset$, since S_1 is not a TRDS for G. Since v_5 is dominated by S_1, we have $\{v_3, v_4\} \subseteq S_1$. But then $(S_1 \setminus \{v_4\}) \cup \{v_6\}$ is a TRDS for G, a contradiction. So we assume that $v_7 \notin S_1$. Again we have $N_G(v_7) \cap S_1 = \emptyset$, since S_1 is not a TRDS for G. Since v_6 is dominated by S_1, we have $\{v_3, v_4, v_5\} \subseteq S_1$. Now $(S_1 \setminus \{v_3, v_4\}) \cup \{v_6\}$ is a TRDS for G, a contradiction. Thus, $k \leq 6$.

Suppose now that $k = 6$. Let S_2 be a $\gamma_{tr}(G + v_2u_2)$-set. If $u_2 \notin S_2$, then $N_G(u_2) \cap S_2 = \emptyset$. If $u_t \in S_2$ then $\{v_3, v_4\} \subseteq S_2$, and so $(S_2 \setminus \{v_4\}) \cup \{v_6\}$ is a TRDS for G, a contradiction. So $u_t \notin S_2$, and so $\{v_3, v_4, v_5\} \subseteq S_2$. It follows that $(S_2 \setminus \{v_3, v_4\}) \cup \{v_6, u_t\}$ is a TRDS for G, a contradiction. We thus assume that $u_2 \in S_2$. Since S_2 is not a TRDS for G, $N_G(u_2) \cap S_2 = \emptyset$. If $u_t \in S_2$, then $\{v_3, v_4\} \subseteq S_2$, and so $(S_2 \setminus \{v_3, v_4\}) \cup \{v_5, v_6\}$ is a TRDS for G. This contradiction yields that $u_t \notin S_2$. But v_5 is dominated by S_2. So $\{v_3, v_4\} \subseteq S_2$. This time $\{u_{i-1} \pmod t\text{: } u_i \in S_2\} \cup \{v_1, v_2, v_5\}$ is a TRDS for G, a contradiction. We deduce that $k \leq 5$.

Next suppose that $k = 5$. Let S_3 be a $\gamma_{tr}(G + v_4u_2)$-set. If $\{v_4, u_2\} \cap S_3 = \emptyset$, then either $N_G(u_2) \subseteq S_3$ or $N_G(v_4) \subseteq S_3$. If $N_G(v_4) \subseteq S_3$, then $S_3 \setminus \{v_3\}$ is a TRDS for G, while if $N_G(u_2) \subseteq S_3$ then $(\{u_{i-1} \pmod t\text{: } u_i \in S_3\}) \cup (S_3 \cap \{v_1, v_2, v_3\})$ is a TRDS for G. Both are contradiction. So $\{v_4, u_2\} \cap S_3 \neq \emptyset$. If $\{v_4, u_2\} \subseteq S_3$, then $v_5 \notin S_3$ and $(S_3 \setminus \{v_3, v_4\}) \cup \{v_5, v_t\}$ is a TRDS for G, which again is a contradiction. We deduce that $|\{v_4, u_2\} \cap S_3| = 1$. If $v_4 \in S_3$, then $N_G[u_2] \cap S_3 = \emptyset$, and so $(S_3 \setminus \{v_3, v_4\}) \cup \{u_t, v_5\}$ is a TRDS for G, a contradiction. It remains to assume that $u_2 \in S_3$. This time $\{u_{i-1} \pmod t\text{: } u_i \in S_3\} \cup \{v_1, v_2\}$ is a TRDS for G, a contradiction. Thus, $k \leq 4$.

Suppose now that $k = 4$. It is straightforward to see that for $t \in \{3, 4, 5\}$, $\gamma_{tr}(G) = \gamma_{tr}(G+v_2v_4)$, and for $t = 6$, $\gamma_{tr}(G) = \gamma_{tr}(G+v_3u_3)$. So we assume that $t \geq 7$. Let S_4 be a $\gamma_{tr}(G+u_4u_{t-1})$-set. We first show that $\{u_4, u_{t-1}\} \cap S_4 \neq \emptyset$. Suppose that $\{u_4, u_{t-1}\} \cap S_4 = \emptyset$. Then $N_G(u_4) \subseteq S_4$ or $N_G(u_{t-1}) \subseteq S_4$. If $N_G(u_4) \cup N_G(u_{t-1}) \subseteq S_4$, then $\{v_3, v_4, u_2, u_3, u_t\} \subseteq S_4$, and $(S_4 \setminus \{v_3, v_4\}) \cup \{u_4, u_{t-1}\}$ is a TRDS for G which is a contradiction. So either $N_G(u_4) \nsubseteq S_4$ or $N_G(u_{t-1}) \nsubseteq S_4$. Assume that $N_G(u_4) \nsubseteq S_4$, and so $N_G(u_{t-1}) \subseteq S_4$. It follows that $\{v_3, v_4\} \subseteq S_4$. Then $S_4 \setminus \{u_3\}$ is a TRDS for G, a contradiction. Similarly, $N_G(u_{t-1}) \nsubseteq S_4$ which produces a contradiction. We deduce that $\{u_4, u_{t-1}\} \cap S_4 \neq \emptyset$. If $u_4 \notin S_4$, then $u_{t-1} \in S_4$ and $N_G[u_4] \cap S_4 = \emptyset$. Since u_3 is dominated by S_4, $\{v_3, v_4, u_2, u_t\} \subseteq S_4$. This time $(S_4 \setminus \{v_3, v_4\}) \cup \{u_3\}$ is a TRDS for G, a contradiction. So $u_4 \in S_4$, and by a similar argument $u_{t-1} \in S_4$. It follows that either $N_G(u_4) \cap S_4 = \emptyset$, or $N_G(u_{t-1}) \cap S_4 = \emptyset$. If $N_G(u_4) \cap S_4 = \emptyset$, then $\{v_3, v_4, u_t\} \subseteq S_4$, and $(S_4 \setminus \{v_4\}) \cup \{u_3\}$ is a TRDS for G, a contradiction. So $N_G(u_4) \cap S_4 \neq \emptyset$, and by symmetry $N_G(u_{t-1}) \cap S_4 \neq \emptyset$. But then S_4 is a TRDS for G, a contradiction. Thus, $k \leq 3$.

Suppose that $k = 3$. It is straightforward to see that for $t = 3$, $\gamma_{tr}(G) = \gamma_{tr}(G + v_2u_2)$, and for $t = 4$, $\gamma_{tr}(G) = \gamma_{tr}(G + v_2u_2)$. So we suppose that $t \geq 5$. Let S_5 be a $\gamma_{tr}(G + v_2u_5)$-set. If $u_5 \in S_5$, then $N_G(u_5) \cap S_5 = \emptyset$. It follows that $\{u_2, v_3\} \subseteq S_5$, and $\{u_{i-4} \pmod t\colon u_i \in S_5\} \cup \{v_1, v_2\}$ is a TRDS for G, a contradiction. So $u_5 \notin S_5$. But then $\{u_2, u_3\} \subseteq S_5$. Again $\{u_{i-4} \pmod t\colon u_i \in S_5\} \cup \{v_1, v_2\}$ is a TRDS for G, a contradiction. We conclude that $k = 2$.

For $t = 3$, it is straightforward to see that either G is obtained from the corona of C_3 by removing at most one pendant edge, or G is obtained from $K_{1,n}$ for an integer $n \geq 5$ by joining two leaves. In both cases $G \in \mathcal{E}$. So assume that $t \geq 4$. We show that v_2 is the unique remote vertex of G. Suppose that v_2 is not the unique remote vertex of G. By Propositions 2 we may assume that u_2 is also a remote vertex. By Proposition 3, $deg(v_2) = deg(u_2) = 3$. Let x be a leaf adjacent to u_2 and let S_6 be a $\gamma_{tr}(G + u_2u_t)$-set. It follows that S_6 is a TRDS for G, a contradiction. We deduce that v_2 is the unique remote vertex of G. For $t \in \{4, 5, 6\}$ it is a routine matter to see that $\gamma_{tr}(G) = \gamma_{tr}(G+u_2u_t)$, and so G is not γ_{tr}-edge critical. Let $t \geq 7$, and let S_7 be a $\gamma_{tr}(G+v_2u_6)$-set. If $u_6 \in S_7$, then $N_G(u_6) \cap S_7 = \emptyset$. This implies that $\{u_2, u_3\} \subseteq S_7$ and $(S_7 \setminus \{u_2, u_3\}) \cup \{u_4, u_5\}$ is a TRDS for G, a contradiction. So $u_6 \notin S_7$. It follows that $N_G[u_6] \cap S_7 = \emptyset$. But then $\{u_2, u_3, u_4\} \subseteq S_7$, and $(S_7 \setminus \{u_2, u_3\}) \cup \{u_5\}$ is a TRDS for G, a contradiction. ■

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