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Forbidden-minors for splitting binary gammoids

Y. M. Borse

Department of Mathematics
University of Pune, Pune
India
ymborse@math.unipune.ernet.in

Abstract

In this paper, we obtain a forbidden-minor characterization of the class of binary gammoids M such that, for every pair of elements x, y of M, both the splitting matroid $M_{x,y}$ and the element splitting matroid $M'_{x,y}$ are binary gammoids.

1 Introduction

Given a matroid M, let $E(M)$ and $r(M)$ denote the ground set and the rank of M, respectively. If G is a graph, then $M(G)$ denotes the circuit matroid of G. We say that two elements x and y of a matroid M are in series if $\{x, y\}$ is a 2-cocircuit. We refer to Oxley [8] for undefined notation and terminology.

Fleischner [5] introduced the *splitting operation* for a graph as follows. Let G be a connected graph and v be a vertex of G of degree at least three and $x = vv_1$, $y = vv_2$ be edges of G. Let $G_{x,y}$ be the graph obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ which is adjacent to v_1 and v_2. For practical purposes, we denote the new edges $v_{x,y}v_1$ and $v_{x,y}v_2$ by x and y, respectively (see Figure 1). We say that the graph $G_{x,y}$ is obtained from G by splitting away the edges x and y from the vertex v.

Fleischner [5] used this splitting operation to characterize the Eulerian graphs. Raghunathan et al. [9] extended the splitting operation from graphs to binary matroids as follows.

Definition 1.1. Let M be a binary matroid and let A be a matrix over $GF(2)$ that represents M. Suppose x and y are distinct elements of M. Let $A_{x,y}$ be the matrix obtained from A by adjoining the row that is zero everywhere except for entries of 1 in the columns corresponding to x and y. Let $M_{x,y}$ be the vector matroid of the matrix $A_{x,y}$. We say that $M_{x,y}$ is obtained from M by splitting away the elements x and y.

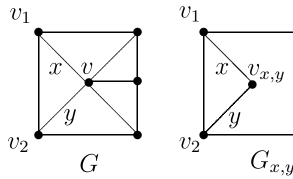


Figure 1

The circuits of the splitting matroid $M_{x,y}$ are characterized as follows.

Lemma 1.2 [9]**.** *Let M be a binary matroid and suppose $\{x, y\} \subseteq E(M)$. Let $\mathcal{C}$ and $\mathcal{C}_{x,y}$ be the collections of circuits of M and $M_{x,y}$, respectively. Then $\mathcal{C}_{x,y} = \mathcal{C}_0 \cup \mathcal{C}_1$, where*
$\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C$ *or* $x \notin C, y \notin C\}$; *and*
$\mathcal{C}_1 = \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, x \in C_1, y \in C_2, C_1 \cap C_2 = \phi$ *and* $C_1 \cup C_2$ *contains no member of* $\mathcal{C}_0\}$.

The *element splitting operation* for a graph with respect to a pair of adjacent edges was introduced by Azadi [1]. Figure 2 shows the graph $G'_{x,y}$ that is obtained from the graph G by element splitting the edges x and y. Thus the element splitting operation combines, in a sense, the Fleischner's above splitting operation [5] and the Tutte's point splitting operation [13]. Azadi [1] further extended this operation from graphs to binary matroids as follows.

Definition 1.3. Let M be a binary matroid and let A be a matrix over $GF(2)$ that represents M. Suppose x and y are distinct elements of M. Let $A'_{x,y}$ be the matrix obtained from A by adjoining the row that is zero everywhere except for entries of 1 in the columns corresponding to x and y, and then adjoining the column which is zero everywhere except for the entry of 1 in the last row. Let $M'_{x,y}$ be the vector matroid of the matrix $A'_{x,y}$. We say that $M'_{x,y}$ is obtained from M by element splitting the elements x and y.

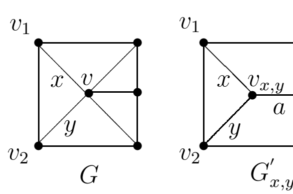


Figure 2

Azadi [1] characterized the circuits of the element splitting matroid $M'_{x,y}$ as follows.

Lemma 1.4 [1]**.** *Let M be a binary matroid and suppose $\{x, y\} \subseteq E(M)$ and $a \notin E$. Let $\mathcal{C}$ and $\mathcal{C}'_{x,y}$ be the collections of circuits of M and $M'_{x,y}$, respectively. Then $\mathcal{C}'_{x,y} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C$ or $x, y \notin C\}$;*
$\mathcal{C}_1 = \{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C},\ x \in C_1, y \in C_2,\ C_1 \cap C_2 = \phi$ *and* $C_1 \cup C_2$ *contains no member of* $\mathcal{C}_0\}$; $\mathcal{C}_2 = \{C \cup \{a\} : C \in \mathcal{C}$ *and contains exactly one of* x *and* $y\}$.

Various properties of the splitting matroid and the element splitting matroid are obtained in [2], [3], [4], [7], [9], [10], [11] and [12].

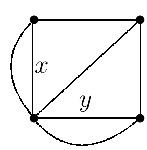
The class of *gammoids* is a well known class of matroids. Binary gammoids are characterized as follows.

Theorem 1.5 [8]. *The following statements are equivalent for a matroid M:*
(i) M is a graphic gammoid;
(ii) M is a regular gammoid;
(iii) M is a binary gammoid;
(iv) M has no minor isomorphic to $U_{2,4}$ or $M(K_4)$.

The splitting operation on a graphic matroid and a cographic matroid may not yield a graphic matroid and a cographic matroid, respectively. Shikare, and Waphare [12] characterized the class of graphic matroids whose splitting matroids, with respect to any pair of elements, are graphic. Borse, Shikare and Dalvi [2] obtained such characterization for the class of cographic matroids. The same authors also characterized the class of graphic matroids and the class of cographic matroids whose element splitting matroids, with respect to any pair of elements, are graphic and cographic, respectively (see [3] and [4]).

The splitting matroid and the element splitting matroid of a binary gammoid with respect to a pair of elements may not be a binary gammoid. In this paper, we obtain a characterization of the class of binary gammoids M such that, for every pair of elements x, y of M, both the splitting matroid $M_{x,y}$ and the element splitting matroid $M'_{x,y}$ are binary gammoids. The following two theorems are the main results of this paper.

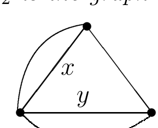
Theorem 1.6. *The splitting operation, by any pair of elements, on a binary gammoid M yields a binary gammoid if and only if M has no minor isomorphic to the circuit matroid $M(G_1)$, where G_1 is the graph depicted in Figure 3.*



G_1

Figure 3

Theorem 1.7. *The element splitting operation, by any pair of elements, on a binary gammoid M yields a binary gammoid if and only if M has no minor isomorphic to the circuit matroid $M(G_2)$, where G_2 is the graph depicted in Figure 4.*



G_2

Figure 4

2 The splitting of binary gammoids

In this section, we prove Theorem 1.6. Note first that if M is a binary gammoid, then the splitting matroid $M_{x,y}$ is also binary for any pair x, y of elements of M. Suppose $M_{x,y}$ is not a binary gammoid for some x, y. Then, by Theorem 1.5, $M_{x,y}$ has $M(K_4)$ as a minor.

Lemma 2.1. *Let M be a binary gammoid and suppose $M_{x,y}$ is not a binary gammoid for some elements x and y of M. Then there is a minor N of M containing x and y such that no two elements of N are in series and $N_{x,y}/\{x\} \cong M(K_4)$ or $N_{x,y}/\{x, y\} \cong M(K_4)$.*

Proof. The proof is similar to the proof of Theorem 2.3 of [12]. □

Corollary 2.2. *Let M be a binary gammoid. For any $x, y \in E(M)$, the matroid $M_{x,y}$ is a binary gammoid if and only if M has no minor N as stated in Lemma 2.1.*

Lemma 2.3. *Let M be a binary gammoid such that no two elements of M are in series. Suppose $M_{x,y}/\{x\} \cong M(K_4)$ or $M_{x,y}/\{x, y\} \cong M(K_4)$ for some elements x, y of M. Then the following statements hold:*
(i) M is graphic;
(ii) M is connected;
(iii) $r(M_{x,y}) = r(M) + 1$;
(iv) M has at most two pairs of parallel elements;
(v) if $M_{x,y}/\{x, y\} \cong M(K_4)$, then M has at most one pair of parallel elements.

Proof. (i) follows from Theorem 1.5. The proofs of (ii) and (iii) are similar to the proofs of corresponding statements in Theorem 2.6(iii) and Proposition 2.2(i) of [12], respectively. By Lemma 1.2, every circuit of M containing both x and y, or neither of them is preserved in $M_{x,y}$. Since $M(K_4)$ does not contain a 2-circuit, each 2-circuit of M contains x or y. Thus (iv) follows. Suppose $M_{x,y}/\{x, y\} \cong M(K_4)$ and M has two pairs of parallel elements. If $\{x, y\}$ is a 2-circuit of M, then $M \setminus \{x, y\} \cong M_{x,y}/\{x, y\} \cong M(K_4)$, a contradiction. This implies that M has disjoint 2-circuits each containing either x or y. Hence, by Lemma 1.2, there is a 4-circuit in $M_{x,y}$ containing both x and y. Therefore $M_{x,y}/\{x, y\}$ contains a 2-circuit, a contradiction. Thus (v) is proved. □

Theorem 1.6 follows from Corollary 2.2 and the following lemma.

Lemma 2.4. *Let M be a binary gammoid. Then no two elements of M are in series and also $M_{x,y}/\{x\} \cong M(K_4)$ or $M_{x,y}/\{x, y\} \cong M(K_4)$ for some elements x, y of M if and only if M is isomorphic to $M(G_1)$, where G_1 is the graph of Figure 3.*

Proof. Suppose M is isomorphic to $M(G_1)$, where G_1 is the graph of Figure 3. Then $M(G_1)_{x,y}/\{x\} \cong M(K_4)$. Further, no two elements of $M(G_1)$ are in series because no vertex of G_1 has degree two.

Conversely, suppose no two elements of M are in series and also $M_{x,y}/\{x\} \cong M(K_4)$ or $M_{x,y}/\{x, y\} \cong M(K_4)$ for some elements x, y of M. By Lemma 2.3(ii), M is connected, and by Theorem 1.5, M is graphic and it does not have $M(K_4)$ as a minor. Let G be a connected graph corresponding to M. Then G is loopless, 2-connected and has minimum degree at least 3.

Case (i). $M_{x,y}/\{x\} \cong M(K_4)$.
Since $r(M(K_4)) = 3$, $r(M_{x,y}) = 4$. Therefore, by Lemma 2.3(iii), $r(M) = 3$. Further, $|E(M)| = 7$. We conclude that G has 4 vertices and 7 edges. Hence G is not simple. Suppose G has only one pair of parallel edges. Then G can be obtained from a simple graph with 4 vertices and 6 edges, that is, from K_4 by putting an edge in parallel. Thus M has $M(K_4)$ as minor, a contradiction. Hence, by Lemma 2.3(iv), G has exactly two pairs of parallel edges. Therefore G can be obtained from a 2-connected simple graph with 4 vertices and 5 edges by adding two edges in parallel to its simplification. By Harary [6, p. 217], every connected simple graph of 4 vertices and 5 edges is isomorphic to the graph of Figure 5.

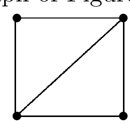


Figure 5

Hence G is isomorphic to one of the two graphs of Figure 6.

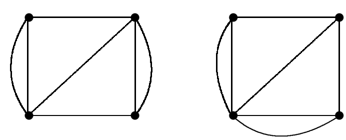


Figure 6

It is easy to see that the circuit matroid of each graph of Figure 6 is isomorphic to the circuit matroid $M(G_1)$, where G_1 is the graph of Figure 3.

Case (ii). $M_{x,y}/\{x, y\} \cong M(K_4)$.
As $r(M(K_4)) = 3$, $r(M_{x,y}) = 5$. Hence $r(M) = 4$ and $|E(M)| = 8$. Then G has 5 vertices, 8 edges and has the degree sequence $(4, 3, 3, 3, 3)$. Suppose G is simple. By Harary [6, p. 217], every 2-connected simple graph of the degree sequence $(4, 3, 3, 3, 3)$ is isomorphic to the graph of Figure 7. We discard this graph because it has K_4 as a minor.

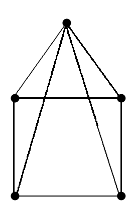


Figure 7

Therefore G is not simple. By Lemma 2.3(v), G has exactly one pair of parallel edges. So G can be obtained from a simple 2-connected graph with degree sequence $(4, 3, 3, 2, 2)$ or $(3, 3, 3, 3, 2)$ by adding an edge in parallel. By Harary [6, p. 217], every 2-connected simple graph with degree sequence $(4, 3, 3, 2, 2)$ or $(3, 3, 3, 3, 2)$ is isomorphic to the graph of Figure 8(i) or 8(ii), respectively. Since the graph of Figure 8(ii) has K_4 as a minor, we discard it. Also, we get a simple graph by adding an edge between the two vertices of degree 2 of the graph of Figure 8(i). Hence G cannot arise from this graph also. □

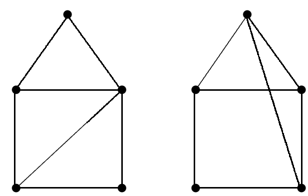

(i) (ii)

Figure 8

3 The element splitting of binary gammoids

In this section, we prove Theorem 1.7.

Lemma 3.1. *Let M be a binary gammoid and suppose $M'_{x,y}$ is not a binary gammoid for some elements x and y of M. Then there is a minor N of M containing x and y such that no two elements of N are in series and one of $N'_{x,y}\setminus\{a\}/\{x\}$, $N'_{x,y}\setminus\{a\}/\{y\}$, $N'_{x,y} \setminus \{a\}/\{x, y\}$, $N'_{x,y}$, $N'_{x,y}/\{x\}$ or $N'_{x,y}/\{x, y\}$ is isomorphic to $M(K_4)$.*

Proof. The proof similar to the proof of Lemma 2.3 of [3]. □

A binary gammoid M is called minimal with respect to $M(K_4)$ if no two elements of M are in series and also M has elements x and y such that one of $M'_{x,y} \setminus \{a\}/\{x\}, M'_{x,y} \setminus \{a\}/\{x, y\}, M'_{x,y}, M'_{x,y}/\{x\}$ or $M'_{x,y}/\{x, y\}$ is isomorphic to $M(K_4)$.

Corollary 3.2. *Let M be a binary gammoid. For any $x, y \in E(M)$, the matroid $M'_{x,y}$ is binary gammoid if and only if M has no minor N that is minimal with respect to $M(K_4)$.*

Lemma 3.3. *Let a binary gammoid M be minimal with respect to the matroid $M(K_4)$. Then the following statements hold: (i) M is graphic; (ii) $r(M'_{x,y}) = r(M)+1$ and (iii) M has neither loops nor coloops.*

Proof. (i) is obvious from Theorem 1.5. (ii) follows from Definition 1.3 and Lemma 1.4. The proof of (iii) is similar to the proof of Lemma 2.6(i) of [3]. □

In the following lemma, we characterize the minimal gammoids with respect to $M(K_4)$.

Lemma 3.4. *Let M be a binary gammoid. Then M is minimal with respect to the matroid $M(K_4)$ if and only if M is isomorphic to one of the two matroids $M(G_i)$ for $i = 1, 2$, where G_1 and G_2 are the graphs of Figure 3 and Figure 4, respectively.*

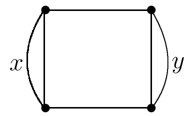


Figure 9

Proof. Suppose M is isomorphic to one of the two matroids $M(G_i)$ for $i = 1, 2$, where G_1 and G_2 are the graphs of Figure 3 and Figure 4, respectively. Then no two elements of M are in series. Also $M(G_1)'_{x,y} \setminus \{a\}/\{x\} \cong M(K_4)$, and $M(G_2)'_{x,y} \cong M(K_4)$. Hence M is minimal with respect to $M(K_4)$.

Conversely, suppose M is a minimal matroid with respect to $M(K_4)$. By Theorem 1.5, M is graphic and does not have $M(K_4)$ as a minor. Let G be a connected graph corresponding to M. Then, by Lemma 3.3(iii), G has minimum degree at least 3. Let x and y be elements of M such that one of $M'_{x,y} \setminus \{a\}/\{x\}$, $M'_{x,y} \setminus \{a\}/\{x, y\}$, $M'_{x,y}/\{x\}$, $M'_{x,y}$ or $M'_{x,y}/\{x, y\}$ is isomorphic to $M(K_4)$.

It is easy to see that $M'_{x,y}\setminus\{a\} = M_{x,y}$. Hence, if $M'_{x,y}\setminus\{a\}/\{x\}$ or $M'_{x,y}\setminus\{a\}/\{x, y\}$ is isomorphic to $M(K_4)$, then, by Lemma 2.4, G is isomorphic to the graph G_1 of Figure 3.

Suppose $M'_{x,y} \cong M(K_4)$. Since $r(M(K_4)) = 3$, $r(M'_{x,y}) = 3$. Hence, by Lemma 3.3(ii), $r(M) = r(M'_{x,y}) - 1 = 2$ and $|E(M)| = |E(M'_{x,y})| - 1 = 6 - 1 = 5$. Consequently, G is a connected graph with 3 vertices and 5 edges and has minimum degree at least 3. By Lemma 1.4, every 2-circuit of M containing both x and y, or neither of them is preserved in $M'_{x,y}$. As $M'_{x,y} \cong M(K_4)$, every 2-circuit of M contains either x or y but not both. This implies that G is isomorphic to the circuit matroid $M(G_2)$, where G_2 is the graph of Figure 4.

Suppose $M'_{x,y}/\{x\} \cong M(K_4)$. Then $r(M) = 3$ and $|E(M)| = 6$. Thus G is a graph with 4 vertices and 6 edges and has minimum degree at least 3. If G is simple, then it is isomorphic to K_4, a contradiction. Hence G is not simple. As above, every 2-circuit of G contains either x or y but not both. The graph of Figure 9 is the only graph with these properties. Hence G is isomorphic to the graph of Figure 9. Let C be a 2-circuit of M containing x. Then, by Lemma 1.4, $C \cup \{a\}$ is a 3-circuit in $M'_{x,y}$. Therefore $M'_{x,y}/\{x\}$ contains a 2-circuit and hence it is not isomorphic to $M(K_4)$, a contradiction.

Suppose $M'_{x,y}/\{x, y\} \cong M(K_4)$. Then $r(M) = 4$ and $|E(M)| = 7$. Thus G is a graph with 5 vertices and 7 edges. Hence G has a vertex of degree less than 3, a contradiction. This completes the proof. □

Proof of Theorem 1.7. Let M be a binary gammoid. From Corollary 3.2 and Lemma 3.4, it follows that $M'_{x,y}$ is a binary gammoid for every pair $\{x, y\}$ of elements of M if and only if M has no minor isomorphic to any of the matroids $M(G_i)$ for $i = 1, 2$, where G_1 and G_2 are the graphs of Figure 3 and Figure 4, respectively. It is easy to see that $M(G_2)$ is a minor of $M(G_1)$. Thus $M'_{x,y}$ is a binary gammoid if and only if M has no minor isomorphic to the matroid $M(G_2)$. □

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