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Disjoint triangles and pentagons in a graph

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Abstract

Let n, s, and t be integers with $s \geq 1$, $t \geq 0$, and $n = 3s + 5t$. Let G be a graph of order n such that the minimum degree of G is at least $(n + s + t)/2$. Then G contains $s + t$ independent subgraphs such that s of them are triangles and t of them are pentagons.

1 Introduction

We consider only finite simple graphs and use standard terminology and notation from Bollabás [1] except as indicated. Let G be a graph. A set of graphs is said to be independent if no two of them share a vertex. Determining the maximum number of independent cycles in a graph has been a subject of great interest. Corrádi and Hajnal [2] proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k independent cycles. This result has inspired much research in the theory of independent cycles in graphs. Enomoto and Wang [4, 7] improved this result by showing the same conclusion under the condition $d(x) + d(y) \geq 4k - 1$ for each pair of non-adjacent vertices x and y of G. El-Zahar [3] conjectured the following:

Conjecture 1 *If G is a graph of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3$ $(1 \leq i \leq k)$ and the minimum degree of G is at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$, then G contains k independent cycles of lengths $n_1, n_2, \ldots, n_k$, respectively.*

He proved this conjecture for $k = 2$ although it remains unproven in general. When $n_1 = n_2 = \cdots = n_k = 4$, the conjecture reduces to the conjecture of Erdős and Faudree [5]. Wang added further support to El-Zahar's conjecture when he proved Theorem 2 below. Cycles of lengths 3, 4, and 5 are referred to as triangles, quadrilaterals, and pentagons, respectively. We also refer the reader to [8, 9] for other recent results concerning cycles in graphs.

Theorem 2 [*10*] *Let s and t be two integers with $s \geq 1$ and $t \geq 0$. Let G be a graph of order $n = 3s + 4t$ such that the minimum degree of G is at least $(n + s)/2$. Then G contains $s + t$ independent cycles such that s of them are triangles and t of them are quadrilaterals.*

We continue on this track by proving the following theorem.

Theorem 3 *Let s and t be two integers with $s \geq 1$ and $t \geq 0$. Let G be a graph of order $n = 3s + 5t$ such that the minimum degree of G is at least $(n + s + t)/2$. Then G contains $s + t$ independent cycles such that s of them are triangles and t of them are pentagons.*

This result contributes a partial solution to El-Zahar's conjecture in the case $n_1 = \cdots = n_s = 3$ and $n_{s+1} = \cdots = n_{s+t} = 5$. As demonstrated in El-Zahar's paper [3], the minimum degree condition in Theorem 3 is sharp by observing the graph $K_{\lfloor (n-k+1)/2 \rfloor, \lceil (n-k+1)/2 \rceil} + K_{k-1}$ where $k = s + t$.

We use the following terminology and notation. Let G be a graph. Let $u \in V(G)$. The neighborhood of u in G is denoted by $N(u)$. Let H be a subgraph of G or a subset of $V(G)$ or a sequence of vertices of G. We define $N_H(u)$ to be the set of neighbors of u contained within H, and let $e(u, H) = |N_H(u)|$. Clearly, $N_G(u) = N(u)$ and $e(u, G)$ is the degree of u in G. If X is a subgraph of G or a subset of $V(G)$ or a sequence of vertices of G, we define $e(X, H) = \sum_u e(X, H)$ where u runs over all the vertices of X. If X and H are vertex disjoint, then $E(X, H)$ is the set of edges of G between X and H. For a subset U of $V(G)$, $G[u]$ denotes the subgraph of G induced by U. We use C_i to denote a cycle of length i for all integers $i \geq 3$. For a cycle C of G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C, and we use $\tau(C)$ to denote the number of chords of C in G. Let P be a path of order n_1 and L be a cycle of order n_2 such that they are independent. We say that (P, L) is optimal if $\tau(L) \geq \tau(L')$ for any cycle L' of order n_2 and any path P' of order n_1 contained in $G[V(P \cup L)]$, such that they are independent. If S is a set of subgraphs of G, we write $G \supseteq S$. For an integer $k \geq 1$ and a graph G', we use kG' to denote k independent graphs isomorphic to G'. If $G_1, \ldots, G_r$ are r graphs and $k_1, \ldots, k_r$ are positive integers, we use $k_1G_1 \uplus \cdots \uplus k_rG_r$ to denote a set of $k_1 + \cdots + k_r$ independent graphs which consist of k_1 copies of $G_1, \ldots,$ k_{r-1} copies of G_{r-1}, and k_r copies of G_r. For two graphs H_1 and H_2, the union of H_1 and H_2 is still denoted by $H_1 \cup H_2$ as usual, that is, $H_1 \cup H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. If $C = x_1x_2 \ldots x_kx_1$ is a cycle, then the operations on the subscripts of the x_i's will be taken modulo k in $\{1, 2, \ldots, k\}$. If P and L are two independent subgraphs of G such that P is a path of order at least 2, we define $e^*(P, L)$ to be $e(uv, L)$ where u and v are the two end-vertices of P.

2 Lemmas

The first lemma is a theorem by Wang [9] and the other lemmas are technical lemmas dealing with the construction of disjoint triangles and pentagons in a given graph $G = (V, E)$.

Lemma 4 *Let G be a graph of order $n \geq 5k + 2$, where k is a positive integer. If the minimum degree of G is at least $\lceil (n + k)/2 \rceil$ then G contains k pentagons and a path of order $n - 5k$ such that they cover all the vertices of G.*

Lemma 5 *Let C be a triangle in G. Let x and y be two distinct vertices of G such that x and y are not on C. If $e(xy, C) \geq 5$ then there exists a vertex $z \in V(C)$ such that $C - z + x$ is a triangle and yz is an edge of G.*

Proof. Trivial. ∎

Lemma 6 *Let R be a pentagon in G. Let x and y be two distinct vertices of G such that x and y are not on R. If $e(xy, R) \geq 7$ then there exists a vertex $z \in V(R)$ such that either (a) holds or, (b) and (c) hold below.*

(*a*) *$G[(V(R) - \{z\}) \cup \{x\}]$ contains a pentagon R' such that $zy \in E$ and $\tau(R') \geq \tau(R) - 1$.*

(*b*) *$G[(V(R) - \{z\}) \cup \{y\}]$ contains a pentagon R'' such that $zx \in E$ and $\tau(R'') \geq \tau(R)$.*

(*c*) *If there exists no vertex $z \in V(R)$ such that (a) holds then $e(xy, R) = 7$.*

Proof. Let $R = z_1z_2z_3z_4z_5z_1$. We assume statement (*a*) does not hold to prove statements (*b*) and (*c*) hold. If $e(x, R) \geq 4$ then it is easy to see that (*a*) holds. So we assume $e(x, R) \leq 3$. It is also easy to see that statements (*b*) and (*c*) hold if $e(x, R) = 2$ and $e(y, R) = 5$. So we can assume $e(x, R) = 3$. Then, without loss of generality $xz_1, xz_3 \in E$. Since we assumed (*a*) does not hold we may assume that either $yx_2 \notin E$ or $\{xz_2, z_2z_4, z_2z_5\} \subset E$. If $yx_2 \notin E$ then $N_R(y) = \{z_1, z_3, z_4, z_5\}$. Hence $R - z_2 + y$ is a pentagon R', $xx_2 \in E$, and $\tau(R') = \tau(R)$. In other words, (*b*) and (*c*) hold. Thus we may assume $yx_2 \in E$ and hence $\{xz_2, z_2z_4, z_2z_5\} \subset E$. Without loss of generality $yz_1 \in E$. Let $R'' = z_1yz_2z_4z_5z_1$ and $xz_3 \in E$. Thus we may assume $\tau(R'') \leq \tau(R) - 1$. Hence, without loss of generality, $N_R(y) = \{z_1, z_2, z_3, z_4\}$ and $\{z_1z_3, z_3z_5\} \subset E$ or else (*b*) and (*c*) hold and we have nothing to show. But then $z_1xz_2z_3z_5z_1$ is a pentagon R''', $yx_4 \in E$, $\tau(R''') \geq \tau(R) - 1$. This is statement (*a*), a contradiction. This concludes the proof of the lemma. ∎

Lemma 7 *Let P be a path of order 4 and R a pentagon in G such that P and R are independent. If $e(P, R) \geq 13$ then $G[(V(P \cup R)) - \{x\}]$ contains a triangle and a pentagon such that they are independent for some $x \in V(P)$.*

Proof. Let $P = x_1x_2x_3x_4$ and $R = z_1z_2z_3z_4z_5z_1$. We assume without loss of generality that $e(x_1x_2, R) \geq 7$. We begin by assuming there exists a vertex $z_i \in V(R)$, say $i = 1$, such that $z_1 \in N_R(x_3) \cap N_R(x_4)$. Then $z_1x_3x_4z_1$ is a triangle T and $e(x_1x_2, z_2z_3z_4z_5) \geq 5$. We can assume that $G[\{x_2, x_3, z_2, z_3, z_4, z_5\}]$ does not contain a pentagon or else we have nothing more to show. This means $e(x_j, z_2z_5) \leq 1$ for $j = 1, 2$ and hence $e(x_1x_2, z_3z_4) \geq 3$. Without loss of generality, say $e(x_1x_2, z_3) = 2$. Then $e(x_j, z_5) = 0$ for $j = 1, 2$ for otherwise one of the pentagons $z_3z_4z_5x_1x_2z_3$ or $z_3z_4z_5x_2x_1z_3$ is disjoint from T and we are done. So $e(x_1x_2, z_2z_4) \geq 3$. This means one of the pentagons $R' = z_2z_3z_4x_1x_2z_2$ or $R'' = z_2z_3z_4x_2x_1z_2$ is disjoint from T, our desired result.

We may now assume $N_R(x_3) \cap N_R(x_4) = \emptyset$. This implies $e(x_3x_4, R) \leq 5$ and hence $e(x_1x_2, R) \geq 8$. Suppose $e(x_1x_2, R) = 10$. Then $e(x_3, R) = 0$ for otherwise $z_ix_2x_3z_i$ is a triangle and $R - z_i + x_1$ is a pentagon for some $i \in \{1, \ldots, 5\}$. Hence, $e(x_4, R) \geq 3$ which means there exists vertices z_j, z_{j+1}, say $j = 1$, such that $x_4z_1z_2x_4$ is a triangle and $z_3z_4z_5x_1x_2z_3$ is a pentagon. So we can assume $e(x_1x_2, R) \leq 9$.

Suppose $e(x_1x_2, R) = 9$. This means $e(x_3x_4, R) \geq 4$. Suppose $e(x_3, R) > 0$ and $e(x_4, R) > 0$. There exists vertices z_j, z_{j+2}, say $j = 1$, such that $z_1x_3, z_3x_4 \in E$. This means $z_1z_2z_3x_4x_3z_1$ is a pentagon R''' and clearly $G[\{x_1, x_2, z_4, z_5\}] \supseteq C_3$. So we assume $e(x_3, R) = 0$ or $e(x_4, R) = 0$. Suppose $N_R(x_1) \cap N_R(x_2) = \{z_1, z_2, z_3, z_4\}$. Without loss of generality $z_1, z_3 \in N_R(x_k)$ for some $k \in \{3, 4\}$ and so $z_1x_kz_3Rz_1$ is a pentagon and $x_1x_2z_2x_1$ is a triangle.

We finally assume $e(x_1x_2, R) = 8$ and hence $e(x_3x_4, R) = 5$. Since $e(x_3x_4, R) = 5$ there exists vertices z_i, z_{i+1}, say $i = 1$, such that $e(x_j, z_1z_2) = 2$ for some $j \in \{3, 4\}$. We also know that $e(x_1x_2, z_3z_4z_5) \geq 4$. If $e(x_1x_2, z_3z_4z_5) \geq 5$ then either $z_3z_4z_5x_1x_2z_3$ or $z_3z_4z_5x_2x_1z_3$ is a pentagon disjoint from the triangle $z_1z_2x_jz_1$. So $e(x_1x_2, z_3z_4z_5) = 4$ and hence $e(x_1x_2, z_1z_2) = 4$ and using the same cycles we see that $e(x_1x_2, z_4) = 2$. Therefore $x_1x_2z_4x_1$ is a triangle. This means that $e(x_3, z_3z_5) \leq 1$ and $e(x_4, z_3z_5) \leq 1$. But since $e(x_3x_4, R) = 5$ and $N_R(x_3) \cap N_R(x_4) = \emptyset$ we can assume without loss of generality that x_3z_3 and $x_3z_5 \in E$. But now $z_1z_2x_1z_1$ is a triangle and $x_3z_3z_4z_5x_4x_3$ is a pentagon and they are disjoint. This proves the lemma. ■

Lemma 8 *Let C be a triangle and P a path of order 3 such that C and P are independent in G. If $e(P, C) \geq 7$ then either $G[V(P \cup C)]$ contains two disjoint triangles or there exists two labelings $C = x_1x_2x_3x_1$ and $P = uvw$ such that $N_C(u) = N_C(w) = \{x_1, x_2\}$ and $e(v, C) = 3$.*

Proof. Wang [7]. ■

Lemma 9 *Let C be a triangle and $P = uvwx$ be a path of order 4 in G such that they are independent. If $e(C, P) \geq 9$ then $G[V(C \cup P) - \{z\}]$ contains two independent triangles for some vertex $z \in V(P)$. Furthermore, if $G[V(C \cup P) - \{y\}]$ does not contain two independent triangles for each end vertex $y \in V(P)$ then $e(u, C) = 0$ for some vertex $u \in \{v, w\}$.*

Proof. Wang [7]. ■

Lemma 10 *Let P be a path of order 3 in G and R a pentagon in G such that they are independent. Suppose, $e^*(P, R) \geq 7$ and (P, R) is optimal. Then either $G[V(P \cup R)] \supseteq C_3 \uplus C_5$ or there exists labelings $P = x_1x_2x_3$ and $R = z_1z_2z_3z_4z_5z_1$ such that $N_R(x_1) = \{z_1, z_2, z_3, z_4\}$, $N_R(x_3) \subseteq N_R(x_1)$, $e(x_2, R) = 0$ and $\tau(R) = 2$ with $\{z_1z_3, z_2z_4\} \subseteq E$.*

Proof. Without loss of generality, we may assume $e(x_1, R) \geq 4$. Assume for a contradiction that $G[V(P \cup R)] \not\supseteq C_3 \uplus C_5$.

We first assume $e(x_1, R) = 5$. We may assume $x_3z_1 \in E$. Let $R' = C - z_1 + x_1$ and let $P' = x_2x_3z_1$. We observe that x_1z_3, x_1z_4 are chords of R' and since (P, R) is optimal, we may assume z_1z_3, z_1z_4 are chords of R. If $\{x_3z_2, x_3z_3\} \cap E \neq \emptyset$ then $z_1z_5z_4z_1$ is a triangle and one of $x_1x_2x_3z_2z_3x_1$ or $x_1x_2x_3z_3z_2x_1$ is a pentagon. So we may assume $\{x_3z_2, x_3z_3\} \cap E = \emptyset$. Similarly, $\{x_3z_4, x_3z_5\} \cap E = \emptyset$. Thus $e(x_3, R) = 1$ and so $e^*(P, R) = 6$, a contradiction.

We may now assume $e(x_1, R) = 4$ and $e(x_3, R) \geq 3$. Without loss of generality., assume $N_R(x_1) = \{z_1, z_2, z_3, z_4\}$. Suppose $x_3z_5 \in E$. Then $R' = x_1z_1z_2z_3z_4x_1$ is a pentagon with chords x_1z_2 and x_1z_3 and $P' = x_2x_3z_5$ is a path of order 3. Hence, we may assume z_5z_2 and z_5z_3 are chords of R since (P, R) is optimal. If $\{x_3z_1, x_3z_2\} \cap E \neq \emptyset$ then $z_3z_5z_4z_3$ is a triangle and one of $x_1x_2x_3z_2z_1x_1$ or $x_1x_2x_3z_1z_2x_1$ is a pentagon. So we may assume $\{x_3z_1, x_3z_2\} \cap E = \emptyset$. Similarly, $\{x_3z_3, x_3z_4\} \cap E = \emptyset$. Thus $e(x_3, R) = 1$ and so $e^*(P, R) = 5$, a contradiction. Therefore, we may assume $N_R(x_3) \subseteq N_R(x_1)$ and without loss of generality $\{x_3z_1, x_3z_2\} \subseteq E$. This means $z_3z_5 \notin E$ for otherwise $z_3z_4z_5z_3$ is a triangle and $z_1z_2Pz_1$ is a pentagon. Clearly $x_2z_2 \notin E$ for otherwise $x_2x_3z_2x_2$ is a triangle and $C - z_2 + x_1$ is a pentagon. Also, $x_2z_4 \notin E$ for otherwise $z_1x_3x_2z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. Suppose $x_2z_5 \in E$. Then $z_5x_2x_3$ is a path of order 3 and $C - z_5 + x_1$ is a pentagon with chords x_1z_2 and x_1z_3. So we may assume z_5z_2 and z_5z_3 are chords of R since (P, R) is optimal. But then $z_2z_1Pz_2$ is a pentagon and $z_3z_4z_5z_3$ is a triangle, a contradiction. So $x_2z_5 \notin E$. Now suppose $x_2z_1 \in E$. Then $x_3z_3 \notin E$ or else $z_1x_2x_1z_4z_5z_1$ is a pentagon and $x_3z_2z_3x_3$ is a triangle. But then $N_R(x_3) = \{z_1, z_2, z_4\}$ and so $x_2x_3z_4z_5z_1x_2$ is a pentagon and $x_1z_2z_3x_1$ is a triangle, a contradiction. Hence, $x_2z_1 \notin E$. We finally assume $x_2z_3 \in E$. Then $x_3z_3 \notin E$ for otherwise $z_1z_2x_1z_4z_5z_1$ is a pentagon and $x_2x_3z_3x_2$ is a triangle. But then $N_R(x_3) = \{z_1, z_2, z_4\}$ and so $z_1z_2x_3z_4z_5z_1$ is a pentagon and $x_1x_2z_3x_1$ is a triangle. Therefore, $e(x_2, R) = 0$.

Clearly $z_1x_1z_3z_4z_5z_1$ is a pentagon with at least one chord, x_1z_4, while $x_2x_3z_2$ is a path of order 3. So we may assume that $z_2z_4 \in E$ or $z_2z_5 \in E$ since (P, R) is optimal. If the latter holds then $z_1z_2z_5z_1$ is a triangle and one of $z_3Pz_4z_3$ or $z_4Pz_3z_4$ is a pentagon. So we may assume the former, that is $z_2z_4 \in E$. If z_1z_4 is a chord of R then $z_1z_4z_5z_1$ is a triangle and $z_3Pz_2z_3$ is a pentagon. So we can assume $z_1z_4 \notin E$ and thus $1 \leq \tau(R) \leq 2$. If $x_3z_3 \in E$ then $x_2x_3z_3$ is a path of order 3 and $C - z_3 + x_1$ is a pentagon with chords x_1z_1 and z_2z_4. If $x_3z_4 \in E$ then $C - z_3 + x_3$ is a pentagon with chords z_2z_4 and x_3z_1 while $x_2x_1z_3$ is a path of order three. Since (P, R) is optimal and $N_R(x_3) \cap \{z_3, z_4\} \neq \emptyset$ we can assume $\tau(R) = 2$ with chords z_1z_3 and z_2z_4. This completes the proof of the lemma. ■

Lemma 11 *Let P be a path of order 3 and let R be a pentagon such that P and R are independent and (P, R) is optimal. If $e(P, R) \geq 10$ then either $G[V(P \cup R)] \supseteq C_3 \uplus C_5$ or there exists labelings $P = x_1x_2x_3$ and $R = z_1z_2z_3z_4z_5z_1$ such that one of the following three statements holds.*

- *(a) $e(x_1, R) = e(x_2, R) = 5$, $e(x_3, R) = 0$ and $\tau(R) = 5$.*
- *(b) $N_R(x_1) = N_R(x_3) = \{z_1, z_2, z_4\}$, $N_R(x_2) = \{z_2, z_3, z_4, z_5\}$, $\tau(R) = 4$, $z_3z_5 \notin E$.*
- *(c) $N_R(x_3) \subseteq N_R(x_1) = \{z_1, z_2, z_4\}$, $\tau(R) = 4$, $z_3z_5 \notin E$.*

Proof. Suppose $G[V(P \cup R)] \not\supseteq C_3 \uplus C_5$. By Lemma 10 we may assume $e^*(P,R) \leq 6$. Therefore, $e(x_2,R) \geq 4$. Without loss of generality, we may assume $\{z_2,z_3,z_4,z_5\} \subseteq N_R(x_2)$ and $e(x_1,R) \geq e(x_3,R)$.

Case 1. $e(x_1,R)=5$. Then $N_R(x_2) \cap N_R(x_3) = \emptyset$, for otherwise, if $\exists v \in N_R(x_2) \cap N_R(x_3)$ then $x_2vx_3x_2$ is a triangle and $R-v+x_1$ is a pentagon. So either $N_R(x_2) = \{z_2,z_3,z_4,z_5\}$ and $N_R(x_3)=\{x_1\}$ or $e(x_1,R)=e(x_2,R)=5$ and $e(x_3,R)=0$. In the former $z_1x_3x_2z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle which is a contradiction. So we may assume the latter. In this case it is easy to see that (a) is satisfied.

Case 2. $e(x_1,R)=4$.

Case 2.1. $N_R(x_1)=\{z_2,z_3,z_4,z_5\}$. If $z_2x_3 \in E$ then $z_1z_2x_3x_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. So $z_2x_3 \notin E$ and similarly $z_5x_3 \notin E$. If $x_3z_1 \in E$ then $z_1x_3x_2z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. Thus, $N_R(x_3) \subseteq \{z_3,z_4\}$. But since $e(x_1,R)=4$ then $e(x_3,R) \geq 1$ and so without loss of generality $x_3z_3 \in E$. However, then we have the pentagon $R-z_3+x_1$ and the triangle $x_2x_3z_3x_2$, a contradiction.

Case 2.2. $N_R(x_1)=\{z_1,z_2,z_3,z_4\}$. If $z_2x_3 \in E$ then $z_1z_2x_3x_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. So $z_2x_3 \notin E$ and similarly $z_5x_3 \notin E$. If $x_3z_1 \in E$ then $z_1x_3x_2z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. Thus, $N_R(x_3) \subseteq \{z_3,z_4\}$. But since $e(x_1,R)=4$ then $e(x_3,R) \geq 1$. If $x_3z_3 \in E$ then $R-z_3+x_1$ is a pentagon and $x_2x_3z_3x_2$ is a triangle. So we may assume $x_3z_4 \in E$ and $x_2z_1 \in E$. But then $z_1x_2x_3z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle, a contradiction.

Case 3. $e(x_1,R)=3$.

Case 3.1. $e(x_2,R)=5$. Suppose $N_R(x_1)=\{z_1,z_2,z_3\}$. We first observe $x_3z_2 \notin E$ for otherwise $x_2x_3z_2x_2$ is a triangle and $z_1x_1z_3Rz_1$ is a pentagon. We also see that $x_3z_4 \notin E$ for otherwise $x_1z_2z_3x_1$ is a triangle and $z_1x_2x_3z_4z_5z_1$ is a pentagon. Similarly, $x_3z_5 \notin E$. Thus $N_R(x_3)=\{z_1,z_3\}$. But then $z_1x_3z_3Rz_1$ is a pentagon and $x_1x_2z_2x_1$ is a triangle, a contradiction. So we can assume without loss of generality that $N_R(x_1)=\{z_1,z_2,z_4\}$. If $x_3z_3 \in E$ then $z_2x_1z_4Rz_2$ is a pentagon and $x_2x_3z_3x_2$ is a triangle. So $x_3z_3 \notin E$ and similarly $x_3z_5 \notin E$. Hence $N_R(x_3) \subseteq N_R(x_1)$. We see that $z_3z_5 \notin E$ for otherwise one of $z_1z_2x_1x_2x_3z_1$ or $z_1z_2x_3x_2x_1z_1$ is a pentagon disjoint from the triangle $z_3z_4z_5z_3$. If $\{z_1,z_2\} \subseteq N_R(x_3)$ then $z_3z_4z_5$ is a path and $z_1Pz_2z_1$ is a pentagon with four chords. This is statement (c). So we can finally assume without loss of generality that $N_R(x_3)=\{z_1,z_4\}$ and it is easy to see that $\tau(R)=4$ which is statement (c). We leave the details to the reader.

Case 3.2. $N_R(x_2)=\{z_2,z_3,z_4,z_5\}$. We first assume $N_R(x_1)=\{z_2,z_3,z_4\}$. Then $x_3z_3 \notin E$ for otherwise $R-z_3+x_1$ is a pentagon and $x_2x_3z_3x_2$ is a triangle. We also see $x_3z_1 \notin E$ for otherwise $x_1z_2z_3x_1$ is a triangle and $z_1x_3x_2z_4z_5z_1$ is a pentagon. So $x_3z_2 \in E$. But then $z_2x_3x_2z_5z_1z_2$ is a pentagon and $x_1z_3z_4x_1$ is a triangle, a contradiction.

We now assume $N_R(x_1)=\{z_2,z_3,z_5\}$. Then $x_3z_4 \notin E$ for otherwise $R-z_4+x_1$ is a pentagon and $x_2x_3z_4x_2$ is a triangle. We also observe that $x_3z_1 \notin E$ for otherwise $z_1x_3x_2z_4z_5z_1$ is a pentagon and $x_1z_2z_3x_1$ is a triangle. Hence $N_R(x_3)=N_R(x_1)$. Let $R'=z_2z_3x_1x_2x_3z_2$ and let $P'=z_4z_5z_1$. We see that $\tau(R')=4$. Thus $\tau(R)=4$.

If z_4z_1 is a chord of R then $G[V(P')]$ is a triangle, a contradiction. Thus we may assume that $z_4z_1 \notin E$. Without loss of generality., we can change the labeling of R so that $N_R(x_1) = N_R(x_3) = \{z_1, z_2, z_4\}$, $N_R(x_2) = \{z_1, z_2, z_3, z_4\}$, $\tau(R) = 4$ with $\{z_1z_3, z_1z_4, z_2z_4, z_2z_5\}$ and we have statement (c).

We now assume $N_R(x_1) = \{z_1, z_2, z_3\}$. We observe that $x_3z_2 \notin E$ for otherwise $R - z_2 + x_1$ is a pentagon and $z_2x_2x_3z_2$ is a triangle. If $\{z_4, z_5\} \subseteq N_R(x_3)$ then $z_4z_5x_3z_4$ is a triangle and $z_1z_2z_3x_2x_1z_1$ is a pentagon. If $\{z_1, z_3\} \subseteq N_R(x_3)$ then $z_2x_1x_2z_2$ is a triangle and $z_1x_3z_3z_4z_5z_1$ is a pentagon. If $\{z_1, z_5\} \subseteq N_R(x_3)$ then $z_1z_5x_3z_1$ is a triangle and $z_2z_3z_4x_2x_1z_2$ is a pentagon. This shows that $e(x_3, R) = 2$, a contradiction.

We finally assume $N_R(x_1) = \{z_1, z_2, z_4\}$. Clearly $x_3z_3 \notin E$ for otherwise $x_2x_3z_3x_2$ is a triangle and $z_2x_1z_4Rz_2$ is a pentagon. Similarly $x_3z_5 \notin E$. Hence $N_R(x_3) = N_R(x_1)$. It is easy to see that statement (b) holds. We leave those details to the reader and the proof of the lemma is complete. ■

Lemma 12 *Let P be a path of order 4 and R a pentagon in G such that P and R are independent and $(P - x, R)$ is optimal for each end vertex $x \in V(P)$. If $e(P, R) \geq 13$ then $G[V(P \cup R) - \{x\}] \supseteq C_3 \uplus C_5$ for some end-vertex $x \in V(P)$.*

Proof. Let $P = x_1x_2x_3x_4$ and $R = z_1z_2z_3z_4z_5z_1$. Suppose $G[V(P \cup R) - \{x\}] \not\supseteq C_3 \uplus C_5$ for any end-vertex $x \in V(P)$. Let $P_1 = x_1x_2x_3$ and let $P_2 = x_2x_3x_4$. Without loss of generality, we may assume $e^*(P_1, R) \geq 7$. Then we may assume $N_R(x_1) = \{z_1, z_2, z_3, z_4\}$, $N_R(x_3) \subseteq N_R(x_1)$, $e(x_2, R) = 0$ and $\tau(R) = 2$ with $\{z_1z_3, z_2z_4\} \subseteq E$ for otherwise, by Lemma 10, we have nothing more to show. This means that $e(x_4, R) = 5$ and $N_R(x_3) = N_R(x_1)$ since $e(P, R) \geq 13$. Let $P_2' = x_2x_3z_3$ and $R' = z_1z_2x_4z_4z_5z_1$. Clearly $\tau(R') = 3$, contradicting the optimality of (P_2, R). This completes the proof of the lemma. ■

3 Proof of Theorem 3

Let n, s, and t be integers with $n = 3s + 5t$, $s \geq 1$, $t \geq 0$. Let G be a graph of order n with $\delta(G) \geq (n + k)/2$ where $k = s + t$. Thus $\delta(G) \geq 2s + 3t$. Suppose, for a contradiction, that G does not contain $s + t$ independent subgraphs such that s of them are triangles and t of them are pentagons. As $\delta(G) \geq (n + k)/2$ and by Lemma 4, G contains t independent pentagons. Let r be the largest number such that there exists r independent triangles in $G - V(\cup_{i=1}^{t} R_i)$ for a set of t independent pentagons $R_1, \ldots, R_t$. By the assumption on G, we have $r \leq s - 1$. We will prove four claims below, from which the theorem will follow.

Claim 1 *There exists a set of $r + t$ independent subgraphs $L_1, \ldots, L_{r+t}$ in G with r of them being triangles and t of them being pentagons such that $G - V(\cup_{i=1}^{t} R_i)$ contains a path of order at least* $\min\{4, n - 3r - 5t\}$.

Proof of Claim 1. Suppose to the contrary, that Claim 1 is false. Among all choices of the sets $\{L_1, \ldots, L_{r+t}\} \subseteq G$ with $\{L_1, \ldots, L_{r+t}\} \supseteq rC_3 \cup tC_5$, we choose $L_1, \ldots, L_{r+t}$ so that the length of a longest path in $G - V(\cup_{i=1}^{r+t} L_i)$ is maximal. Subject to this restriction, we further choose $L_1, \ldots, L_{r+t}$ so that $\sum_{i=1}^{r+t} \tau(L_i)$ is maximal. Let $H = \cup_{i=1}^{r+t} L_i$ and $D = G - V(H)$. Let $P = x_1 x_2 \ldots x_p$ be a longest path in D. By our assumption $p \leq 2$ if $r = s - 1$ and $p \leq 3$ otherwise. Let x_0 be a vertex in $D - V(P)$. We observe that $e(x_0 x_1, L_i) \leq 4$ for any triangle L_i in H, for otherwise, by Lemma 5, there exists a vertex $z \in V(C)$ such that $C - z + x_0$ is a triangle and $x_1 z$ is an edge of G, which contradicts the maximality of P.

We begin by assuming $p = 1$. Then $e(x_0 x_1, H) \geq 4s + 6t > 4r + 6t$. Therefore, there exists a pentagon $L_i \in H$, say $i = 1$, such that $e(x_0 x_1, L_1) \geq 7$. Therefore by Lemma 6, without loss of generality, $\exists z \in V(L_1)$ such that $G\left[(V(L_1) - \{z\}) \cup \{x_0\}\right]$ contains a pentagon L_1' such that $z x_1 \in E$. This contradicts our choice of $L_1, \ldots, L_{r+t}$ so that the longest path in D is maximal.

We now assume $p = 2$. Then $e(x_0 x_1, D) \leq 2$ and so $e(x_0 x_1, D) \geq 4s + 6t - 2 > 4r + 6t$. Thus, there exists a pentagon L_i in H, say $i = 1$, such that $e(x_0 x_1, L_1) \geq 7$. By Lemma 6 there exists a vertex $z \in V(L_1)$ such that $G\left[(V(L_1) - \{z\}) \cup \{u\}\right]$ contains a pentagon L_1' such that zv is a path P' of order 2 for some $u, v \in \{x_0, x_1\}$, with $u \neq v$. We can assume that $u = x_1$, $v = x_0$, and $e(x_0, D) = 0$ for otherwise we again violate the maximality of P. Hence, by Lemma 6, $\tau(L_1') \geq \tau(L_1)$ and $e(x_0 x_1, L_1) = 7$. Similarly if $e(x_0 x_2, L_1) \geq 7$ then $e(x_0 x_2, L_1) = 7$. Hence, $e(x_0 x_2, L_1) \leq 7$. Thus, $e(x_0 x_2, H - V(L_i)) \geq 4s + 6t - 1 - 7 > 4r + 6(t - 1) + 1$. This means there exists an L_i in H, say $i = 2$, such that $e(x_0 x_2, L_2) \geq 7$. As before, we can show that L_2 is a pentagon and $G\left[(V(L_2) - \{y\}) \cup \{x_2\}\right]$ contains a pentagon L_2' such that $y x_0$ is a path P'' of order 2 for some $y \in V(L_2)$. We can now replace L_1, L_2, and P by L_1', L_2', and $P' \cup P''$ respectively. Clearly, $P' \cup P''$ is a path of order 3, contradicting the maximality of P.

We finally assume $p = 3$. Then $r \leq s - 2$ and hence $|V(D)| = 3(s - r) \geq 6$. Suppose x_2 has a neighbor x' in $D - V(P)$. Then $N_D(x') = \{x_2\}$ or else there is a triangle in D or a path of order at least 4 in D, both of which are contradictions. Hence, $e(x_1 x', H) \geq 4s + 6t - 2 \geq 4r + 6t + 6$. As before, $e(x_1 x', L_j) \leq 4$ for each triangle L_j in H and so there exists a pentagon L_i, say $i = 1$, such that $e(x_0 x_1, L_1) \geq 7$. Thus, by Lemma 6, $G\left[(V(L_1) - \{z\}) \cup \{u\}\right]$ contains a pentagon L_1' such that zv is a path P' of order 2 for some $z \in V(L_1)$ and $u, v \in \{x_1, x'\}$ with $u \neq v$. Replacing L_1 with L_1' and replacing u with z we now have a path $P'' = z v x_2 x_3$ of order 4, which contradicts the maximality of P. Hence, we may assume that P is a component of D. We can now repeat the above arguments to show there is a choice of $L_1'', L_2'', \ldots, L_{r+t}''$ in $G - V(P)$ such that $\left\{L_1'', \ldots, L_{r+t}''\right\} \supseteq rC_3 \cup tC_5$ and $G - V(\cup_{i=1}^{r+t} L_i'') - V(P)$ contains a path F of order 3. Let $F = y_1 y_2 y_3$. We again repeat the above arguments showing that there exists a pentagon L_i''', say $i = 1$, such that $e(x_1 y_1, L_1''') \geq 7$. By Lemma 6 again, there exists a vertex $w \in V(L_1''')$ such that without loss of generality, $x_1 w \in E$ and $G\left[V(L_1''') - \{w\} \cup \{y_1\}\right]$ is a pentagon. But then $w x_1 x_2 x_3$ is a path of order 4, contradicting the maximality of P. This proves Claim 1. ■

Claim 2 $r = s - 1$.

Proof of Claim 2. Suppose to the contrary that $r \leq s-2$. By Claim 1, there exists a set of $r+t$ independent subgraphs $L_1, \ldots, L_{r+t}$ such that r of them are triangles, t of them are pentagons and $G - V(\cup_{i=1}^{r+t} L_i)$ contains a path P of order 4. We again choose $L_1, \ldots, L_{r+t}$ so that $\sum_{i=1}^{r+t} \tau(L_i)$ is maximal. Let $H = \cup_{i=1}^{r+t} L_i$, let $D = G - V(H)$ and let $P = x_1x_2x_3x_4$. Clearly $N_D(x_i) \cap N_D(x_{i+1}) = \emptyset$ for $i = 1, 2, 3$ for otherwise D contains a triangle. Hence, $e(x_1x_2, D) \leq 3(s-r)$ and $e(x_3x_4, D) \leq 3(s-r)$. Thus $e(P, H) \geq 8s + 12t - 6(s-r) = 2s + 12t + 6r > 8r + 12t$. If there exists a triangle L_i in H, such that $e(P, L_i) \geq 9$ then by Lemma 9, $G[V(L_i \cup P)] \supseteq 2C_3$ and hence $G \supseteq (r+1)C_3 \uplus tC_5$, a contradiction. Hence $e(P, L_i) \leq 8$ for all triangles $L_i \in H$. This means that there exists a pentagon L_i in H, say $i = 1$, such that $e(P, L_1) \geq 13$. By Lemma 7, $G[V(P \cup L_1)]$ contains a triangle T and a pentagon L_1' such that they are independent. Once again we see that $G \supseteq (r+1)C_3 \uplus tC_5$, a contradiction. This proves the claim. ■

By Claim 1 and Claim 2, there exists a set of $s+t$ independent subgraphs $L_1, \ldots, L_{s+t-1}$ and P in G such that $s-1$ of them are triangles, t of them are pentagons and P is a path of order 3. We choose these independent subgraphs such that

$$\sum_{i=1}^{s+t-1} \tau(L_i) \text{ is maximum.} \tag{1}$$

Let $\mathcal{T}$ be the set of all triangles in $\{L_1, \ldots, L_{s+t-1}\}$ and $\mathcal{R}$ be the set of all pentagons in $\{L_1, \ldots, L_{s+t-1}\}$. Set $H = \cup_{i=1}^{s+t-1} L_i$ and say $P = x_1x_2x_3$. Set $G_i = G\left[V(\cup_{j=1}^{i} L_j) \cup V(P)\right]$ and $H_i = H - V(\cup_{j=1}^{i} L_j)$ for each $i \in \{1, 2, \ldots, k-1\}$.

Claim 3 *For each* $i \in \{1, 2, \ldots s+t-1\}$, *if* $e^*(P, L_i) \geq |V(L_i)| + 2$, *then* $L_i \in \mathcal{T}$.

Proof of Claim 3. Suppose to the contrary that there exists an $L_i \in \mathcal{R}$, say $i = 1$, such that $e^*(P, L_1) \geq 7$. By (1) we can assume (P, L_1) is optimal. Let $L_1 = z_1z_2z_3z_4z_5z_1$. By Lemma 10, we assume without loss of generality, that $N_{L_1}(x_1) = \{z_1, z_2, z_3, z_4\}$, $N_{L_1}(x_3) \subseteq N_{L_1}(x_1)$, $e(x_2, L_1) = 0$ and $\tau(L_1) = 2$ with $\{z_1z_3, z_2z_4\} \subseteq E$ or else $G[V(L_1 \cup P)] \supseteq C_3 \uplus C_5$. Let $L_1' = x_1z_1z_2z_3z_4x_1$. Clearly, $\tau(L_1') = \tau(L_1) + 2$. Now, $e(x_jz_5, L_1 \cup P) \leq 7$ for $j = 2, 3$. Hence, $e(x_jz_5, H_1) \geq 4(s-1) + 6(t-1) + 3$ for $j = 2, 3$. This implies, $e(x_jz_5, H_1) \geq 4(s-1) + 6(t-1) + 3$ for $j = 2, 3$. Hence, there exists $L_r \in H_1$ such that $e(x_jz_5, L_r) \geq |V(L_r)| + 2$ for $j = 2, 3$. Say $r = 2$. If $L_2 \in \mathcal{T}$, then by Lemma 5, $G[V(L_2) \cup \{x_2, x_3, z_5\}] \supseteq C_3 \uplus P_3$. Replacing L_1 with L_1' we obtain a contradiction with (1). So we may assume $L_2 \in \mathcal{R}$. If there is a $j \in \{2, 3\}$ such that $G[V(L_2) \cup \{x_j, z_5\}]$ contains a pentagon C and a path P' of order 2, $\tau(C) \geq \tau(L_2) - 1$, and $x_j \in V(P')$, then we replace L_1, L_2, and P with L_1', C, and $P' + x_2x_3$, and obtain a contradiction with (1). Therefore, we can assume $G[V(L_2) \cup \{x_j, z_5\}]$ does not contain two such independent subgraphs. By Lemma 6, for each $j \in \{2, 3\}$ there exists a vertex $u \in V(L_2)$ such that $G[V(L_2) \cup \{x_j, z_5\} - \{u\}]$ contains a pentagon L_2' with $uz_5 \in E$ and $\tau(L_2') \geq \tau(L_2)$. In particular, this is true for $j = 2$. Furthermore, Lemma 6 tells us that we have $e(x_2z_5, L_2) = 7$. It also clear that

$e(x_3z_5, L_2) \leq 7$. Thus, $e(x_3z_5, G_2) \leq 14$ and so, $e(x_3z_5, H_2) \geq 4(s-1)+6(t-2)+2$. By the above argument, there exists an $L_q \in H_2$, say $q = 3$, such that $L_3 \in \mathcal{R}$ and $e(x_3z_5, L_3) \geq 7$. Furthermore, Lemma 6 tells us that there exists a vertex $u \in V(L_3)$ such that $G[V(L_2) \cup \{x_3, z_5\} - \{u\}]$ contains a pentagon L_3' with $uz_5 \in E$ and $\tau(L_3') \geq \tau(L_3)$. Replacing L_1, L_2, L_3, and P, with L_1', L_2', L_3', and $P' + uz_5$, we again obtain a contradiction with (1). This completes the proof of the claim. ■

Claim 4 *For each* $i \in \{1, 2, \ldots s+t-1\}$, *if* $e(P, L_i) \geq 7$, *then* $L_i \in \mathcal{R}$.

Proof of Claim 4. Suppose to the contrary that there exists an $L_i \in \mathcal{T}$, say $i = 1$, such that $e(P, L_1) \geq 7$. Let $L_1 = z_1z_2z_3z_1$. Then by Lemma 8, we may assume that

$$N_{L_1}(x_1) = N_{L_1}(x_3) = \{z_1, z_2\} \quad \text{and } e(x_2, L_1) = 3.$$

Clearly, $e(x_1x_3, H_1) \geq 4s + 6t - 6 = 4(s-2) + 6t + 2$. Hence, there exists an $L_i \in H_1$, say $i = 2$, such that $e(x_1x_3, L_2) \geq |V(L_2)| + 2$. By Claim 3, we may assume that L_2 is a triangle $y_1y_2y_3y_1$. We also assume without loss of generality, that $N_{L_2}(x_1) \supseteq \{y_1, y_2\}$ and $e(x_3, L_2) = 3$. Since G_2 does not contain $3C_3$ we see that $e(x_2, L_2) = 0$ and $e(y_3, z_1z_2) = 0$. Consequently, if $R_1 = \{z_3, x_2, x_3, y_3\}$, then $e(R_1, G_2) \leq 22$ and so $e(R_1, H_2) \geq 8s + 12t - 22 = 8(s-3) + 12t + 2$. Clearly $G_2 - R \supseteq 2C_3$ for any $R \subseteq R_1$ with $|R| = 3$. Suppose there is a triangle T in $\mathcal{T} \cap H_2$, such that $e(R_1, T) \geq 9$. Then, by Lemma 9, $G[R \cup V(T)] \supseteq 2C_3$, and so G_3 contains $4C_3$, a contradiction. So we may assume that $e(R_1, L_j) \leq 8$ for any $L_j \in \mathcal{T} \cap H_2$. Hence, there exists a pentagon L_i in H_2, say L_3, such that $e(R_1, L_3) \geq 13$. But then by Lemma 12, $G[R - \{v\} \cup V(L_3)] \supseteq C_3 \uplus C_5$ for some $v \in \{z_3, y_3\}$. In either case $G_3 \supseteq 3C_3 \uplus C_5$, a contradiction. This completes the proof of the claim. ■

We are now ready to complete the proof of the theorem. Clearly, $e(P, H) \geq 6s + 9t - 4 \geq 6(s-1) + 9t + 2$. By Claim 4, we may assume $e(P, L_i) \leq 6$ for each $L_i \in \mathcal{T}$. Hence, there exists $L_i \in \mathcal{R}$, say $i = 1$, such that $e(P, L_1) \geq 10$. By Lemma 11, $e(x_1x_3, L_1) \leq 6$ and there exists a labeling of $L_1 = z_1z_2z_3z_4z_5z_1$ such that $z_1z_3 \in E$ and $\{z_2, z_3, z_4, z_5\} \subseteq N_{L_1}(x_2)$. Hence, $e(x_1x_3, G_1) \leq 8$ and thus $e(x_1x_3, H_1) \geq 4s + 6t - 8 = 4(s-1) + 6(t-1) + 2$. This implies that there exists an $L_i \in H_1$, say $i = 2$, such that $e(x_1x_3, L_2) \geq |V(L_2)| + 2$. By Claim 3, we can assume $L_2 \in \mathcal{T}$. So $e(x_1x_3, L_2) \geq 5$ and clearly $G[V(L_2) \cup \{x_1, x_3\}] \supseteq C_5$. Let $L_1' = G[V(L_2) \cup \{x_1, x_3\}]$, $L_2' = z_1z_2z_3z_1$, and $L_{s+t} = x_2z_4z_5x_2$. Therefore, $G \supseteq sC_3 \uplus tC_5$, a contradicition. This completes the proof of the theorem. ■

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