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Note on three-character $(q+1)$-sets in $\mathrm{PG}(3,q)$

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Abstract

We give a combinatorial characterization of twisted cubics in $\mathrm{PG}(3,q)$.

1 Introduction

Let $S_r = S_{r,q} = PG(r,q)$ be a Galois space of dimension r and order q, where $q = p^h$, p a prime. We recall that the number of the points of a d-subspace S_d of S_r is denoted by $\theta_d = \theta_{d,q} = \sum_{i=0}^{d} q^i$, $0 \leq d \leq r$. Moreover the number of d-subspaces S_d of S_r is denoted by $\gamma_{r,d} = \gamma_{r,d,q} = \prod_{i=0}^{d} \frac{\theta_{r-i}}{\theta_{d-i}}$, $0 \leq d \leq r$ and $\gamma_{r,-1} = 1$.

The study of the k-sets of S_r, that is, the sets of k points of S_r, has been founded and deepened by Segre [10].

A useful tool to study a k-set K of S_r is the *d-characters* of K, i.e. the numbers $t_i^d = t_i^d(K)$ of i-secant d-subspaces. Following [11], we call the *degree*, with respect to the dimension d, of K, the greatest integer $g^d = g^d(K)$ such that $t_g^d \neq 0$.

By counting in two different ways the total number of d-subspaces S_d, the number of pairs (P, S_d) where $P \in K$ and S_d is a d-subspace through P, and the number of pairs $(\{P,Q\}, S_d)$ where $\{P,Q\} \subset K$ and S_d is a d-subspace through P and Q, we get the following system of linear equations on integers t_i^d.

$$(1.1)\begin{cases} \sum_{i=0}^{\theta_d} t_i^d = \gamma_{r,d,q} \\ \sum_{i=0}^{\theta_d} i t_i^d = k\gamma_{r-1,d-1,q} \\ \sum_{i=0}^{\theta_d} i(i-1) t_i^d = k(k-1)\gamma_{r-2,d-2,q} \end{cases}$$

A set K is said to be of *class* $[m_1, m_2, \ldots, m_s]_d$ if $t_i^d \neq 0$ implies $i \in \{m_1, m_2, \ldots, m_s\}$. Moreover, a set K of class $[m_1, m_2, \ldots, m_s]_d$ is called of *type* $(m_1, m_2, \ldots, m_s)_d$ if $i \in \{m_1, m_2, \ldots, m_s\}$ implies $t_i^d \neq 0$; see [1], [4], [8], [9], [11] and [12].

A set K is said to be an *s-character set* with respect to the dimension d if exactly s d-characters of K are different from zero; see [3].

A *normal rational curve* C of S_r is an irreducible algebraic variety of dimension 1 which is contained in S_r but not in a proper subspace. A k-arc of S_r is any k-set of linearly independent points of S_r if $k \leq r$ or does not contain $r+1$ linearly dependent points if $k \geq r+1$.

In this paper r is assumed to be three and q odd. A *twisted cubic* C can be represented in its canonical form as follows

$$C = \{P(t) = (t^3, t^2, t, 1), t \in GF(q) \cup \{\infty\}\},$$

where $t = \infty$ gives the point $(1, 0, 0, 0)$. Twisted cubics over finite fields were defined and studied by Segre [10]. Further properties and relation to hyperbolic quadrics were given by Hirschfeld [2], [5], [6] and [7]. The main property of a twisted cubic of S_3 is that it is a maximal arc [10], namely it is a set of $q+1$ points of S_3, no four of which are coplanar. Segre shows that in S_3, any $(q+1)$-arc is a twisted cubic; see [10]. In this scheme of things we prove the following result.

Theorem. *In S_3, a $(q+1)$-set of class $[a, b, c]_1$ such that $g^2 = g^1 + 1$, is a twisted cubic.*

2 The Proof of the Theorem

We prove the theorem in several steps. Consider a $(q+1)$-set K of S_3. It is well-known that K has at least two characters different from zero, with respect to lines; see [12]. We get

Step 2.1. *A $(q+1)$-set K of S_3 has at least three characters different from zero.*

Proof. Suppose on the contrary that K is a $(q+1)$-set of S_3 of type $(m, n)_1$ with $0 \le m \le n$. The system of linear equations (1.1) becomes the following:

$$\begin{cases} t_m + t_n = (q^2+1)(q^2+q+1) \\ mt_m + nt_n = (q+1)(q^2+q+1) \\ m(m-1)t_m + n(n-1)t_n = (q+1)q \end{cases}$$

If $m = 0$, then from the 2$^{\text{nd}}$ and the 3$^{\text{rd}}$ equations we have $q = (n-1)(q^2+q+1) > q$. If $m > 0$, then from the 1$^{\text{st}}$ and the 2$^{\text{nd}}$ equations we have $0 \le (m-1)t_m + (n-1)t_n = q(q^2+q+1)(1-q) < 0$.
In both cases we have a contradiction. □

In view of Step 2.1, we will investigate the 3-character $(q+1)$-sets. From now on let us assume that K is a 3-character $(q+1)$-set.

Step 2.2. *A 3-character $(q+1)$-set K of S_3 is of type $(0, 1, c)_1$.*

Proof. Let K be a $(q+1)$-set of S_3 of type $(a, b, c)_1$ with $0 \le a < b < c$. The system of linear equations (1.1) becomes the following:

$$\begin{cases} t_a + t_b + t_c = (q^2+1)(q^2+q+1) \\ at_a + bt_b + ct_c = (q+1)(q^2+q+1) \\ a(a-1)t_a + b(b-1)t_b + c(c-1)t_c = (q+1)q \end{cases}$$

By subtracting the first equation from the second one we get $(a-1)t_a + (b-1)t_b + (c-1)t_c = q(q^2+q+1)(1-q)$ which is less then zero. Since $0 \le a < b < c$, if

$(a-1) \geq 0$ then $(a-1)t_a + (b-1)t_b + (c-1)t_c > 0$, a contradiction. Therefore $(a-1) < 0$ which implies $a = 0$. Taking into account the third equation of the system we get

$$c(c-b)t_c = (q+1)[(q+1)^2 - b(q^2+q+1)]. \tag{2.1}$$

Since $c(c-b)t_c > 0$ then $(q+1)^2 - b(q^2+q+1) > 0$. So $q + (1-b)(q^2+q+1) > 0$, which implies $b = 1$.

□

From now on let us assume that K is a $(q+1)$-set of type $(0,1,c)_1$.

Step 2.3. *If K is a 3-character $(q+1)$-set of S_3, then K is a set of type $(0,1,p^t+1)_1$ where*

1. $0 \leq t \leq h$
2. $t \neq 0$ *implies* $t \mid h$ *and* h/t *is an odd integer.*

Proof. In view of Step 2.2, K is a $(q+1)$-set of type $(0,1,c)_1$.

Let P be a point of K and let us denote by n the number of c-secant lines through P. Counting in two different ways the number of pairs (Q,r) where r is a line through P and Q is a point of $r \cap K - \{P\}$, we get $(c-1)n = (q-1) = p^h$. Therefore $(c-1) \mid p^h$. Thus $c = p^t + 1$ where $0 \leq t \leq h$. Since $b = 1$ from (2.1) we have $c(c-1)t_c = q(q+1)$, that is, $p^t(p^t+1)t_c = p^h(p^h+1)$. So $(p^t+1)t_c = p^{h-t}(p^h+1)$, which implies that $(p^t+1) \mid p^{h-t}(p^h+1)$. Since (p^t+1) and p^{h-t} are coprime, we get $(p^t+1) \mid (p^h+1)$. The last condition is equivalent to (2). □

Step 2.4. *In view of (1) Step 2.3, the extreme cases are:*

- $t = 0$ *which implies that K is a $(q+1)$-set of type $(0,1,2)_1$, i.e. a cap, a set of points no three of which are collinear.*
- $t = h$ *which implies that k is a $(q+1)$-set of type $(0,1,q+1)_1$, i.e. a line.*

Remark. *In $PG(3,p^{2^n})$ a 3-character $(q+1)$-set is either a line or a cap.*

Step 2.5. *If K is a set of type $(0,1,c)_1$ of $PG(3,q)$, then, for each plane π of S_3, the set $K \cap \pi$ is a set of class $[0,1,c]_1$ of π.*

Step 2.6. *If $2 < c < (q+1)$ then $g^2 > c+1$.*

Proof. Let r be a c-secant line of K. Since $c < (q+1)$, there is at least one point P of K not on r. Let π denote the plane through P and r. As the set $K \cap \pi$ is a set of class $[0,1,c]_1$, then each line through P and a point of r is a c-secant line. So $\mid K \cap \pi \mid \geq (c^2 - c + 1)$. Since $2 < c$, we have $g^2 \geq \mid K \cap \pi \mid > c+1$. □

Step 2.7. *A 3-character $(q+1)$-set K such that $g^2 = g^1 + 1$ is a set of type $(0,1,2)_1$.*

Proof. In view of Step 2.3, K is a $(q+1)$-set of type $(0, 1, p^t + 1)_1$ where $0 \leq t \leq h$. We have to prove that $t = 0$ necessarily. On the contrary let us assume that $t > 0$, so $(p^t + 1) > 2$. Since $g^2 = g^1 + 1$, we have that K is not a line. Therefore $(p^t + 1) < (q + 1)$ and $t \neq h$. Moreover $2 < (p^t + 1) < (q + 1)$, and from Step 2.6 we have that $g^2 > c + 1 = g^1 + 1$, a contradiction. □

Finally, the Theorem follows from the previous steps and by observing that $g^2 = 3$ and so K is a $(q + 1)$-arc.

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