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Open neighborhood locating-dominating sets

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Abstract

For a graph G that models a facility, various detection devices can be placed at the vertices so as to identify the location of an intruder such as a thief or saboteur. Here we introduce the open neighborhood locating-dominating set problem. This deals with problems in which the intruder at a vertex can interfere with the detection device located there. We seek a minimum cardinality vertex set S with the property that for each vertex v its open neighborhood $N(v)$ has a unique non-empty intersection with S. Such a set is an *OLD-set* for G. Among other things, we describe minimum density *OLD-sets* for various infinite grid graphs.

1 Introduction

For studies involving safeguards applications for graphical models of facilities or for multiprocessor networks, various types of protection sets have been studied where the objective is to precisely locate an "intruder" such as a thief, saboteur, or fire, or a faulty processor. It is generally assumed that a detection device located at a vertex v in a graph $G = (V, E)$ modeling the system can detect an intruder only

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if the intruder is at v or a vertex location adjacent to v. The open neighborhood of v is $N(v) = \{w \in V(G) : vw \in E(G)\}$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$.

Locating sets (for which a protection device can determine the distance to an intruder) were introduced in Slater[14] and subsequently by Harary and Melter[8] where they were called metric bases. The concepts of locating and dominating were merged in [16, 17]. See also [3, 5, 6, 13, 18]. When a protection device at vertex v can distinguish between there being an intruder at v or at a vertex in $N(v)$, but which vertex in $N(v)$ can not be pinpointed, then one is interested in having a *locating-dominating set*. When only the presence of an intruder in $N[v]$ can be detected, with no information as to which vertex in $N[v]$ contains the intruder, one is interested in *identifying codes* as introduced by Karpovsky, Chakrabarty and Levitin [11]. Haynes, Henning and Howard [9] add the condition that the locating-dominating set or identifying code S not have any isolated vertices. See also [1, 2, 4]. A bibliography of papers concerned with locating-dominating sets and identifying codes, currently listing over 150 papers, is maintained by Lobstein [12].

In this paper we consider situations in which an intruder at a location v can prevent the detection device at v from detecting the fault. That is, a detection device at a vertex v can only determine the presence of an intruder in $N(v)$. A vertex set $S \subseteq V(G)$ is an *open-locating-dominating set*, an *OLD-set* for G, if and only if for each vertex $w \in V(G)$ there is at least one vertex v in $S \cap N(w)$ (that is, S is an open-dominating set) and for any pair of distinct vertices w and x we have $N(w) \cap S \neq N(x) \cap S$. The *open-locating-dominating number* $OLD(G)$ is the minimum cardinality of an *OLD-set* for G. An *OLD-set* for G of order $OLD(G)$ will be called an *OLD(G)-set*.

For the graph H in Figure 1, let $S = \{v_1, v_2, v_3\}$. Then $N(v_1) \cap S = \{v_2, v_3\}$, $N(v_2) \cap S = \{v_1, v_3\}, N(v_3) \cap S = \{v_1, v_2\}, N(v_4) \cap S = \{v_3\}$, and $N(v_5) \cap S = \{v_1\}$. Because all five sets are distinct, S is an open-locating-dominating set. In fact, $OLD(H) = |S| = 3$.

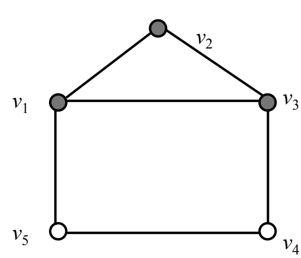


Figure 1: $OLD(H) = |\{v_1, v_2, v_3\}| = 3$

In general, call a collection $\Re = \{S_1, S_2, \ldots, S_t\}$ with $S_i \subseteq V(G)$ *distinguishing* if for each $w \in V(G)$ some $S_i \in \Re$ contains w and for each pair of distinct vertices w and x we have $\{S_i : w \in S_i\} \neq \{S_i : x \in S_i\}$ (that is, some S_i contains exactly one of w and x). Thus $\Re_i \subseteq V(G)$ is an identifying code if $\Re_1 = \{N[v] : v \in S\}$ is distinguishing, a locating-dominating set if $\Re_2 = \{N(v) : v \in S\} \cup \{\{v\} : v \in S\}$ is distinguishing, and an open-locating-dominating set if $\Re_3 = \{N(v) : v \in S\}$ is distinguishing.

In the context of coding theory, Honkala, Laihonen, and Ranto [10] consider various types of codes for F_2^n with $F_2 = \{0, 1\}$, that is, for the *n-cube* Q_n. What they called "identifying codes with non-transmitting faulty vertices" is what we would call an $OLD(Q_n)$*-set*.

Clearly every graph has a locating-dominating set, but if $w \neq x$ and $N[w] = N[x]$ then G does not have an identifying code.

Observation 1. *A graph G has an open-locating-dominating set if and only if the minimum degree of G satisfies $\delta(G) \geq 1$ and, whenever $w \neq x$, we have $N(w) \neq N(x)$.*

Observation 2. *If $OLD(G) = h$, then $|V(G)| \leq 2^h - 1$.*

Proof. This follows from the fact that for each vertex v we have a distinct non-empty subset $N(v) \cap S$ for an *OLD-set* S. ■

Let C_h be the cycle on $h \geq 5$ vertices, $V(C_h) = \{v_1, v_2, \ldots, v_h\}$ and $E(C_h) = \{v_1v_2, v_2v_3, \ldots, v_{h-1}v_h, v_hv_1\}$. Note that $N(v_i) = \{v_{i-1}, v_{i+1}\}$ (mod h), and the open neighborhoods $N(v_1), \ldots, N(v_h)$ are distinct. We can add $2^h - 1 - h$ additional vertices, each of which has a distinct open neighborhood in $V(C_h)$, to obtain a graph G_h with $|V(G_h)| = 2^h - 1$ and $OLD(G_h) = h$.

As noted in Garey and Johnson [7], Problem 3-SAT is NP-complete.

3-SAT

INSTANCE: Collection $C = \{c_1, c_2, \ldots, c_M\}$ of clauses on set $U = \{u_1, u_2, \ldots, u_N\}$ such that $|c_i| = 3$ for $1 \leq i \leq M$.

QUESTION: Is there a satisfying truth assignment for C?

Open-Locating-Dominating (OLD)

INSTANCE: Graph $G = (V, E)$ and positive integer $K \leq |V|$.

QUESTION: Is $OLD(G) \leq K$?

Theorem 3. *Problem OLD is NP-complete.*

Proof. Clearly $OLD \in NP$. We show a polynomial time reduction from 3-SAT to *OLD*. Given U and C, for each u_i construct the graph G_i on 21 vertices shown in Figure 2(a), and for each clause c_j construct the graph H_j shown in Figure 2(b).

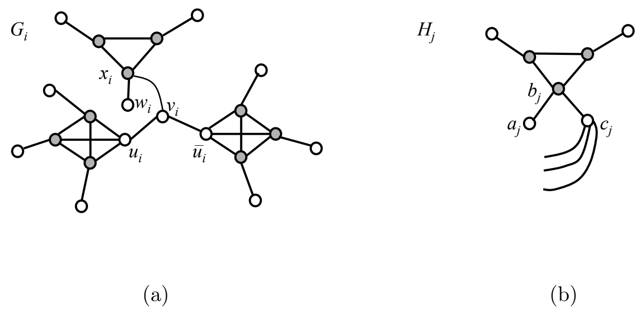


Figure 2: Variable and clause graphs G_i and H_j, $1 \leq i \leq N$, $1 \leq j \leq M$

Finally, to complete the construction of graph G, for $1 \leq j \leq M$ if clause $C_j = \{u_{j,1}, u_{j,2}, u_{j,3}\}$ where each $u_{j,t}$ is some u_i or $\bar{u}_i$, let clause vertex c_j be adjacent to variable vertices $u_{j,1}, u_{j,2}$ and $u_{j,3}$. Note that G has $21N+7M$ vertices and $27N+10M$ edges and can be constructed from C in polynomial time.

If $S \subseteq V(G)$ is any *OLD(G)-set*, then S must contain every vertex adjacent to an endpoint because S is open-dominating. That is, S contains the $9N + 3M$ darkened vertices of Figure 2. Because $N(w_i) \cap S = \{x_i\} \neq N(v_i) \cap S$, it must be the case that $(N(v_i) \cap S) \cap \{u_i, \bar{u}_i\} \neq \emptyset$. Likewise, $N(a_j) \cap S = \{b_j\} \neq N(c_j) \cap S$, so $S \cap \{u_{j,1}, u_{j,2}, u_{j,3}\} \neq \emptyset$.

It is now easy to see that C has a satisfying truth assignment if and only if $OLD(G) = (9N + 3M) + N = 10N + 3M$. ■

2 Trees

Throughout this section we let T_n be a tree of order $|V(T_n)| = n \geq 2$. Letting $\Im$ be the collection of all trees T_n with $n \geq 2$ in which no two endpoints have the same neighbor, it is easy to see that $OLD(T_n)$ is defined if and only if $T_n \in \Im$.

First, we consider paths P_n on n vertices. We have $OLD(P_2) = 2$ and $P_3 \notin \Im$. Suppose $n \geq 4, V(P_n) = \{v_1, v_2, \ldots, v_n\}$ with $v_i, v_{i+1} \in E(P_n)$ for $1 \leq i \leq n-1$.

Let D be an OLD-set for P_n. If $v_i \notin D$ then $N(v_{i+1}) \cap D \neq \emptyset$ implies that $v_{i+2} \in D$. Now $N(v_{i+1}) \cap D = \{v_{i+2}\} \neq N(v_{i+3}) \cap D$ implies that $N(v_{i+3}) \cap D = \{v_{i+2}, v_{i+4}\}$. That is, if $v_i \notin D$ then $(\{v_{i-4}, v_{i-2}, v_{i+2}, v_{i+4}\} \cap V(P_n)) \subseteq D$. Note, in particular, that $\{v_2, v_4, v_{n-1}, v_{n-3}\} \subseteq D$. If $v_1 \notin D$, then $\{v_2, v_3, v_4, v_5\} \subseteq D$; if $v_3 \notin D$ then $\{v_1, v_2, v_4, v_5\} \subseteq D$; and if $v_5 \notin D$ then $\{v_1, v_2, v_3, v_4\} \subseteq D$. Also, if $\{v_1, v_2, v_3, v_4, v_5\} \subseteq D$, then $D - v_1$, would also be an OLD-set. We thus have the following proposition.

Proposition 4. *If D is an OLD-set for P_n for $n \geq 5$, then*

(i) $|D \cap \{v_1, v_2, v_3, v_4, v_5\}| = 4$ *and exactly one of* v_1, v_3, v_5 *is not in* D*;*

(ii) *if* $D \cap \{v_i, v_{i+1}\} = \emptyset$, *then* $5 \le i \le n-5$ *and*

$$\{v_{i-4}, v_{i-3}, v_{i-2}, v_{i-1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\} \subseteq D; \quad \textit{and}$$

(iii) *if* $D \cap \{v_{i-1}, v_i, v_{i+1}\} = \{v_{i-1}, v_{i+1}\}$ *then* $3 \le i \le n-2$ *and*

$$\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \subseteq D.$$

Lemma 5. *If* D *is an OLD-set for* P_n *with* $n \ge 6$, *then*

$$|\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\} \cap D| \ge 4 \textit{ for } \ 1 \le i \le n-5.$$

Proof. If any two consecutive vertices in $\{v_i, v_{i+1}, \ldots, v_{i+5}\}$ are not in D, then the other four are in D by Proposition 4(ii). Thus each pair $\{v_i, v_{i+1}\}, \{v_{i+2}, v_{i+3}\}$, $\{v_{i+4}, v_{i+5}\}$ can be assumed to have at least one element in D, and if D contains both elements in one pair then there is a total of at least four in D. Without loss of generality, assume $v_{i+2} \in D$ and $v_{i+3} \notin D$. But now $v_{i+4} \in D$ and $D \cap \{v_{i+2}, v_{i+3}, v_{i+4}\} = \{v_{i+2}, v_{i+4}\}$ implies that $\{v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}\} \subseteq D$. ■

Lemma 6. $OLD(P_n) \ge \lceil (2/3)n \rceil$.

Proof. Let D be an $OLD(P_n)$-set. We will associate two vertices in D with each $v_i \notin D$. If $v_1 \notin D$, then $\{v_2, v_3, v_4, v_5\} \subseteq D$ by Proposition 4 (i), so we associate v_2 and v_3 with v_1. Similarly, if $v_n \notin D$ we associate v_{n-1} and v_{n-2} with v_n. For $3 \le i \le n-2$, if $D \cap \{v_{i-1}, v_i, v_{i+1}\} = \{v_{i-1}, v_{i+1}\}$ then we can associate v_{i+1} and v_{i+2} with v_i by Proposition 4 (iii), and if $\{v_i, v_{i+1}\} \cap D = \emptyset$ we can associate v_{i-1} and v_{i-2} with v_i and v_{i+2} and v_{i+3} with v_{i+1}. ■

Proposition 7. $OLD(P_{6k}) = 4k$, *and the unique* $OLD(P_{6k})$*-set is*
$$\{v_2, v_3, v_4, v_5, v_8, v_9, v_{10}, v_{11}, \ldots, v_{6k-4}, v_{6k-3}, v_{6k-2}, v_{6k-1}\} = S.$$

Proof. By Lemma 6, $OLD(P_{6k}) \ge 4k$, and the given set S is clearly an OLD-set for P_{6k}, so $OLD(P_{6k}) = 4k$. Since any OLD-set for P_n contains $\{v_2, v_4, v_{n-3}, v_{n-1}\}$, the unique $OLD(P_6)$-set is $\{v_2, v_3, v_4, v_5\}$. We complete the proof by induction on k. Assume $k \ge 2$. Note that any $OLD(P_{6k})$-set D must have $|D \cap \{v_{6j+1}, v_{6j+2}, \ldots, v_{6j+6}\}| = 4$ for $0 \le j \le k-1$. Because exactly one of v_1, v_3, v_5 is not in D, we have $v_6 \notin D$. Thus $D \cap \{v_7, v_8, \ldots, v_{6k}\}$ is an $OLD(P_{6k-6})$-set and, by induction, $D \cap \{v_7, v_8, \ldots, v_{6k}\}$ must be $\{v_8, v_9, v_{10}, v_{11}, v_{14}, v_{15}, v_{16}, v_{17}, \ldots, v_{6k-4}, v_{6k-3}, v_{6k-2}, v_{6k-1}\}$. Now $D \cap \{v_6, v_7\} = \emptyset$ implies that $\{v_2, v_3, v_4, v_5\} \subseteq D$ by Proposition 4(ii), and so $D = S$ as required. ■

Proposition 8. $OLD(P_{6k+3}) = 4k+3$ *for* $k \ge 1$.

Proof. One can easily verify that $\{v_2, v_3, \ldots, v_8\}$ is an $OLD(P_9)$-set and $OLD(P_9) = 7$. We proceed by induction on k. First, $\{v_2, v_3, \ldots, v_8, v_{11}, v_{12}, v_{13}, v_{14}, v_{17}, v_{18}, v_{19},$

$v_{20}, \ldots, v_{6k-1}, v_{6k}, v_{6k+1}, v_{6k+2}\}$ is an OLD-set for P_{6k+3}, so $OLD(P_{6k+3}) \leq 4k+3$. By Lemma 6 we have $OLD(P_{6k+3}) \geq 4k+2$. To see that $OLD(P_{6k+3}) = 4k+3$ we consider the three cases of Proposition 4(i).

Case 1. Assume $v_3 \notin D$. In this case, $\{v_1, v_2\} \subseteq D$, and $D - \{v_1, v_2\}$ is an OLD-set for $P^* = v_4, v_5, \ldots, v_{6k+3} = P_{6k+3} - \{v_1, v_2, v_3\}$ containing $\{v_4, v_5\}$. By Proposition 7, $|D \cap V(P^*)| \geq 4k+1$, and so $OLD(P_{6k+3}) \geq 4k+3$.

Case 2. Assume $v_5 \notin D$. Then $|D \cap \{v_1, v_2, v_3\}| = 3$ and each six consecutive vertices of P^* have at least four vertices in D, which implies that $D \geq 4k+3$.

Case 3. Assume $v_1 \notin D$. If $v_6 \notin D$ then $D^* = D - \{v_2, v_3, v_4, v_5\}$ is an OLD-set for $P^{\#} = v_7, v_8, \ldots, v_{6k+3}$. By induction, $|D^*| \geq 4(k-1)+3$, and so $|D| = |D^*|+4 \geq 4k+3$. If $v_6 \in D$ and $v_7 \notin D$, then by Lemma 6 we have $|D \cap \{v_8, v_9, \ldots, v_{6k+3}\}| \geq \lceil (2/3)(6k-4) \rceil$ and $|D| \geq 5 + \lceil 4k - 8/3 \rceil = 4k+3$. If $\{v_6, v_7\} \subseteq D$, then $D^* = D - \{v_2, v_3\}$ is an OLD-set for P^* containing v_4, so $|D^*| \geq 4k+1$ by Proposition 7 and $|D| \geq 4k+3$. ■

Proposition 9. $OLD(P_{6k+4}) = 4k+4$.

Proof. $\{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_{13}, v_{14}, v_{15}, v_{16}, \ldots, v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4}\}$ is an OLD-set for P_{6k+4}, so $OLD(P_{6k+4}) \leq 4k+4$. Let D be an $OLD(P_{6k+4})$-set. By Proposition 4(i) we have

$$|D \cap \{v_1, v_2, v_3, v_4, v_5\}| = |D \cap \{v_{6k}, v_{6k+1}, v_{6k+2}, v_{6k+3}, v_{6k+4}\}| = 4,$$

and by Lemma 5 we have $|D \cap \{v_6, v_7, \ldots, v_{6k-1}\}| \geq 4k-4$, so $|D| \geq 4k+4$. ■

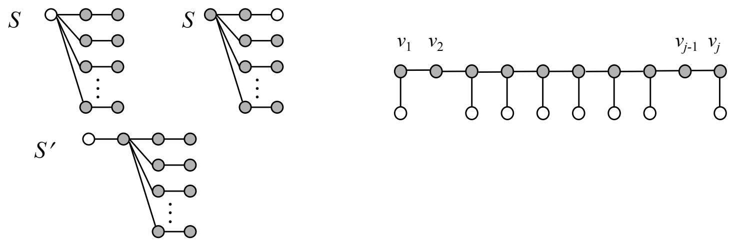
Proposition 10. $OLD(P_{6k+5}) = 4k+4; OLD(P_{6k+2}) = 4k+2;$ *and* $OLD(P_{6k+1}) = 4k+1$.

Proof. By Lemma 6, $OLD(P_{6k+5}) \geq 4k+4, OLD(P_{6k+2}) \geq 4k+2$, and $OLD(P_{6k+1}) \geq 4k+1$. $\{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{6k+1}, v_{6k+2}, v_{6k+4}, v_{6k+5}\}$ is an OLD-set for P_{6k+5}, so $OLD(P_{6k+5}) = 4k+4$.

Now $\{v_1, v_2, v_4, v_5, v_7, v_8, \ldots, v_{6k-2}, v_{6k-1}, v_{6k+1}, v_{6k+2}\}$ is an OLD-set for P_{6k+2} , so $OLD(P_{6k+2}) = 4k+2$. And $\{v_2, v_3, v_4, v_5, v_6, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}, v_{17}, v_{18}, \ldots, v_{6k-3}, v_{6k-2}, v_{6k-1}, v_{6k}\}$ is an OLD-set for P_{6k+1}, so $OLD(P_{6k+1}) = 4k+1$. ■

n	$OLD(P_n)$
$6k$	$4k$
$6k+1$	$4k+1$
$6k+2$	$4k+2$
$6k+3$	$4k+3$
$6k+4$	$4k+4$
$6k+5$	$4k+4$
$6k+6$	$4k+4$

Table 1: Values for $OLD(P_n)$



(a) (b)

Figure 3: (a) $OLD(T_n) = n - 1$ (b) $OLD(T_n) \to 1/2(n)$

In Slater [16] it is shown that the locating-dominating number *LD(G)* satisfies $n/3 < LD(T_n) \leq n - 1$. For the tree $S(K_{1,t-1})$ of order $n = 2t - 1$ obtained by subdividing each edge of the star $K_{1,t-1}$, and for the tree S' obtained from $S(K_{1,t-1})$ by adding an endpoint attached to the center vertex, for $t \geq 3$ we have $OLD(S(K_{1,t-1})) = 2t - 2 = n - 1$ and $OLD(S') = 2t - 1 = n - 1$ as shown in Figure 3(a). For the tree T_{2j-2} of order $n = 2j - 2$ in Figure 3(b) we have $OLD(T_{2j-2}) = j > n/2$, and $\lim_{j\to\infty} OLD(T_{2j-2})/(2j - 2) = 1/2$.

Theorem 11. *Let* $T_n \in \Im$ *with* $n \geq 5$, *then* $\lceil n/2 \rceil + 1 \leq OLD(T_n) \leq n - 1$.

Proof. For the upper bound, if v, w, x, y is a path in T_n with $\deg v = 1$ and $\deg w = \deg x = 2$, or if v, w, x, y, z is a path with $\deg v = \deg z = 1$ and $\deg w = \deg y = 2$, or if v, w, x, y is a path with $\deg v = \deg y = 1$ and $\deg x = 2$ and $\deg w \geq 3$, then $V(T_n) - v$ is an OLD-set for T_n.

For the lower bound, the proof is by induction. One can easily verify that $OLD(T_n) \geq \lceil n/2 \rceil + 1$ for $n \leq 7$. Assume $n \geq 8$, and let S be an $OLD(T_n)$-set. Suppose there exists interior vertex $v \notin S$. If $\deg v \geq 3$, let the components of $T_n - v$ have orders $n_1, n_2, \ldots, n_t$. (Clearly, each $n_i \geq 2$ or else S is not an open-dominating set.) Then $|S| \geq (n_1/2 + 1/2) + (n_2/2 + 1/2) + \ldots + (n_t/2 + 1/2) \geq 1/2(n_1 + n_2 + \ldots + n_t + 1) + 1/2$ by induction. Hence, $OLD(T_n) \geq \lceil n/2 \rceil + 1$. If $\deg v = 2$, let the components of $T_n - v$ have orders n_1 and n_2. If either n_1 or n_2 is even, say n_1 is even, then $|S| \geq (n_1/2 + 1) + (n_2/2 + 1/2) = 1/2(n_1 + n_2 + 1) + 1 = n/2 + 1$, so $OLD(T_n) \geq \lceil n/2 \rceil + 1$. If n_1 and n_2 are both odd, then $n = n_1 + n_2 + 1$ is also odd. Now $|S| \geq (n_1/2 + 1/2) + (n_2/2 + 1/2) = (n_1 + n_2 + 1)/2 + 1/2 = n/2 + 1/2$, so $|S| \geq \lceil n/2 \rceil + 1$.

Suppose every interior vertex of T_n is in S. Since no support vertex is adjacent to two or more endpoints, the number e of endpoints satisfies $e \leq n/2$. If n is odd, we are done. It remains to show that we can not have exactly $n/2$ endpoints, none of which is in S. Assume to the contrary, and let T_n be the corona $T^* \circ K_1$. Since $n \geq 8$, $|V(T^*)| \geq 4$. Let u be an endpoint of T^* adjacent to $v \in V(T^*)$, and let x and y be the endpoints of T_n adjacent to u and v, respectively. Then $N(u) \cap S = N(y) \cap S = \{v\}$, a contradiction. ■

3 Infinite Grids

Percentage parameters for locally-finite, countably infinite graphs were defined in Slater [19]. For the current parameter we have $OLD\%(G) =$ $MIN\{\lim_{k\to\infty} |N_k[v] \cap S|/|N_k\{v\}| : S$ is an OLD-set for $G\}$ where the minimum is taken over all v in $V(G)$, and $N_k[v] = \{w \in V(G) : d(v,w) \leq k\}$ is the set of vertices at distance at most k from v.

Likewise, in Slater[19] the "share" $sh(v; D)$ of vertex v in a dominating set D is defined. Similarly we define the *open-share* $sh^o(v; D)$ of a vertex v in an open-dominating set D as a measure of the amount of domination done by v. For example, in Figure 4, because $N(v_1) \cap \{v_2, v_3\} = \{v_2, v_3\}$, each of the vertices v_2 and v_3 is considered to have a 1/2 share in dominating vertex v_1. Similarly, because $N(v_4) \cap \{v_2, v_3\} = \{v_3\}$, vertex v_3 has a whole share in dominating vertex v_4.

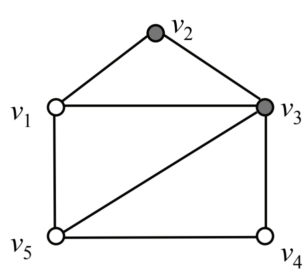
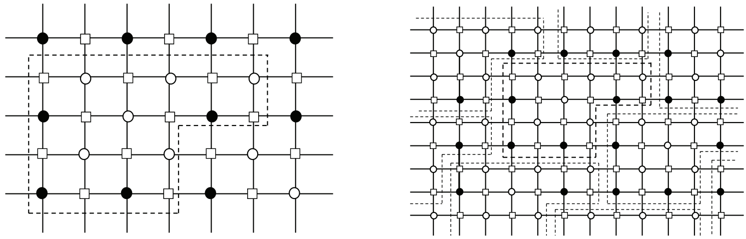


Figure 4: $sh^o(v_2; \{v_2, v_3\}) = 1/2 + 0 + 1 = 3/2$

Here we define the open-share of a vertex v in open-dominating set D to be $sh^o(v; D) = \Sigma_{w\in N(v)} 1/|N(w) \cap D|$. For finite graphs G with an open-dominating set D we have $\Sigma_{v\in D} sh^o(v; D) = |V(G)|$ and $|D| \geq |V(G)|/MAX_{v\in V} sh^o(v; D)$. Further, if D is an open-neighborhood locating-dominating set and $v \in D$, then $sh^o(v; D) \leq 1 + 1/2(\deg(v) - 1)$. Similar to a result in [19] is the next theorem.

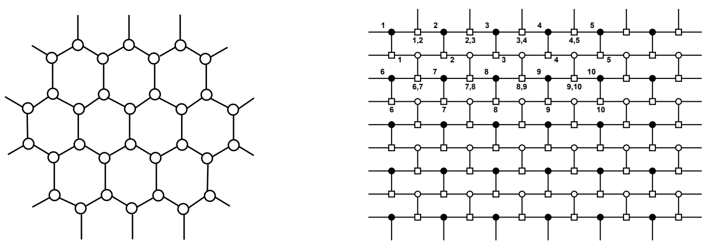
Theorem 12. *If G is regular of degree r, then $OLD(G) \geq (2/(1 + r))|V(G)|$. If a countably infinite graph G is regular of degree r, then $OLD\%(G) \geq 2/(1 + r)$.*

In this section we show that, for the infinite square grid $Z \times Z$ which is regular of degree four, we have $OLD\%(Z \times Z) = 2/5$; for the infinite hexagonal grid HX which is regular of degree three, we have $OLD\%(HX) = 2/4 = 1/2$; and for the infinite triangular TR which is regular of degree six, we have $2/7 \leq OLD\%(TR) \leq 1/3$.



(a) (b)

Figure 5: An $OLD(Z \times Z)$-tiling



(a) (b)

Figure 6: Infinite hexagonal grid HX

For the infinite grid $Z \times Z$ that is regular of degree 4, by Theorem 12, $OLD\%(Z \times Z) \geq 2/(1+4) = 2/5$. To show that $OLD\%(Z \times Z) \leq 2/5$, we observe that $Z \times Z$ is bipartite and partition $V(Z \times Z)$ into two groups, say square vertices and round vertices as shown Figure 5(b). If we use 4/10 of the round vertices by repeating the tile pattern shown in Figures 5(a,b), all of the square vertices are open-dominated and located. Similarly 2/5 of the square vertices can open-dominate and locate the round vertices, so we have $OLD\%(Z \times Z) \leq 2/5$ which results in the following theorem.

Theorem 13. *For infinite square grid $Z \times Z$, $OLD\%(Z \times Z) = 2/(1+4) = 2/5$.*

For the infinite hexagonal grid HX that is regular of degree 3, by Theorem 12, $OLD\%(HX) \geq 2/(1+3) = 1/2$. To show that $OLD\%(HX) \leq 1/2$, we observe that HX is bipartite and partition $V(HX)$ into square vertices and round vertices. The set containing 1/2 of the round vertices, as shown in Figure 6(b), dominates and locates all of the square vertices. Similarly 1/2 of the square vertices can open-dominate and locate the round vertices, so we have $OLD\%(HX) \leq 1/2$, which results in the following theorem.

Theorem 14. *For infinite hexagonal grid HX, $OLD\%(HX) = 2/(1+3) = 1/2$.*

For the infinite triangular grid TR that is regular of degree 6, by Theorem 12, $OLD\%(TR) \geq 2/(1+6) = 2/7$. The set of darkened vertices shown in Figure 7 is an OLD-set and its cardinality is 1/3, so we have $OLD\%(TR) \leq 1/3$ which results in the following theorem. To date, we have not determined the exact value for $OLD\%(TR)$.

Theorem 15. *For the infinite triangular TR which is regular of degree six, we have $2/7 \leq OLD\%(TR) \leq 1/3$.*

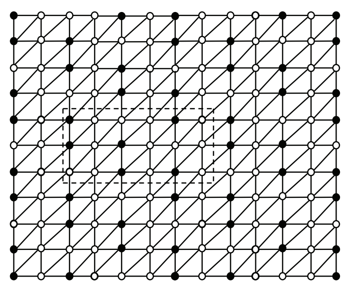


Figure 7: Infinite triangular grid TR

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