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Existence of $(K_4 - e)$-GDDs of type 2^n

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Abstract

Suppose K_v is the complete undirected graph with v vertices. Let $K_4 - e$ be the graph obtained from K_4 by removing one edge. A $(K_4 - e)$ group divisible design (briefly $(K_4 - e)$-GDD) of type h^n is a triple $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a partition of X into groups each of size h, $\mathcal{B}$ is an edge-disjoint decomposition of the edge set of $K_{h,h,\ldots,h}$, the multipartite complete undirected graph with $\mathcal{G}$ as the partition of the vertex set X, into copies of $K_4 - e$. It is proved in this paper that there exists a $(K_4 - e)$-GDD of type 2^n if and only if $n \equiv 0, 1 \pmod 5$ and $n \geq 5$. Consequently, $K_{2n} - F$ has a $(K_4 - e)$-decomposition if and only if $n \equiv 0, 1 \pmod 5$ and $n \geq 5$, where F is a one-factor of K_{2n}.

1 Introduction

Let K_v be the complete undirected graph with v vertices. Let $K_4 - e$ be the *graph* obtained from K_4 on the vertex set $\{a, b, c, d\}$ by removing one edge. We shall use $\{a, b, c; d\}$ to denote the $K_4 - e$ on the vertex set $\{a, b, c, d\}$ missing the edge $\{c, d\}$. The reader may refer to [6] for more details about graph theory.

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties: (i) X is a finite set of points, (ii) $\mathcal{G}$ is a partition of X into subsets called groups, (iii) $\mathcal{B}$ is a set of subsets (called *blocks*) of X, such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly one block.

The type of a *GDD* is the multset $\{|G|, G \in \mathcal{G}\}$. We shall use an "exponential" notation to describe types: so type $g_1^{u_1} g_2^{u_2} \ldots g_k^{u_k}$ denotes u_i occurrences of g_i, $1 \leq i \leq k$, in the multiset.

Let K-GDD denote a *group divisible design* with block sizes from a set of integers K. If $K = \{k\}$, then a $\{k\}$-GDD of type n^k is called a *transversal design* and is denoted by TD(k, n). It is well-known that the existence of a TD(k, n) is equivalent to the existence of $k - 2$ *mutually orthogonal Latin squares* (MOLS) of order n. A

$\{k\}$-GDD of type 1^v is called a *balanced incomplete block design*, denoted here by BIBD$(k,1;v)$. We will make use of the following results.

Lemma 1.1 [3] *(1) There exists a TD$(5,n)$ for every integer $n \geq 4$ except for $n = 6$ and possibly for $n = 10$.*
(2) There exists a TD$(6,n)$ for every integer $n \geq 5$ except for $n = 6$ and possibly for $n = 10, 14, 18, 22$.

Lemma 1.2 [1, 4, 5] *(1) There exists a BIBD$(5,1;v)$ if and only if $v \equiv 1, 5 \pmod{20}$ and $v \geq 5$.*
(2) There exists a BIBD$(6,1;v)$ if and only if $v \equiv 1, 6 \pmod{15}$ and $v \geq 6$ except for $v = 16, 21, 36$ and possibly for $v = 51, 61, 81, 166, 226, 231, 256, 261, 286, 316,$ 321,346, $351, 376, 406, 411, 436, 441$,471, $501, 561, 591, 616, 646, 651, 676, 771, 796, 801$.

A $(K_4 - e)$ *group divisible design* (denoted by $(K_4 - e)$-GDD) of type h^n is a triple $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a partition of X into groups (*holes*) of size h each, $\mathcal{B}$ is an edge-disjoint decomposition of the edge set of $K_{h,h,\ldots,h}$ (the multipartite complete undirected graph with $\mathcal{G}$ as the partition of the vertex set X) into copies of (*blocks*) $K_4 - e$.

A $(K_4 - e)$-design of order n is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is an edge-disjoint decomposition of the edge set of K_n with vertex set X, into copies of (*blocks*) $K_4 - e$. It is well-known (see Bermond and Schönheim [2] for example) that a $(K_4 - e)$-design of order n exists for all $n \equiv 0, 1 \pmod 5$ and $n \geq 6$. It is easy to see that a (K_4-e)-GDD of type 1^n is precisely a $(K_4 - e)$-design of order n.

Obviously we have the following necessary conditions for the existence of a (K_4-e) group divisible design of type h^n.

Lemma 1.3 *If there exists a (K_4-e) group divisible design of type h^n, then $hn(hn-h) \equiv 0 \pmod 5$ and $n \geq 3$.*

By Lemma 1.3, the necessary conditions for the existence of a $(K_4 - e)$-GDD of type 2^n are $n \equiv 0, 1 \pmod 5$ and $n \geq 5$. In this paper, we shall use direct and recursive constructions to show that these necessary conditions are also sufficient.

2 Basic construction techniques

In this section, we will introduce some basic techniques for constructing a $(K_4 - e)$ *group divisible design* of type 2^n. Firstly, we need the following definition. (The reader may refer to [3, 5] for more notation and definitions which are not presented in this paper.) A *pairwise balanced design* (PBD) $B(K,1;v)$ is a pair $(X, \mathcal{B})$, where X is a v-set (of points) and $\mathcal{B}$ is a pair of distinct points of X (*blocks*) with sizes in K such that every pair of distinct points of X is contained in exactly one block of $\mathcal{B}$. If $K = \{k\}$, then a $B(K,1;v)$ is a $BIBD(k,1;v)$. It is well-known that we have the following Constructions 2.1 and 2.2.

Construction 2.1 *(Weighting) Let $(X,\mathcal{G},\mathcal{B})$ be a GDD, and let $w: X \to Z^+\cup\{0\}$ be a weight function on X. Suppose that for each block $B\in\mathcal{B}$, there exists a (K_4-e)-GDD of type $\{w(x): x\in B\}$. Then there is a (K_4-e)-GDD of type $\{\sum_{x\in G} w(x): G\in\mathcal{G}\}$.*

Construction 2.2 *(Breaking up Groups) If there exist a (K_4-e)-GDD of type $2^n(2g)^t$ and a (K_4-e)-GDD of type 2^g, then there is a (K_4-e)-GDD of type 2^{n+gt}.*

Construction 2.3 *If there exists a PBD$(\{5,6\},1;n)$, then there exists a (K_4-e)-GDD of type 2^n.*

Proof: By Construction 2.1, we need only to prove that there exists a (K_4-e)-GDD of type 2^v for $v\in\{5,6\}$. In fact, let $X_1=Z_8\cup\{\infty_i \mid i=0,1\}$, the group set $G_1=\{\{i,4+i\}|0\le i\le 3\}\cup\{\{\infty_0,\infty_1\}\}$, $B_1=\{\{0,1,3;\infty_0\}+i \pmod 8|0\le i\le 7\}$, where $\infty_0+i=\infty_{i(mod2)}$; then (X_1,B_1) is a (K_4-e)-GDD of type 2^5.

Similarly, let $X_2=Z_{12}$, the group set $G_2=\{\{i,6+i\}\mid 0\le i\le 5\}$, $B_2=\{\{0,1,3;5\}+i \pmod{12} \mid 0\le i\le 11\}$; then (X_2,B_2) is a (K_4-e)-GDD of type 2^6. This completes the proof. ■

Construction 2.4 *For every integer $n\ge 2$, there exists a (K_4-e)-GDD of type $2^n(4n-4)^1$.*

Proof: Let the vertex set be $Z_{2n}\cup\{\infty_i \mid 0\le i\le 4n-5\}$. Let $F_0,F_1,\ldots,F_{2n-2}$ be the 1-factorization of the complete graph K_{2n} on the vertex set Z_{2n}. Suppose that $F_i=\{\{a^i_{2j},a^i_{2j+1}\}\mid 0\le j\le n-1\}$ for $0\le i\le 2n-2$. The group set $G=F_{2n-2}\cup\{\{\infty_i \mid 0\le i\le 4n-5\}\}$, and $B=\cup_{i=0}^{2n-3}\{\{a^i_{2j},a^i_{2j+1},\infty_{2i};\infty_{2i+1}\}\mid 0\le j\le n-1\}$. It is readily verified that (X,B) is the required (K_4-e)-GDD. This completes the proof. ■

By Constructions 2.1 and 2.2 (breaking up groups), we have the following Construction.

Construction 2.5 *Suppose that there exists a TD$(6,m)$. Let $0\le t\le m$. Then there exists a (K_4-e)-GDD of type $(2m)^5(2t)^1$. Moreover, if there exist a (K_4-e)-GDD of type 2^m and a (K_4-e)-GDD of type 2^t, then there exists a (K_4-e)-GDD of type 2^{5m+t}.*

3 Existence of (K_4-e)-GDDs of type 2^n

In this section, we will investigate the existence of a (K_4-e)-GDD of type 2^n. First, we have the following results.

Lemma 3.1 *For every $n\in\{5,6,21,25,31,41,45,46,61,65,66,76,81,85,91,96,101,125,126,151\}$, there exists a (K_4-e)-GDD of type 2^n.*

Proof: For $n = 5, 6$, the desired $(K_4 - e)$-GDDs can be found in the proof of Construction 2.3.

For every $n \in \{31, 46, 66, 76, 91, 96, 126, 151\}$, there exists a BIBD$(6, 1; n)$ by Lemma 1.2. Then there exists a $(K_4 - e)$-GDD of type 2^n by Construction 2.3.

For every $n \in \{21, 25, 41, 45, 61, 65, 81, 85, 101, 125\}$, there exists a BIBD$(5, 1; n)$ by Lemma 1.2. Then there exists a $(K_4 - e)$-GDD of type 2^n by Construction 2.3. This completes the proof. ■

Lemma 3.2 *If* $n \in \{11, 36, 51, 56, 86\}$*, then there exists a* $(K_4 - e)$*-GDD of type* 2^n*.*

Proof: For each $n \in \{11, 36, 51, 56, 86\}$, let the vertex set be Z_{2n}, the group set $G = \{\{0, n + i\} \mid 0 \le i \le n - 1\}$. The desired design can be obtained by adding 1 (modulo $2n$) successively to the following base blocks.

$n = 11 : \{0, 1, 3; 7\}, \{0, 4, 9; 12\}$.

$n = 36 : \{27, 39, 0; 11\}, \{0, 9, 19; 24\}, \{6, 24, 56; 55\}, \{0, 17, 38; 42\}, \{0, 1, 3; 5\}, \{0, 35, 43; 46\}, \{0, 6, 13; 20\}$.

$n = 51 : \{57, 86, 32; 33\}, \{0, 6, 13; 43\}, \{49, 64, 68; 85\}, \{0, 12, 28; 42\}, \{65, 66, 68; 55\}, \{0, 23, 45; 69\}, \{4, 54, 45; 62\}, \{0, 32, 5; 63\}, \{0, 14, 34; 40\}, \{0, 17, 35; 55\}$.

$n = 56 : \{36, 103, 59; 111\}, \{43, 75, 10; 13\}, \{63, 65, 58; 79\}, \{0, 1, 4; 13\}, \{39, 81, 15; 54\}, \{0, 9, 20; 19\}, \{31, 69, 9; 10\}, \{0, 17, 35; 58\}, \{6, 31, 67; 79\}, \{0, 55, 81; 83\}, \{0, 6, 40; 49\}$.

$n = 86 : \{4, 105, 121; 32\}, \{52, 108, 0; 161\}, \{10, 12, 17; 18\}, \{0, 1, 4; 159\}, \{111, 138, 126; 87\}, \{0, 9, 19; 161\}, \{18, 63, 40; 109\}, \{0, 17, 35; 151\}, \{19, 45, 113; 117\}, \{0, 54, 138; 29\}, \{54, 90, 157; 147\}, \{0, 30, 61; 62\}, \{112, 154, 23; 25\}, \{0, 37, 76; 77\}, \{82, 129, 32; 17\}, \{0, 58, 102; 106\}, \{0, 33, 82; 92\}$. ■

Lemma 3.3 *There exists a* $(K_4 - e)$*-GDD of type* 2^n *for every* $n \in \{15, 20, 35, 40, 50, 60, 70, 95\}$*.*

Proof: For each $n \in \{15, 20, 35, 40, 50, 60, 70, 95\}$, let the vertex set be $Z_{2n-2} \cup \{\infty_0, \infty_1\}$, and the group set $G = \{\{0, n+i-1\} \mid 0 \le i \le n-2\} \cup \{\{\infty_0, \infty_1\}\}$. The desired design can be obtained by adding 1 (modulo $2n$) successively to the following base blocks, where $\infty_0 + i = \infty_{i (mod 2)}$.

$n = 15 : \{0, 1, 7; \infty_0\}, \{0, 5, 8; 9\}, \{0, 2, 12; 13\}$.

$n = 20 : \{0, 1, 16; \infty_0\}, \{0, 10, 3; 8\}, \{0, 5, 11; 17\}, \{0, 4, 13; 18\}$.

$n = 35 : \{0, 1, 16; \infty_0\}, \{17, 64, 42; 4\}, \{0, 7, 17; 18\}, \{16, 30, 61; 60\}, \{0, 9, 28; 42\}, \{0, 2, 5; 6\}, \{0, 12, 32; 39\}$.

$n = 40 : \{0, 1, 16; \infty_0\}, \{9, 20, 2; 30\}, \{0, 2, 5; 74\}, \{22, 35, 72; 70\}, \{0, 8, 20; 46\}, \{5, 58, 24; 29\}, \{0, 33, 47; 55\}, \{0, 9, 26; 36\}$.

$n = 50 : \{0, 1, 16; \infty_0\}, \{65, 69, 86; 47\}, \{0, 2, 5; 9\}, \{33, 94, 81; 80\}, \{0, 8, 20; 31\}$,

$\{38, 57, 13; 97\}, \{0, 28, 60; 74\}, \{14, 43, 85; 8\}, \{0, 10, 36; 43\}, \{0, 11, 41; 45\}.$

$n = 60 : \{0, 1, 16; \infty_0\}, \{3, 64, 10; 75\}, \{0, 2, 5; 114\}, \{23, 32, 51; 9\},$
$\{0, 8, 21; 106\}, \{46, 68, 93; 2\}, \{0, 26, 53; 55\}, \{33, 63, 106; 95\},$
$\{0, 31, 70; 69\}, \{28, 68, 104; 105\}, \{0, 100, 17; 33\}, \{0, 10, 34; 60\}.$

$n = 70 : \{0, 1, 16; \infty_0\}, \{6, 58, 48; 47\}, \{0, 57, 17; 18\}, \{48, 126, 55; 71\},$
$\{0, 2, 5; 134\}, \{93, 102, 25; 74\}, \{0, 8, 20; 125\}, \{9, 36, 65; 62\},$
$\{0, 22, 46; 113\}, \{3, 110, 75; 37\}, \{0, 30, 63; 62\}, \{2, 38, 86; 76\},$
$\{0, 45, 59; 89\}, \{0, 37, 80; 87\}.$

$n = 95 : \{0, 1, 16; \infty_0\}, \{39, 175, 51; 126\}, \{0, 42, 123; 86\}, \{90, 149, 32; 79\},$
$\{0, 45, 93; 96\}, \{36, 38, 42; 33\}, \{0, 54, 111; 110\}, \{153, 166, 184; 174\},$
$\{0, 7, 17; 75\}, \{102, 124, 77; 148\}, \{0, 9, 23; 169\}, \{108, 138, 170; 11\},$
$\{0, 26, 53; 55\}, \{74, 107, 173; 174\}, \{0, 50, 154; 85\}, \{71, 110, 151; 31\},$
$\{0, 43, 63; 115\}, \{0, 69, 105; 106\}, \{0, 38, 98; 112\}.$ ■

Lemma 3.4 *There exists a $(K_4 - e)$-GDD of type 2^n for every $n \in \{10, 16, 26, 30, 55, 75, 80, 90, 100\}$.*

Proof: For $n = 10, 16$, taking $(K_4 - e)$-GDDs of types $2^4 12^1$ and $2^6 20^1$ respectively (for existence of these, see Construction 2.4), breaking up groups, we then obtain the desired designs. For $n = 30, 55, 75, 80, 90, 100$, taking a TD(6, 5), a TD(5, 11), a TD(5, 15), a TD(5, 16), a TD(6, 15) and a TD(5, 20) respectively, giving weight 2, we then obtain $(K_4 - e)$-GDDs of types 10^6, 22^5, 30^5, 32^5, 30^6 and 40^5, respectively. Breaking up groups, we then obtain the desired $(K_4 - e)$-GDDs. For $n = 26$, delete 4 points in one group of a TD(6, 5) to obtain a $\{5, 6\}$-GDD of type $5^5 6^1$. Give all the points of the GDD weight 2; then we obtain a $(K_4 - e)$-GDD of type $10^5 2^1$. The desired design can be obtained by using Construction 2.2. This completes the proof. ■

Now we are in a position to prove our main results.

Theorem 3.5 *There exists a $(K_4 - e)$ group divisible design of type 2^n if and only if $n \equiv 0, 1 \pmod 5$ and $n \geq 5$.*

Proof: The necessity follows from Lemma 1.3. The sufficiency can be proved by induction on n. For each $n \in T = \{t \mid t \equiv 0, 1 \pmod 5,\ 5 \leq t \leq 101\} \cup \{125, 126, 151\}$, there exists a $(K_4 - e)$-GDD of type 2^n by Lemmas 3.1–3.4. When $n \notin T$ and $n \equiv 0, 1 \pmod 5$, we apply Construction 2.5. We list the suitable parameters such that $n = 5m + t$ below.

$$
\begin{array}{lll}
n = 25k = 5m + t & : m = 5(k-1), k \geq 6, & t = 25; \\
n = 25k + 1 = 5m + t & : m = 5(k-1), k \geq 7, & t = 26; \\
n = 25k + 5 = 5m + t & : m = 5k, k \geq 3, & t = 5; \\
n = 25k + 6 = 5m + t & : m = 5k, k \geq 3, & t = 6; \\
n = 25k + 10 = 5m + t & : m = 5k + 1, k \geq 2, & t = 5; \\
n = 25k + 11 = 5m + t & : m = 5k + 1, k \geq 2, & t = 6; \\
n = 25k + 15 = 5m + t & : m = 5k + 1, k \geq 2, & t = 10; \\
n = 25k + 16 = 5m + t & : m = 5k + 1, k \geq 3, & t = 11; \\
n = 25k + 20 = 5m + t & : m = 5k + 1, k \geq 3, & t = 15; \\
n = 25k + 21 = 5m + t & : m = 5k + 1, k \geq 3, & t = 16.
\end{array}
$$

This completes the proof. ■

The existence of a $(K_4 - e)$ group divisible design of type 2^n is equivalent to the existence of a $(K_4 - e)$-decomposition of $K_{2n} - F$, where F is a one-factor of K_{2n}. So we have the following corollary.

Corollary 3.6 *Let F be a one-factor of K_{2n}. Then $K_{2n} - F$ has a $(K_4 - e)$-decomposition if and only if $n \equiv 0, 1 \pmod 5$ and $n \geq 5$.*

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