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# Vertex-magic labeling of non-regular graphs

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**Abstract**

In this paper, we are studying vertex-magic total labelings (VMTLs) of simple graphs. By now much is known about methods for constructing VMTLs for regular graphs. Here we are studying non-regular graphs. We show how to construct labelings for several families of non-regular graphs, including graphs formed as the disjoint union of two other graphs already possessing VMTLs. We focus on conditions which make these VMTLs strong, so that previously known methods can then be used to build larger graphs from these which will themselves have VMTLs. In the second part of the paper, we investigate ways of describing how far a graph may be from being regular but still possess a VMTL.

## 1 Introduction

There has been a considerable amount of progress recently in showing how to construct vertex-magic total labelings (VMTLs) for quite a variety of graphs. Most of the graphs shown to possess VMTLs are regular, an observation that prompted the second author to conjecture [10] that, apart from two small exceptions $K_2$ and $2C_3$, all regular graphs are vertex-magic. The important paper by McQuillan [12] introduced a method for proving that a large family of cubic graphs was vertex-magic, without actually having to construct the labeling. This method was further developed by the first author [3] to show that large families of regular graphs were vertex-magic. Subsequent work by the authors have built further on this method [4]. For example, we now know that all odd-regular graphs of order up to 17 are vertex-magic, and asymptotically *almost all* regular graphs of any odd order are vertex-magic.

Much less work has been done on non-regular graphs. The early papers [9] and [7] dealt with trees and complete bipartite graphs respectively. In both cases, constructions were given for various infinite families of graphs. Equally important, conditions were discovered under which it is impossible for such graphs to admit a VMTL. In this paper we construct labelings for several infinite families of non-regular graphs, and then examine conditions on order and degree which must be satisfied by every vertex-magic graph.

The method embodied in Theorem 2.1 of [3] begins with a *strong* VMTL of some graph $G$ (one in which the largest labels are assigned to the vertices) and adds an arbitrary 2-factor in such a way that we create a strong VMTL for the new graph $G^*$ of the same order but larger size. The work described in [4] focusses on cases where the initial graph $G$ is *regular* in order to produce labelings for additional *regular* graphs, since we were motivated by the conjecture described above. When $G$ is not regular, if we can find a strong VMTL for $G$, then we can still carry out the construction, thereby creating strong VMTLs for more non-regular graphs of the same order, but larger size. In some cases this process can be iterated in a new way, as we will show.

## 2 Non-regular graphs from Mutations

In [5] we studied a process we called *mutation* which provides a way of generating VMTLs for non-regular graphs. Mutation begins with a VMTL for one graph and transforms it into either a different VMTL for the same graph or a VMTL for a different graph by swapping the adjacencies of a set of edges incident with one vertex with another set of edges incident with a different vertex. If the initial VMTL $\lambda$ is strong, then all the VMTLs resulting from mutating $\lambda$ will also be strong.

Mutation is a remarkably fruitful procedure for generatingVMTLs for new graphs. If we begin with a regular graph and swap two sets of edges of different sizes, then the resulting graph will not be regular, so we can use mutation as a way of producing many labelings for many non-regular graphs.

Without wanting to reproduce too much of what has appeared in [5], we illustrate in Figure 1 an example where a strong VMTL of an odd cycle is mutated via a two-for-one swap into a strong VMTL for a kite. Then a 2-factor can be added, producing a graph of larger size with a strong VMTL.

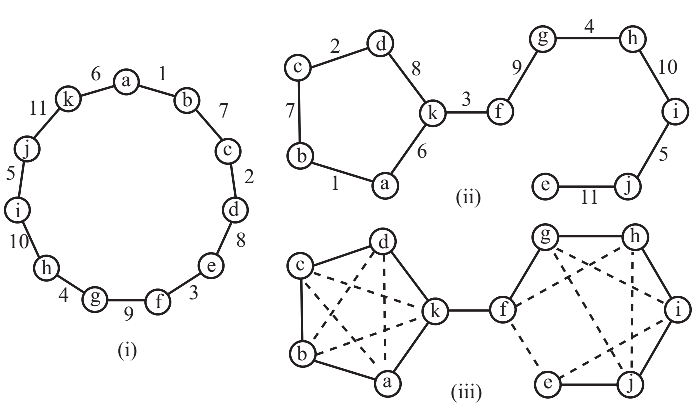


Figure 1: Mutating $C_{11}$ to a $(5,6)$-kite then adding a 2-factor

There are altogether 6 regular cubic graphs of order 8 and all have strong labelings (these are illustrated in Figure 4 of [5]). It is easy to check that every one of these labelings permits a 3-for-2 mutation and these mutations produce strong VMTLs for graphs with degree sequence $2 \cdot 3^6 \cdot 4$. A mutation of the 3-cube is illustrated in Figure 2. It is also easy to check that each of the strong labelings of these regular graphs permits a 2-for-1 mutation, and thus we have another set of graphs with degree sequence $2 \cdot 3^6 \cdot 4$ having strong VMTLs. We note that not all the resulting graphs need be different, although the labelings are always different. In many cases these new VMTLs can be further mutated to produce more non-regular graphs.

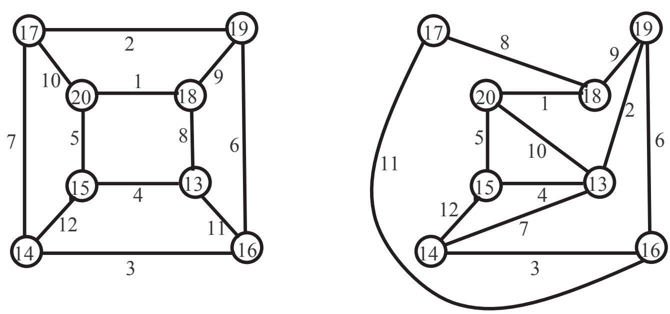


Figure 2: Mutating the 3-cube to a non-regular graph

There are 19 connected cubic graphs of order 10 and all have strong VMTLs. These are illustrated in [5]. Again it is easily checked that each of these VMTLs admits both a 3-for-2 mutation and a 2-for-1 mutation, thus in each case producing a strong VMTL for a non-regular graph with degree sequence $2 \cdot 3^8 \cdot 4$. As before these may not all be different graphs but will be different labelings, and in many cases able to be mutated through many further steps.

## 3 Union of regular graphs with strong VMTLs

If a union of two graphs possesses a strong VMTL then we can apply Theorem 2.1 of [3] to build VMTLs of further graphs by adding 2-factors to the union. We consider the case where the two graphs are regular but of different degree so that the union is not regular.

If two regular graphs $G$ and $H$ both possess a strong VMTL then in some cases a strong VMTL for $G \cup H$ can be formed by adding a constant to the vertex-labels of one graph and adding a different constant to both edge- and vertex-labels of the other. The following theorem describes one condition under which this can occur:

**Theorem 3.1.** *Let $G$ be a $d$-regular graph of order $v$ and $H$ a $t$-regular graph of order $u$ each having a strong VMTL. If $vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2$ then $G \cup H$ possesses a strong VMTL.*

*Proof.* Since $G$ is $d$-regular of order $v$ it has $\frac{1}{2}vd$ edges, and similarly $H$ has $\frac{1}{2}ut$ edges. We construct the VMTL for $G \cup H$ as follows: add $\frac{1}{2}vd$ to every edge- and vertex-label of $H$ and add $\frac{1}{2}ut + u$ to the vertex labels of $G$. It is routine to check that the labels range from 1 to $v + u + \frac{1}{2}vd + \frac{1}{2}ut$ and that the largest appear on the vertices. Calculating the magic constant for this labeling from $G$ and then from $H$, we get:

$$k_G = \tfrac{1}{2}ut + u + \tfrac{1}{v}(\tfrac{1}{2}vd(\tfrac{1}{2}vd + 1) + \tfrac{v}{2}(\tfrac{1}{2}vd + 1 + \tfrac{1}{2}vd + v))$$
$$k_H = (t + 1)\tfrac{1}{2}vd + \tfrac{1}{u}(\tfrac{1}{2}ut(\tfrac{1}{2}ut + 1) + \tfrac{u}{2}(\tfrac{1}{2}ut + 1 + \tfrac{1}{2}ut + u))$$

For the labeling to be a VMTL, we need $k_G = k_H$. Equating the expressions $k_G$ and $k_H$ and simplifying gives:

$$vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2 \quad (1)$$

Thus if this relationship holds the labeling is a VMTL, and since the largest labels are still on the vertices of $G \cup H$, the labeling will be a strong VMTL as required. □

Of course this condition is very restrictive. However, one special case is especially simple and of particular interest; namely where $G$ is a quartic and $H$ is a 2-regular graph (and where both graphs are of odd order, so that a strong VMTL exists). In this case, equation 1 simplifies to $u = v + 2$. For example, $K_5$ and $C_7$ satisfy these conditions and thus $K_5 \cup C_7$ possesses a strong VMTL. Similarly, letting $G = (C_7)^C$ and letting $H$ be any one of the graphs $C_9$, $C_3 \cup C_6$, $C_4 \cup C_5$ or $C_3 \cup C_3 \cup C_3$ will give a graph $G \cup H$ with a strong VMTL since each of these 2-regular graphs possesses a strong VMTL (see [4]).

We emphasize that since the unions labelled using this construction are not regular then neither are any graphs of greater size built from them by adding 2-factors and applying Theorem 2.1 of [3].

The magic constants for strong VMTLs of cubic graphs, such as those produced by McQuillan's construction, are too small to permit the direct application of Theorem 3.1 to unite such a graph with another regular graph having a strong VMTL. For example, as shown in Section 4.1 we would require $e > 1.71v$ to add even a cycle. However, if we add any 2-factor to the cubic graph, we obtain a graph where such a union is possible, and the theorem can be applied.

For example, if we add a 2-factor to any such cubic graph of order $4t$ we obtain a 5-regular graph $G(4t, 10t)$ for which we can construct a strong VMTL with a magic constant of $37t + 3$. This graph would now satisfy the conditions of the theorem if we wished to proceed further by uniting it with a cycle of order $14t + 3$. We obtain a strong VMTL of $G(4t, 10t) \cup C_{14t+3}$, which is a graph with degree sequence $2^{14t+3}5^{4t}$.

We can repeat the process: if we add a 2-factor to this graph then the union of the resulting graph with a cycle satisfies the conditions of Theorem 3.1 and thus also possesses a strong VMTL. We explore this process of building a sequence of graphs of increasing size, order and irregularity further in Section 4.1.

# 4 Unions of two graphs, one regular and one with a strong VMTL

We can also consider a more general case of a union of two graphs. This time, we require that one possesses a strong VMTL and the other is regular and possesses a (not necessarily strong) VMTL.

Let $G(v, e)$ be the (possibly non-regular) graph with a strong VMTL with magic constant $k_G$ and let $H(u, \frac{1}{2}ru)$ be an $r$-regular graph possessing a VMTL with magic constant $k_H$. Then if we add $e$ to each edge- and vertex-label of $H$ and add $u + \frac{1}{2}ru$ to each of the vertex labels of $G$, the result will be a VMTL providing:

$$k_H + (r + 1)e = k_G + u + \tfrac{1}{2}ru. \tag{2}$$

Note that since $r \geq 2$ and we have $k_H > u + \frac{ru}{2}$, then we need $k_G > (r + 1)e$ for the equation to hold.

A simple example is the union of the cycle $C_{4t}$ with a 4-regular graph of order $4t - 1$. In [14], there is a labeling of $C_{4t}$ with a magic constant of $\frac{1}{2}(20t + 4)$. A strong VMTL of an $r$-regular graph $G(v, \frac{1}{2}rv)$ has a magic constant

$$\frac{1}{2}(\frac{1}{2}r^2v + rv + r + v + 1)$$

and hence a 4-regular graph of order $4t - 1$ has a magic constant of $26t - 4$. Then equation 2 gives us

$$\begin{aligned} k_H + (r + 1)e &= \tfrac{1}{2}(20t + 4) + 3(8t - 2) = 34t - 4, \\ k_G + u + f &= (26t - 4) + 4t + 4t = 34t - 4. \end{aligned}$$

Hence the labeling is a VMTL as required. Since it is not a strong VMTL we cannot build labelings of further graphs from it by using the method of Theorem 2.1 of [3].

For the theme of this paper, it is important to emphasize that the graph with the strong VMTL does not need to be regular. Also, if both graphs have strong labelings then it may be possible to iterate the process.

For example, a quartic graph of order $2n-1$ and a cycle of order $2n+1$ together satisfy the conditions. If we can find a strong VMTL of their union we can then add a 2-factor to obtain a graph $G$ having degree sequence $4^{2n+1}6^{2n-1}$. $G$ is a connected graph with $10n - 1$ edges and a magic constant of $39n - 3$. We can then form the union of $G$ with a cycle of order $18n - 3$ to satisfy equation 2 and again add a 2-factor. This new graph of order $22n - 3$ will have a strong VMTL and degree sequence $4^{18n-3}6^{2n+1}8^{2n-1}$. We will show in the next section that we can continue this process to obtain graphs that are ever more irregular but still having strong VMTLs.

## 4.1 Unions where the second graph is an odd cycle

If we restrict our attention to the case $G_0(v, e) \cup C_u$ where $G_0$ has a strong VMTL then we can obtain some interesting results. Note firstly that $k_G = \frac{1}{v}(e(e+1) + \frac{v}{2}(2e+v+1))$

and $k_H = \frac{1}{2}(5u+3)$. From the expression for $k_G$ it follows that $2e(e+1)/v$ is an integer. Equation 2 gives us:

$$\frac{1}{2}(5u+3) + 3e = \tfrac{1}{v}(e(e+1) + \tfrac{v}{2}(2e+v+1)) + 2u$$

which can be rearranged to give:

$$u = v - 2 - 4e + \tfrac{1}{v}(2e(e+1)).$$

Since $u$ is greater than 2, we have

$$v + \tfrac{1}{v}(2e(e+1)) > 4e + 4$$

which will be the case if

$$e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2 + 4v + 1}$$

or, approximating, if $e > 1.71v$.

Secondly, we require $u$ to be an odd integer. With $k_G$ as above, we get

$$2k_G - 6e - 3 = v - 4e - 2 + \frac{1}{v}(2e(e+1)).$$

The left-hand side of this equation is an odd integer while the right-hand side is equal to $u$, hence $u$ is odd as required. Further any cycle of odd order possesses a strong VMTL. Note that we made no assumptions about the order, size or structure of $G$ (although we have already shown that $e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2+4v+1}$). We specified only that it possessed a strong VMTL. So we have shown that for any graph with a strong VMTL with $e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2+4v+1}$ there is a cycle of odd order that satisfies equation 2. We summarize these results in the following theorem:

**Theorem 4.1.** *Let $G_0(v,e)$ be any graph having a strong VMTL. Then $G_0(v,e) \cup C_u$ will also possess a strong VMTL providing $e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2+4v+1}$ and $u = v - 2 - 4e + 2e(e+1)/v$.*

We can now use this labeling as a basis for building more complex graphs that also possess strong VMTLs. We do this by adding one or more 2-factors to $G_0(v,e) \cup C_u$ by the method of Theorem 2.1 of [3] to obtain a graph $G_1$ of order $v+u$ and size $e + u + q(v+u)$ (where $q \geq 1$ is the number of 2-factors added). Note that the resulting graphs may not even be connected.

Given that $e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2+4v+1}$, it follows that:

$$e + u + q(v+u) > -\tfrac{1}{2} + (v+u) + \tfrac{1}{2}\sqrt{2(v+u)^2 + 4(v+u) + 1}$$

and $G_1$ will also have a strong VMTL. As a result we can repeat the process by forming the union of $G_1$ with another cycle. We can thus obtain an infinite sequence of graphs

$$G_0 \subset G_1 \subset \cdots \subset G_i \subset \cdots$$

in which each succeeding graph has a larger degree set than the graph from which it was constructed. If $G_0$ has vertices of $s$ different degrees then $G_i$ will have vertices of $s+i$ different degrees. It should also be noted that at each iteration we can choose how many 2-factors to add to the graph so that in effect we have a branching sequence of graphs, no two of which will possess the same degree sequence or degree set.

These results lead to the following theorem:

**Theorem 4.2.** *There exist graphs possessing VMTLs which have minimum degree 2 and arbitrarily large maximum degree.*

*Proof.* Let $G_0(v,e)$ be a graph with a strong VMTL in which $e > -\frac{1}{2} + v + \frac{1}{2}\sqrt{2v^2+4v+1}$. Then as shown above we can construct an infinite sequence of graphs

$$G_0 \subset G_1 \cdots \subset G_i \subset \cdots$$

for which

$$\Delta(G_0) < \Delta(G_1) \cdots < \Delta(G_i) < \cdots$$

i.e. we have a strictly increasing sequence of maximum degrees. For each of the graphs $G_i$ we can find a cycle $C_{u_i}$ such that $G_i \cup C_{u_i}$ possesses a VMTL, and therefore $\delta(G_i) = 2$. The result follows. □

This result will be relevant in the next section.

## 5 Constraints on Vertex Degrees

Clearly, one of the most important problems in the study of vertex-magic total labelings is to find easily stated conditions that apply to any graph and tell us whether or not the graph admits a VMTL. For example, what straightforward conditions on the degrees of a graph are there that preclude the existence of a VMTL?

In [2], we showed that the complete bipartite graph $K_{n,n+i}$ has no VMTL for $i \geq 2$. The proof is a simple counting argument that calculates the maximum possible magic constant from the set of lower degree vertices and shows it must be smaller than the minimum possible magic constant calculated from the set of higher degree vertices. In [9], we used a similar argument on trees to show that if the ratio of leaves to internal vertices was too high (more than about $\sqrt{3}$) then the tree had no VMTL. Similar elementary counting arguments have been used elsewhere to obtain similar results.

**Problem.** *If there is too much variation among the degrees in a graph, then the graph has no VMTL. Find a more precise formulation of this statement that applies to all graphs, in terms of the degrees of the vertices of the graph.*

There will be limitations on how strong such a statement can be because as we illustrate in Section 5.3 the degree sequence does not determine whether or not a graph has a VMTL. How to formulate such a condition will be interesting because, for example, it must take into account the result of Theorem 4.2.

Alan Beardon has published the only result so far that goes beyond our simple counting arguments of [2] and [9]. He derived a constraint on the maximum degree in terms of both the number of vertices and the number of edges. We are able to build on this result to show that if $G$ possesses a VMTL then both the minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$ are constrained by the *average* degree $D$ of the vertices in the graph.

## 5.1 Constraint on the maximum degree

In [1], Beardon showed that a graph with $v$ vertices and $e$ edges cannot possess a VMTL unless the degree $d$ of each one of its vertices satisfies the following condition:

$$v^2 + 2e^2 + 4ev + 2e - (vd^2 + (5v-2)d + v) \geq 0.$$

This inequality may be rearranged as:

$$d \leq \frac{1}{2v}\left(-5v + 2 + \sqrt{21v^2 - 20v + 4 + 4v^3 + 8ve^2 + 16ev^2 + 8ev}\right).$$

We can use this inequality to determine a simple relationship between the order of a graph, the average degree $\overline{d}$ and the degree of any single vertex as follows. Firstly, we note that

$$2 + \sqrt{21v^2 - 20v + 4 + 4v^3 + 8ve^2 + 16ev^2 + 8ev} < (2v + 4e + 6)\sqrt{v}$$

so for each degree $d$ in $G$

$$\begin{aligned} d &< \frac{1}{2v}(-5v + (2v + 4e + 6)\sqrt{v}) \\ &= \overline{d}\sqrt{v} + \sqrt{v} - (\frac{5}{2} - \frac{3}{\sqrt{v}}) \\ &< \overline{d}\sqrt{v} + \sqrt{v} - 1, \end{aligned}$$

for $v > 4$. So we have:

$$\frac{d+1}{\overline{d}+1} < \sqrt{v},$$

for $v > 4$. If $v \leq 4$ it is relatively trivial to determine whether a graph has a VMTL.

Since this inequality is true for any vertex degree $d$, it is true for the maximum degree $\Delta$ and thus we have proved the following theorem:

**Theorem 5.1.** *Let $V$ be any graph of order $v > 4$ possessing a VMTL. If $\Delta$ is the maximum degree and $\overline{d}$ is the average degree in $G$, then*

$$\frac{\Delta+1}{\overline{d}+1} < \sqrt{v}.$$

Take as a simple example the cycle $C_v$ with $s$ new pendant vertices adjoined to a single vertex. Then according to the theorem the resulting graph will not have a VMTL if $s \geq 3\sqrt{v} - 3$.

### 5.2 Constraint on the number of independent vertices

The following theorem demonstrates a second way in which we can quantify the kind of relationship proposed in the Problem.

**Theorem 5.2.** *A graph $G(v, e)$ with average vertex degree $\overline{d}$ and an independent set of $m$ vertices with degree sum $s$ does not have a VMTL if $\overline{d} \geq 2\overline{s}+2-\frac{1}{v}(s+\frac{1}{2}(m+2))$ (where $\overline{s} = \frac{s}{m}$).*

*Proof.* If we assign the largest labels to the independent set of vertices and their incident edges then the sum of these labels divided by $m$ must be greater than or equal to the smallest possible magic constant $k$, otherwise $G$ could not have a VMTL. Hence for $G$ to have a VMTL we would require:

$$\frac{s+m}{2m}((v+e)+(v+e+1-s-m)) \geq \frac{1}{v}(\frac{v}{2}(v+2e+1)+e(e+1))$$

or rearranging the inequality,

$$\frac{1}{2m}(sv-m+\sqrt{(s^2+4ms+2m^2)v^2-2v(ms^2+2sm^2+m^3)+m^2}) \geq e.$$

But

$$(\frac{s}{m}+1)v-\frac{s}{2}-\frac{m+2}{4} > \frac{1}{2m}(sv-m+\sqrt{(s^2+4ms+2m^2)v^2-2v(ms^2+2sm^2+m^3)+m^2}).$$

So $G$ cannot have a VMTL if $e \geq (\frac{s}{m}+1)v - \frac{1}{2}s - \frac{1}{4}(m+2)$. Since $\overline{d} = \frac{2e}{v}$, $G$ cannot have a VMTL if $\overline{d} \geq 2\overline{s}+2-\frac{1}{v}(s+\frac{1}{2}(m+2))$ as required. □

**Corollary 5.1.** *A graph $G(v, e)$ with average vertex degree $\overline{d}$ and an independent set of $m$ vertices with degree sum $s$ does not have a VMTL if $\overline{d} \geq 2\overline{s}+2$.*

This theorem suggests that for a graph to possess a VMTL its smallest vertex degrees cannot deviate too widely from the average degree of the entire graph.

As a simple example, consider a graph $G(v, \frac{dv}{2})$, i.e. of average degree $d$. If we add a single leaf to any vertex we obtain $G^*(v+1, \frac{dv+2}{2})$ and if $d \geq \frac{2}{v}+4$ then $G^*$ will not possess a VMTL. Thus, if we add a leaf to any $(4+r)$-regular graph with $r \geq 1$ then the resulting graph cannot possess a VMTL.

### 5.3 VMTLs and degree sequences

It might be reasonable to expect that if two graphs have the same degree sequence then either they would both possess VMTLs or neither of them would. However, as our example shows, degree sequence alone does not determine the existence or otherwise of a VMTL. Consider the graphs shown in Figure 5.3 where the numbers identify the graphs in [13].

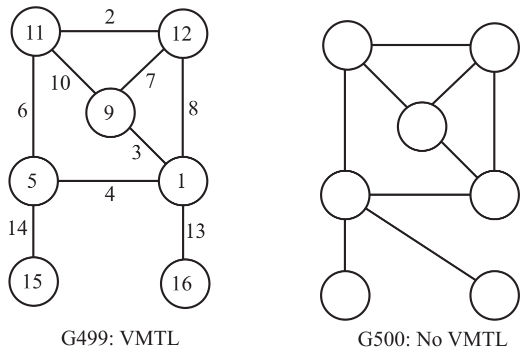


Figure 3: Two graphs with the same degree sequence

To see why the second graph does not possess a VMTL, let
$\Sigma_v$ be the sum of the labels of the vertex of degree 4 and its two non-stem edges,
$\Sigma_s$ be the sum of the labels on the stems and
$\Sigma_l$ be the sum of the labels on the leaves.
Then
$\Sigma_v + \Sigma_s = k$ and
$\Sigma_s + \Sigma_l = 2k$
But $\min(\Sigma_v) \geq 6$, hence $\Sigma_s \leq k - 6$ and $\Sigma_l \geq k + 6$. Since $\max(\Sigma_l) = 31$, $k \leq 25$. However

$$\min(k) \geq \frac{1}{7}(9(10) + \frac{7}{2}(10 + 16)) = 25.857 > 25,$$

so the graph cannot have a VMTL.

This analysis indicates that the existence of a VMTL depends on the structure of the graph, not just its degree sequence. This seems to say that we will never have a completely definitive answer to our *Problem*. How close we can come will be most interesting.

## Acknowledgements

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