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On the spectra of the fullerenes that contain a nontrivial cyclic-5-cutset*

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Abstract

A fullerene, which is a 3-connected cubic plane graph whose faces are pentagons and hexagons, is cyclically 5 edge-connected. For a fullerene of order n, it is customary to index the eigenvalues in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is known that the largest eigenvalue is 3. Let $R = \{f_i \mid i \in \mathbb{Z}_l\}$ be a set of l faces of a fullerene F such that f_i is adjacent to f_{i+1}, $i \in \mathbb{Z}_l$, via an edge e_i. If the edges in $\{e_i \mid i \in \mathbb{Z}_l\}$ are independent, then we say that R forms a ring of l faces. In this paper we show that if a fullerene contains a nontrivial cyclic-5-cutset, then it has

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$2r-2$ eigenvalues that can be arranged in pairs $\{\mu, -\mu\}$ $(1 < \mu < 3)$, where r is the number of the rings of five faces. Meanwhile 1 is one of its eigenvalues and $\lambda_{r+1} \geq 1$.

1 Introduction

All graphs in this paper are simple and connected. Let G be a graph with vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The characteristic polynomial $f_G(x)$ of G is the characteristic polynomial of the adjacency matrix $A(G)$ of G. That is, $f_G(x) = \det(A(G) - xI)$, where I is the identity matrix. Analogously, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G, and to form the spectrum of this graph. Since $A(G)$ is symmetric, all its eigenvalues are real. We always order them in the non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

A *fullerene graph* (or *a fullerene* for short) is a 3-connected cubic plane graph whose faces are pentagons and hexagons. By Euler's formula the number of pentagons equals 12. From a chemical point of view, fullerenes correspond to carbon 'sphere'-shaped molecules, the important class of molecules which are interesting synthesized [3, 7] structures with subtle electronic properties [1, 3, 6, 7, 9, 10, 12–17]. Meanwhile, the chemical stability is determined by the spectrum of the molecular graph. So for chemicophysical interest we study the eigenvalues of the fullerenes.

Let us describe the leapfrog operation on a cubic planar graph as follow, which was first introduced by Fowler and Steer in [15]. If X is a cubic planar graph with n vertices and $m = 3n/2$ edges, then its line graph $L(X)$ is a planar 4-regular graph with m vertices. The leapfrog graph $F(X)$ is formed by taking each vertex of $L(X)$ and splitting it into a pair of adjacent vertices in such a way that every triangular face around a vertex of X becomes a six-cycle; once again, a drawing such as Fig. 1 is the easiest way to visualize this. Then $F(X)$ is a cubic planar graph on $2m$ vertices with n faces of length 6. In particular, if X is a fullerene, then so is $F(X)$.

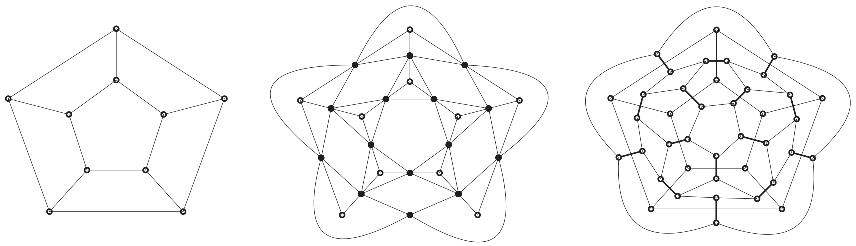

Figure 1: A leapfrog operation on a cubic graph. The highlighted edges in the leapfrog graph (in the right) indicates the selected splitting of the solid vertices in the middle graph, which is induced from the line graph of the left cubic graph.

It was proved rigorously that a fullerene derived from the leapfrog operation has exactly half of its eigenvalues positive and exactly half of its eigenvalues negative [11].

In this paper, we will study fullerenes with nontrivial cyclic-5-cutsets. By results in [8] such fullerenes contain two disjoint antipodal pentacaps, that is, two antipodal faces whose neighboring faces are also pentagonal which are connected via rings of five hexagonal faces, see Fig. 2. A graph is said to be *cyclically k-edge-connected* (or

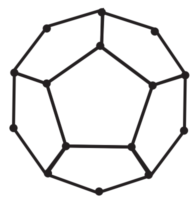

Figure 2: The pentacap.

cyclically k-connected for short), if at least k edges must be removed to disconnect it into two components, each containing a cycle. Such a set of k edges is called a *cyclic-k-edge cutset* (or *cyclic-k-cutset* for short) and moreover, it is called a *trivial cyclic-k-cutset* if at least one of the resulting two components induces a single k-cycle.

Obviously, since each fullerene contains a pentagonal face (actually 12 pentagonal faces), it contains a trivial cyclic-5-cutset. Let $R = \{f_i \mid i \in \mathbb{Z}_l\}$ be a set of l faces of a fullerene F such that f_i is adjacent to f_{i+1}, $i \in \mathbb{Z}_l$, via an edge e_i. If the edges in $\{e_i \mid i \in \mathbb{Z}_l\}$ are independent, then we say that R forms a *ring of l faces*. In this paper, we will show that a fullerene admitting nontrivial cyclic-5-cutset has $2r-2$ eigenvalues that can be arranged in pairs $\{\mu, -\mu\}$ $(1 < \mu < 3)$, where r is the number of the rings of five faces. Meanwhile, 1 is one of its eigenvalues and $\lambda_{r+1} \geq 1$.

2 Structure of the fullerenes admitting nontrivial cyclic-5-cutsets

A detail description of cyclic-5-cutsets in fullerenes is given in [8]. Observe that the edges in a cyclic-5-cutset of a fullerene are independent and that the edges in the cyclic-5-cutset together with their coincident faces form a ring of five faces (i.e., $\{f_i \mid 1 \leq i \leq 5\}$ in Fig. 3).

Lemma 2.1 ([8]) *Let F be a fullerene admitting a nontrivial cyclic-5-cutset. Then F contains a pentacap (see Fig. 2). More precisely, it contains two disjoint antipodal pentacaps.*

According to Lemma 2.1, only the subgraph shown in Fig. 4 can occur between the two disjoint antipodal pentacaps. That is, in each hexagonal faces there exist two vertices, the first of which has a neighbor inside the ring which the hexagon belongs to and the second of which has a neighbor outside the ring which the hexagon belongs to. Fig. 5 shows a fullerene admitting a nontrivial cyclic-5-cutset with 3 five faces

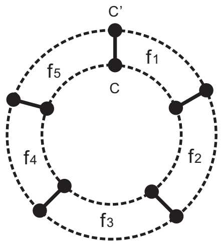

Figure 3: The local structure of a fullerene.

rings. Here, one of them consists of five hexagonal faces. The other two are five pentagonal rings, as illustrated with shadow.

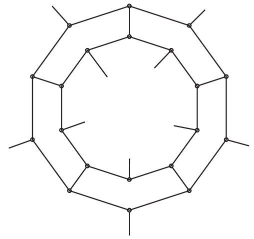

Figure 4: A structure of ring of five hexagonal faces in a fullerene admitting a nontrivial cyclic-5-cutset.

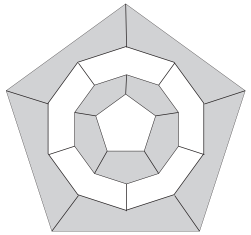

Figure 5: A fullerene with a nontrivial cyclic-5-cutset and three rings of five faces.

3 Interlacing and some results on the eigenvalues

Suppose M is a real symmetric $n\times n$ matrix. Let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ denote its eigenvalues in non-increasing order. Suppose A is a real symmetric $n \times n$ matrix and B is a real symmetric $m \times m$ matrix, where $m \leq n$. We say that the eigenvalues of B *interlace* the eigenvalues of A if

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A), \text{ for } i = 1, \dots, m.$$

Suppose A is an $n \times n$ matrix. For $m \leq n$, an $m \times m$ matrix B is obtained from A by deleting $n - m$ rows and $n - m$ corresponding columns. Then B is called a *principal submatrix* of A.

Lemma 3.1 **([4])** *Let A be a real symmetric $n \times n$ matrix and let B be a principal submatrix of A of order $m \times m$. Then, for $i = 1, \dots, m$,*

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A).$$

Suppose rows and columns of

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ A_{2,1} & \cdots & A_{2,m} \\ \cdots & \cdots & \cdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}$$

are partitioned according to a partition $\pi = \{X_1, \dots, X_m\}$ of the set $\{1, \dots, n\}$ with the $m \times m$ *characteristic matrix* $\tilde{S}$ (that is, $\tilde{s}_{i,j} = 1$ if $i \in X_j$ and 0 otherwise). Each X_j is called a *cell* of this partition. The *quotient matrix* of A according to π is an $m \times m$ matrix $\tilde{B}$ whose entries are the average row sums of the blocks of A. More precisely,

$$(\widetilde{B})_{i,j} = \frac{1}{|X_i|}\underline{\mathbf{1}}^\top A_{i,j}\underline{\mathbf{1}} = \frac{1}{|X_i|}(\widetilde{S}^\top A\widetilde{S})_{i,j}$$

($\underline{\mathbf{1}}$ denotes the all-one vector of suitable length). The partition is called *regular* (or *equitable*) if each block $A_{i,j}$ of A has constant row (and column) sum, that is $A\widetilde{S} = \widetilde{S}\widetilde{B}$.

Lemma 3.2 **([5])** *Suppose $\widetilde{B}$ is the quotient matrix of a symmetric partitioned matrix A. Then the eigenvalues of $\widetilde{B}$ interlace the eigenvalues of A.*

Let X be a graph. We say that a partition π of $V(X)$ with cells $C_1, \cdots, C_m$ is *equitable* if the number of neighbors lying in C_j of a vertex u in C_i is a constant b_{ij}, independent of u. An equivalent definition is that the subgraph of X induced by each cell is regular, and the edges joining any two distinct cells form a semiregular bipartite graph. The graph with m cells of π as its vertices and b_{ij} arcs from the ith to the jth cells is called the *quotient* of X over π, and it is denoted by X/π. Therefore, the entries of the adjacency matrix of this quotient are given by $A(X/\pi) = (b_{ij})$.

One important class of equitable partitions arises from automorphisms of a graph. The orbits of any group of automorphisms of X form an equitable partition. An example is given by the group of rotations of order 5 acting on the Petersen graph. Two orbits of this group action, namely the 5 "inner" vertices and the 5 "outer" vertices, form an equitable partition π_1 with quotient matrix

$$A(X/\pi_1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Lemma 3.3 ([4]) *If π is an equitable partition of a graph X, then the characteristic polynomial of $A(X/\pi)$ divides the characteristic polynomial of $A(X)$.*

A *three-diagonal matrix* of order n of the form

$$\widetilde{C} = \begin{pmatrix} a_1 & b_2 & & & \\ c_2 & a_2 & b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_n \\ & & & c_n & a_n \end{pmatrix}$$

is called a *Jacobi three-diagonal matrix*, where $b_i c_i > 0$ for $2 \leq i \leq n$.

Let P_j, $j = 1, 2, \ldots, 2r$, be the j-th order sequential principal submatrix formed by the first j rows and columns of the matrix $\widetilde{C} - \lambda I$, and let $p_j(\lambda) = \det(P_j)$. Let $p_0(\lambda) \equiv 1$. It is easy to get that

$$\begin{cases} p_1(\lambda) = a_1 - \lambda; \\ p_i(\lambda) = (a_i - \lambda)p_{i-1}(\lambda) - b_i c_i p_{i-2}(\lambda), \quad i = 2, \ldots, n \end{cases} \tag{3.1}$$

Moreover, let $\alpha_n(\lambda)$ be the number of pairs, such that $p_i(\lambda)$ and $p_{i+1}(\lambda)$ have the same sign for a real number λ, where $i = 0, 1, \ldots, n-1$. If $p_i(\lambda) = 0$, then according to the proof of [18, Chapter 2: Theorem 3.2], $p_{i-1}(\lambda) \neq 0$. So we regulating the sign of $p_i(\lambda)$ is identical with $p_{i-1}(\lambda)$ when $p_i(\lambda) = 0$.

Lemma 3.4 ([18]) *Let $\widetilde{C}$ be the Jacobi three-diagonal matrix. Keeping the notations defined above, then for a real number λ, the number of its eigenvalues in $[\lambda, \infty)$ is equal to $\alpha_n(\lambda)$.*

4 The main result

Let $r \geq 3$ be the number of five faces rings in a fullerene F admitting nontrivial cyclic-5-cutset. By the observation below Lemma 2.1, we obtain that F is uniquely determined which is described as in Fig. 6. Note that F contains at least one 5 faces ring whose faces are hexagons. We partition the vertices of F into $2r$ cells. Vertices in the same partition are labeled by the same number. That is, grouping the five vertices of pentagon at the tip into the first cell label them by "1" in (see Fig 6). The vertices labeled by "2" are grouped into the second cell. Vertices labeled by "3" are grouped into the third cell, and so on. Thus we have a partition π for $V(F)$ containing $2r$ cells. This partition is equitable as the partition sets are orbits of an automorphism of a fullerene of order 5. According to this partition, the quotient matrix $\widetilde{B}$ of the adjacency matrix of F is as follows:

$$\widetilde{B} = \begin{pmatrix} 2 & 1 & & & & & \\ 1 & 0 & 2 & & & & \\ & 2 & 0 & 1 & & & \\ & & 1 & 0 & 2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 2 & 0 & 1 \\ & & & & & 1 & 2 \end{pmatrix} \tag{4.1}$$

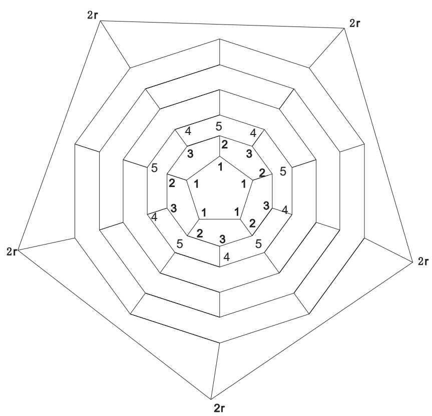

Figure 6: Partition of the vertex set in a fullerene admitting a nontrivial cyclic-5-cutset.

Observe that the quotient matrix $\widetilde{B}$ of a fullerene F admitting a nontrivial cyclic-5-cutset with respect to this particular equitable partition is a Jacobi three-diagonal matrix.

Theorem 4.1 *Let F be a fullerene of order n admitting a nontrivial cyclic-5-cutset. Then*

(a) 1 is one of its eigenvalues and $\lambda_{r+1}(F) \geq 1$, where r is the number of the rings of five faces ($r \geq 3$), and

(b) F has $2r-2$ eigenvalues that can be grouped in pairs $\{\mu, -\mu\}$, where $1 < \mu < 3$.

PROOF. Let $\widetilde{B}$ be the quotient matrix of the adjacency matrix of F described in (4.1). Let the polynomial $p_j(\lambda)$ be the jth principal minor of $\det(\widetilde{B} - \lambda I)$. It is clear that $p_0(\lambda) = 1$, $p_1(\lambda) = -\lambda + 2$, $p_2(\lambda) = \lambda^2 - 2\lambda - 1$ and $p_3(\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 8$. Suppose first that j is odd and greater than 3. By (3.1) we have

$$\begin{aligned} p_j(\lambda) = &- \lambda p_{j-1}(\lambda) - 4p_{j-2}(\lambda) = -\lambda(-\lambda p_{j-2}(\lambda) - p_{j-3}(\lambda)) - 4p_{j-2}(\lambda) \\ &= (\lambda^2 - 4)p_{j-2}(\lambda) + \lambda p_{j-3}(\lambda) = (\lambda^2 - 5)p_{j-2}(\lambda) + p_{j-2}(\lambda) + \lambda p_{j-3}(\lambda) \\ &= (\lambda^2 - 5)p_{j-2}(\lambda) - 4p_{j-4}(\lambda). \end{aligned}$$

Suppose now that j, $2 < j < 2r$, is even and greater than 2 and less than $2r$, ($r \geq 3$). By a similar computation, we also get that

$$\begin{aligned} p_j(\lambda) =& -\lambda p_{j-1}(\lambda) - p_{j-2}(\lambda) = -\lambda(-\lambda p_{j-2}(\lambda) - 4p_{j-3}(\lambda)) - p_{j-2}(\lambda) \\ &= (\lambda^2 - 1)p_{j-2}(\lambda) + 4\lambda p_{j-3}(\lambda) = (\lambda^2 - 5)p_{j-2}(\lambda) + 4p_{j-2}(\lambda) + 4\lambda p_{j-3}(\lambda) \\ &= (\lambda^2 - 5)p_{j-2}(\lambda) - 4p_{j-4}(\lambda). \end{aligned}$$

Thus $p_j(\lambda)$ satisfies the following recurrence relation.

$$p_j(\lambda) = (\lambda^2 - 5)p_{j-2}(\lambda) - 4p_{j-4}(\lambda), \text{ for } 4 \leq j \leq 2r - 1. \tag{4.2}$$

with the initial conditions $p_0(\lambda) = 1$, $p_1(\lambda) = -\lambda + 2$, $p_2(\lambda) = \lambda^2 - 2\lambda - 1$ and $p_3(\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 8$.

Let $q_k = p_{2k}(1) = p_{2k}$ for all k, $0 \leq k \leq r - 1$. Then Equation (4.2) becomes $q_k = -4q_{k-1} - 4q_{k-2}$ with $q_0 = 1$ and $q_1 = -2$. Solving this equation gives $p_{2k} = q_k = (-2)^k$.

Similarly, let $\overline{q}_k = p_{2k+1}(1) = p_{2k+1}$ for all k, $0 \leq k \leq r - 1$. Then Equation (4.2) becomes $\overline{q}_k = -4\overline{q}_{k-1} - 4\overline{q}_{k-2}$ with $\overline{q}_0 = 1$ and $\overline{q}_1 = -2$. Thus we have $p_{2k+1} = \overline{q}_k = (-2)^k$.

In particular, $p_{2r-2} = (-2)^{r-1}$ and $p_{2r-1} = (-2)^{r-1}$. By (3.1) we get $p_{2r} = (2-1)p_{2r-1} - p_{2r-2} = 0$ and hence $\lambda = 1$ is an eigenvalue of $\widetilde{B}$. It is easy to count that $\alpha_n(1) = r+1$. By Lemma 3.4, the number of eigenvalues of $\widetilde{B}$ in $[1, \infty)$ is $r+1$. Thus we have $\lambda_{r+1}(\widetilde{B}) = 1$. By Lemma 3.4, $\lambda_{r+1}(F) \geq \lambda_{r+1}(\widetilde{B}) = 1$. Hence we have part (a).

Denote the matrix $\widetilde{B}$ of order $2r$ by $\widetilde{B}_{2r}$, here $r \geq 3$. In this part, we will prove that for an arbitrary r,

$$\det(\widetilde{B}_{2r} - \lambda I) = (3 - \lambda)(1 - \lambda)(a_1^2 - \lambda^2)(a_2^2 - \lambda^2) \cdots (a_{r-1}^2 - \lambda^2),$$

where a_i, $i = 1, 2, \ldots, r - 1$, are eigenvalues of $\widetilde{B}_{2r}$ and $1 < a_i < 3$.

We will prove it by induction on r. For $r = 3$ or $r = 4$, the result holds by direct computation. Assume the result holds for any positive integer less than r, for $r \geq 5$. Then

$$\begin{aligned} &\det(\widetilde{B}_{2r} - \lambda I) \\ &\quad = (2 - \lambda)p_{2r-1}(\lambda) - p_{2r-2}(\lambda) \\ &\quad = (2 - \lambda)[(\lambda^2 - 5)p_{2r-3}(\lambda) - 4p_{2r-5}(\lambda)] - [(\lambda^2 - 5)p_{2r-4}(\lambda) - 4p_{2r-6}(\lambda)] \\ &\quad = (2 - \lambda)(\lambda^2 - 5)p_{2r-3}(\lambda) - (\lambda^2 - 5)p_{2r-4}(\lambda) - 4(\lambda - 2)p_{2r-5}(\lambda) + 4p_{2r-6}(\lambda) \\ &\quad = (\lambda^2 - 5)\det(-\lambda I - \widetilde{B}_{2r-2}) - 4\det(\lambda I - \widetilde{B}_{2r-4}) \\ &\quad = (\lambda^2 - 5)(3 - \lambda)(1 - \lambda)(b_1^2 - \lambda^2)(b_2^2 - \lambda^2) \cdots (b_{r-2}^2 - \lambda^2) \\ &\qquad -4(3 - \lambda)(1 - \lambda)(c_1^2 - \lambda^2)(c_2^2 - \lambda^2) \cdots (c_{r-3}^2 - \lambda^2) \\ &\quad = (\lambda - 3)(\lambda - 1)\big[(\lambda^2 - 5)(b_1^2 - \lambda^2)(b_2^2 - \lambda^2) \cdots (b_{r-2}^2 - \lambda^2) \\ &\qquad -4(c_1^2 - \lambda^2)(c_2^2 - \lambda^2) \cdots (c_{r-3}^2 - \lambda^2)\big], \end{aligned}$$

where b_i, $i = 1, 2, \ldots, r-2$ are eigenvalues of $\widetilde{B}_{2r-2}$ and c_j, $j = 1, 2, \ldots, r-3$ are eigenvalues of $\widetilde{B}_{2r-4}$.

Since $\det(\widetilde{B}_{2r} - \lambda I)/(\lambda - 3)(\lambda - 1)$ is a polynomial of λ^2 and eigenvalues of $\widetilde{B}$ are real, $\det(\widetilde{B}_{2r} - \lambda I)$ is of the form

$$(3 - \lambda)(1 - \lambda)(a_1^2 - \lambda^2)(a_2^2 - \lambda^2) \cdots (a_{r-1}^2 - \lambda^2),$$

for some nonnegative real numbers a_i's.

It is well known that the multiplicity of 3 is one, then $a_i \neq 3$. According to the proof of (a), $1 \leq a_i < 3$.

By putting $\lambda = -1$ into the equation (4.2), we get that $p_{2k}(-1) = (-2)^k(1 - 2k)$ and $p_{2k+1}(-1) = (-1)^k(3 + 2k)$ for $0 \leq k \leq r - 1$. Then $p_{2r}(-1) = 2(-2)^{r-1}(2r - 1)$. So -1 is not an eigenvalue of $\widetilde{B}$ and hence $a_i \neq 1$. Therefore, all eigenvalues of $\widetilde{B}$ except 1 and 3 can be paired up as $\{\mu, -\mu\}$. Moreover, $1 < \mu < 3$.

By Lemma 3.3 we obtain the part (b) of the theorem. □

5 Conclusion

In this paper we use the interlacing techniques to prove that if F is a fullerene with nontrivial cyclic-5-cutset, then it has $2r - 2$ eigenvalues which can be paired up into the form $\{-\mu, \mu\}$, $(1 < \mu < 3)$, where r is the number of the rings of five faces. The other two eigenvalues are 3 and 1. Table 1 lists all eigenvalues of this kind for the smallest five fullerenes admitting nontrivial cyclic 5-cutsets. With the exception of the $2r$ eigenvalues mentioned above the multiplicities of eigenvalues of these fullerenes are even (at least 2). However, whether all of the fullerenes with nontrivial cyclic-5-cutset satisfy this property or not is an open problem.

Acknowledgment

We would like to thank the referee for giving several valuable comments and suggestions. We also appreciate Professor An Chang for his help and guidance.

Table 1: The spectra of the fullerenes with nontrivial cyclic-5-cutset.

	$n=30$	$r=3$					
3	−2.64575	2.64575	−1.73205	1.73205	1		
2.41183	2.41183	−2.32142	−2.32142	−2.16309	−2.16309	−2.11892	−2.11892
1.79079	1.79079	−1.51181	−1.51181	1.36923	1.36923	−1.12177	−1.12177
0.718012	0.718012	0.666822	0.666822	0.148667	0.148667	0.131653	0.131653
	$n=40$	$r=4$					
3	−2.79793	2.79793	−2.23607	2.23607	−1.47363	1.47363	1
2.48857	2.48857	−2.44501	−2.44501	−2.14686	−2.14686	−2.13808	−2.13808
2.102	2.102	−1.95339	−1.95339	1.48385	1.48385	1.47527	1.47527
−1.38028	−1.38028	−1.22955	−1.22955	1.10178	1.10178	−0.855151	−0.855151
0.572552	0.572552	0.50938	0.50938	0.225619	0.225619	0.189299	0.189299
	$n=50$	$r=5$					
3	−2.86986	2.86986	−2.49721	2.49721	−1.94009	1.94009	−1.32813
1.32813	1						
2.52913	2.52913	−2.50495	−2.50495	2.26419	2.26419	−2.17747	−2.17747
−2.14343	−2.14343	−2.14167	−2.14167	1.83249	1.83249	−1.67147	−1.67147
1.53429	1.53429	−1.48369	−1.48369	1.29173	1.29173	1.25916	1.25916
−1.14772	−1.14772	−1.05699	−1.05699	0.915924	0.915924	−0.707771	−0.707771
0.490874	0.490874	0.425131	0.425131	0.271098	0.271098	0.221156	0.221156
	$n=60$	$r=6$					
3	−2.90931	2.90931	−2.64575	2.64575	−2.23607	2.23607	−1.73205
1.73205	−1.23931	1.23931	1				
2.5532	2.5532	−2.53843	−2.53843	2.36001	2.36001	−2.30547	−2.30547
−2.14273	−2.14273	−2.14238	−2.14238	2.0439	2.0439	−1.93684	−1.93684
1.61803	1.61803	1.56085	1.56085	−1.53296	−1.53296	−1.46373	−1.46373
1.39353	1.39353	−1.30546	−1.30546	1.12873	1.12873	1.10819	1.10819
−0.98277	−0.98277	−0.945322	−0.945322	0.787889	0.787889	−0.618034	−0.618034
0.442544	0.442544	0.37629	0.37629	0.299973	0.299973	0.240979	0.240979
	$n=70$	$r=7$					
3	−2.93324	2.93324	−2.73751	2.73751	−2.42695	2.42695	−2.02729
2.02729	−1.58305	1.58305	−1.18158	1.18158	1		
2.56866	2.56866	−2.55899	−2.55899	2.42147	2.42147	−2.38509	−2.38509
2.17988	2.17988	−2.14259	−2.14259	−2.14252	−2.14252	−2.10605	−2.10605
1.85143	1.85143	−1.73855	−1.73855	1.57653	1.57653	−1.55969	−1.55969
1.45431	1.45431	1.44987	1.44987	−1.39755	−1.39755	−1.30888	−1.30888
1.25824	1.25824	−1.1568	−1.1568	1	1	1	1
−0.869696	−0.869696	−0.864208	−0.864208	0.69727	0.69727	−0.559707	−0.559707
0.412954	0.412954	0.34645	0.34645	0.319061	0.319061	0.254204	0.254204

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