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The 2-star spectrum of stars

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Abstract

Let $\mathcal{K}$ be a family of graphs. A $\mathcal{K}$-decomposition, $\mathscr{D}$, of a graph H (called the *host*) is a partition of the edges of H such that the subgraph induced by each part of the partition (called *blocks*) is isomorphic to an element of $\mathcal{K}$. The *chromatic index* of the decomposition, denoted $\chi'(\mathscr{D})$, is the minimum number of colors required to color each block in the decomposition so that blocks that share a common node in H receive different colors. The $\mathcal{K}$*-spectrum* of H, denoted $Spec_{\mathcal{K}}(H)$, is the set of all values of $\chi'(\mathscr{D})$ over all possible $\mathcal{K}$-decompositions of H. In this paper, we will show that any n-element subset of the positive integers is the spectrum of a tree when decomposing into a family of n trees. We will also look at ways of improving this result. In particular, we examine the problem of whether any n-element subset of positive integers is the spectrum of a star when decomposing into other stars. These results often have a number theoretical flavor to them, as they deal strictly with the parameters involved and not the underlying graphs.

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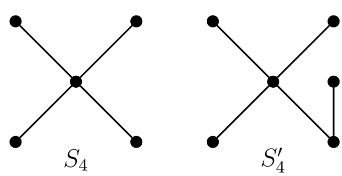
1 Introduction

A *decomposition* $\mathscr{D}$ of a graph $H = (V, E)$ is a partition of the edge set E of H. The graph H is called the *host graph* for the decomposition. For each part $\mathcal{B}$ of the partition, the subgraph of H induced by $\mathcal{B}$ is called a *block* of the partition. Let $\mathcal{K}$ be a family of graphs. When each part of the partition is isomorphic to an element of $\mathcal{K}$, we say that $\mathscr{D}$ is a $\mathcal{K}$-*decomposition* of H. In this case, we will refer to the elements of $\mathcal{K}$ as the *prototypes* for the decomposition. By definition, the blocks are edge-disjoint but in general they share common nodes. The *intersection graph* of the decomposition $\mathscr{D}$, denoted $I(\mathscr{D})$, has a vertex for each block of the partition and two vertices A and B are adjacent if and only if the corresponding blocks share a common node in H. All graphs in this paper are finite, undirected, simple graphs with no isolated vertices.

To help keep the levels of abstraction clear, the vertices of $I(\mathscr{D})$ will be called *vertices* but the vertices of the host graph and the prototypes will be called *nodes*. All other terminology and notation will be consistent with West [5], unless otherwise noted. In particular, the *star* with n edges will be denoted S_n. S'_n will denote the graph derived from S_n by appending a pendant edge to one of the leaves (see Figure 1). This appended edge will be referred to as the *tail*. As usual, K_n will refer to the complete graph on n vertices.

Figure 1: S_4 and S'_4



The *chromatic index* of a decomposition $\mathscr{D}$ is the chromatic number of $I(\mathscr{D})$. Thus the chromatic index of a $\mathcal{K}$-decomposition $\mathscr{D}$, denoted $\chi'(\mathscr{D})$, is the minimum number of colors required to color the blocks in the decomposition so that blocks which share a node in H receive different colors. The $\mathcal{K}$-*chromatic spectrum of* H, denoted $Spec_{\mathcal{K}}(H)$, is the set of all values of $\chi'(\mathscr{D})$ over all possible $\mathcal{K}$-decompositions of H [1, 2].

There are two problems that are often considered with respect to the chromatic spectrum. The first is to determine $Spec_{\mathcal{K}}(H)$ for a given $\mathcal{K}$ and H. The second is to find $\mathcal{K}$ and H such that $S = Spec_{\mathcal{K}}(H)$ for a given $S \subset \mathbb{Z}^+$. In this paper, we will focus on the latter problem. In either case, progress in this area has often been difficult. It is for this reason that results are often interesting.

2 Decomposing into a family of trees

We begin by giving a construction that shows that every n-element subset of the positive integers is the spectrum of a tree when decomposing into a family of n trees.

Theorem 2.1 *Let $S \subset \mathbb{Z}^+$ such that $|S| = n$. There exists a tree, H, and a family of n trees, $\mathcal{K}$, such that $S = Spec_{\mathcal{K}}(H)$.*

Proof. Suppose that $S = \{s_1, ..., s_n\}$ is the desired spectrum, where $1 \leq s_1 < ... < s_n$. It suffices to construct the required H and $\mathcal{K}$. If $n = 1$, then we take $H = S_{s_1}$ and $\mathcal{K} = \{K_2\}$. For $n \geq 2$, take $H = S'_{s_n}$ and $\mathcal{K} = \{K_2, S'_{s_n - s_1 + 1}, ..., S'_{s_n - s_{n-1} + 1}\}$. To achieve s_n, simply decompose H with respect to K_2. Since s_n isomorphic copies of K_2 share a common node, the chromatic index of this decomposition is at least s_n. As the tail of H only shares a common node with one other edge, it may receive the same color as another edge. Hence we have a chromatic index of s_n. For $1 \leq j < n$, s_j can be achieved by decomposing H with respect to K_2 and $S'_{s_n - s_j + 1}$. $S'_{s_n - s_j + 1}$ will partition all but $s_j - 1$ edges of H. The remaining edges can be partitioned by isomorphic copies of K_2. Thus we have s_j blocks sharing a common node in H. As these are all the blocks of the decomposition, the chromatic index is s_j. No two $S'_{s_n - s_j + 1}$ can be used in the same decomposition, as they would share the same tail. Thus these are the only elements in the spectrum. ■

There are several ways in which this result could be improved. First, is to achieve an n-element spectrum using k prototypes, where $k < n$. The second is to use a construction that is of a less artificial nature. For the rest of this paper, we consider this second possibility.

3 Decomposing into stars

Theorem 2.1 states that we can get every set of n positive integers as the spectrum of a tree when decomposing into n prototypes. However, the construction in that theorem uses caterpillars (i.e., the S'_n) which force an orientation. Namely, the tails of these graphs must align. An improvement on this is to use stars which have no inherent orientation and achieve the same result. When the host graph is a star, say $H = S_N$, the only prototypes that make sense are other stars. Define $\mathcal{K} = \{S_{s_1}, ..., S_{s_k}\}$ where $s_i > s_j$ when $i < j$.

Theorem 3.1 *Let $H = S_N$ and $\mathcal{K} = \{S_{s_1}, ..., S_{s_k}\}$. A $\mathcal{K}$-Decomposition of H exists if and only if there is a set of non-negative integers, $x_1, ..., x_k$ that satisfy:*

$$s_1 x_1 + \cdots + s_k x_k = N. \tag{1}$$

Proof. Suppose that there are non-negative integers $x_1, ..., x_k$ that satisfy Equation (1). Note that $e(H) = N$ and $e(S_{s_i}) = s_i$. Thus by taking x_i isomorphic copies of S_{s_i}, we have covered $s_i x_i$ edges of H. Doing this for all values of i, covers $s_1 x_1 + \cdots + s_k x_k = N$ edges.

Suppose that such a partition exists. Since we are dealing with isomorphic copies of graphs, there must be a non-negative number of each, say x_i copies of S_{s_i} for $i = 1, ..., k$. The total edge count must satisfy Equation (1). ■

Theorem 3.2 *If $\mathscr{D}$ is a decomposition of S_N with respect to $\mathcal{K}$, then $\chi(I(\mathscr{D})) = x_1 + ... + x_k$, where the x_i's satisfy Equation (1) and $\mathcal{K} = \{S_{s_1}, ..., S_{s_k}\}$.*

Proof. Note that our decomposition consists of x_i copies of S_{s_i} for $i = 1, ..., k$. Thus the intersection graph generated by the decomposition consists of $x_1 + \cdots + x_k = w$ vertices. Since we are decomposing a star, each of the blocks will share a common node, namely the center of H. Thus, all of the vertices in the intersection graph must be pairwise adjacent. Ergo, $I(\mathscr{D}) \cong K_w$. Thus $\chi'(\mathscr{D}) = \chi(I(\mathscr{D})) = w$. ■

Theorems 3.1 and 3.2 suggest an interesting connection with a classic problem in number theory. Suppose that $\mathcal{K} = \{S_p : p \text{ is prime}\}$. Then $2 \in Spec_{\mathcal{K}}(S_{2n})$ for all $n \in \mathbb{Z}^+$ if and only if Goldbach's Conjecture holds.

Proposition 3.3 *Let $s_2 < s_1$ and N be positive integers. Let $\mathcal{K} = \{S_{s_1}, S_{s_2}\}$. Then $Spec_{\mathcal{K}}(S_N)$ is nonempty if $N \geq s_1 s_2 - s_1 - s_2$ and $\gcd(s_1, s_2)|N$. Further, for all $\chi \in Spec_{\mathcal{K}}(S_N)$, we have:*

$$\frac{N - s_2\beta}{s_1} + \beta \leq \chi \leq \frac{N - s_1\alpha}{s_2} + \alpha$$

where α and β are the least positive integer solutions to $s_1 x \equiv N \pmod{s_2}$ and $s_2 x \equiv N \pmod{s_1}$ respectively. Moreover, $Spec_{\mathcal{K}}(S_N)$ is arithmetic with constant step size $\frac{s_1 - s_2}{\gcd(s_1, s_2)}$.

Proof. Define $g = \gcd(s_1, s_2)$. If $g|N$ and $N \geq s_1 s_2 - s_1 - s_2$, then $Spec_{\mathcal{K}}(H)$ is non-empty by Sylvester [4]. Note that we must find non-negative integers, x_1 and x_2, such that $s_1 x_1 + s_2 x_2 = N$. It is well known (sf. [3]) that the integer solutions of the Linear Diophantine Equation $s_1 x + s_2 y = N$ are of the form $(x, y) = (x_0 + s_2 n/g, y_0 - s_1 n/g)$ where (x_0, y_0) is any particular solution to the equation and $n \in \mathbb{Z}$. Our solutions are restricted to non-negative integers. Hence, we define α and β are the least positive integer solutions to $s_1 x \equiv N \pmod{s_2}$ and $s_2 x \equiv N \pmod{s_1}$ respectively. The solutions of this equation are equivalent to a decomposition of S_N into x_1 copies of S_{s_1} and x_2 copies of S_{s_2} by Theorem 3.1. By Theorem 3.2, this decomposition will have chromatic index:

$$\chi' = x_1 + x_2 = x_0 + y_0 + \frac{(s_2 - s_1)n}{\gcd(s_1, s_2)}.$$

Hence the spectrum will be arithmetic with step size $\frac{s_1 - s_2}{\gcd(s_1, s_2)}$. ■

In order to improve upon Theorem 2.1, we must be able to force the spectrum to have exactly two numbers in it. In other words, given $u < w$ (necessarily positive integers), we must find N, s_1, and s_2 such that there is a unique pair of non-negative integer solutions (x_1, x_2) and (y_1, y_2) to the respective problems:

$$s_1x_1 + s_2x_2 = N \quad \text{and} \quad x_1 + x_2 = u.$$
$$s_1y_1 + s_2y_2 = N \quad \text{and} \quad y_1 + y_2 = w.$$

This differs quite a bit from the classic spectrum problem. In our previous problem, we were given N, s_1, and s_2 and we wished to find the sum of the weights for the representation of N. Now, we are given the sums of the weights and we wish to find the star sizes that give the corresponding representation. We begin with a special case.

Lemma 3.4 *If there exists* $k \in \mathbb{N}$ *such that* $k < \frac{u}{2}$ *and* $\gcd(u-k, w-k) = 1$, *then we can find* N, s_1 *and* s_2 *with* $\mathcal{K} = \{S_{s_1}, S_{s_2}\}$ *such that* $Spec_{\mathcal{K}}(S_N) = \{u, w\}$.

Proof. Assume that there exists $k \in \mathbb{N}$ such that $k < \frac{u}{2}$ and $\gcd(u-k, w-k) = 1$. Take $s_1 = w-k$, $s_2 = u-k$, and $N = u(w-k)$. So our Linear Diophantine equation is $(w-k)x_1 + (u-k)x_2 = u(w-k)$. This has solutions at $(u, 0)$ and $(k, w-k)$, with corresponding chromatic indices u and w respectively. To show uniqueness, note that $(u-k)x_2 \equiv 0 \pmod{w-k}$. Since $\gcd(u-k, w-k) = 1$, it follows that $u-k$ is not a zero-divisor modulo $w-k$. Thus for some $m \in \mathbb{N}$, we have that $x_2 \equiv 0 \pmod{w-k}$ implies $x_2 = m(w-k)$. Substituting this into our original equation yields:

$$x_1 = (1-m)u + mk < (1-m)u + m\frac{u}{2} = \left(1 - \frac{m}{2}\right)u.$$

Note that if $m \geq 2$ our non-negativity assumptions are violated. Hence either $m = 1$ or $m = 0$. If $m = 1$, this solution is $(k, w-k)$. If $m = 0$, then this is the solution $(u, 0)$. Ergo, our solutions are unique. ■

It seems likely that such a k would almost always exist. However, in order to prove the general theorem, we need an additional lemma.

Lemma 3.5 *If* $2u - 2 < w$, *we can find* N, s_1, *and* s_2 *with* $\mathcal{K} = \{S_{s_1}, S_{s_2}\}$ *such that* $Spec_{\mathcal{K}}(S_N) = \{u, w\}$.

Proof. If $u = 1$, then we can apply Lemma 3.4. In all other cases, we take $s_1 = w - u + 1$, $s_2 = 1$, and $N = w$. The rest of the proof is analogous to that of Lemma 3.4. ■

The cases listed above are enough to construct every possible two-element spectrum. To prove this, we must find when a k exists that satisfies Lemma 3.4 and determine how to find this k. To do this, we apply Bertrand's Postulate.

Proposition 3.6 Bertrand's Postulate (c.f. [3]) *If* $n \geq 2$, *there is always a prime number* p *such that* $n < p < 2n$.

Theorem 3.7 *Given any $u < w$, we can find N, s_1, and s_2 with $\mathcal{K} = \{S_{s_1}, S_{s_2}\}$ such that $Spec_{\mathcal{K}}(S_N) = \{u, w\}$.*

Proof. If $w > 2u - 2$, then we can apply the result from Lemma 3.5. Further, if $u < 4$ or u is prime and $w \leq 2u - 2$, we can apply Lemma 3.4. Thus we may assume without loss of generality that $u \geq 4$, u is not prime, and that $w \leq 2u - 2$.

By Bertrand's Postulate (Proposition 3.6), there is a prime $p \in (\lceil \frac{u}{2} \rceil, u)$. Let $u - k = p$. This implies that $k = u - p < \frac{u}{2}$. If $\gcd(u - k, w - k) = 1$, then we can apply Lemma 3.4. So assume $\gcd(u - k, w - k) \neq 1$. Since $u - k$ is prime and $w - k > u - k$, it follows that $w - k = m(u - k)$, where $m \in \mathbb{Z} \cap \{m \geq 2\}$. If $m \geq 3$, then:

$$w - k \geq 3(u - k) = 3u - 3k \Rightarrow w \geq 3u - 2k > 3u - 2\tfrac{u}{2} = 2u.$$

This is contrary to our assumption that $w \leq 2u - 2$. If $w - k = 2(u - k)$, then $w = 2u - k = u + p$. Let $\gcd(u, w) = \gcd(u, u + p) = g$. This implies that there exist $m, n \in \mathbb{Z} \cap (2, \infty)$ such that $u = mg$ and $u + p = ng$. But, $u + p = mg + p = ng$. This implies that $(n - m)g = p$. Hence g divides p, but p is prime. If $g = 1$, then we can apply Lemma 3.4. Hence we may assume that $g = p$. However, since $p > \frac{u}{2}$, it follows that $u = mp > 2\frac{u}{2} = u$, a contradiction. ■

4 Open Problems

Not every three-element subset of the integers forms an arithmetic progression. Thus, it follows from Proposition 3.3 that not every three-element spectrum can be achieved via a decomposition of a star into two stars. Because of the difficulty of the Frobenius problem for three or more variables, it may be difficult to determine the spectrum of a star when decomposing into a family of three or more stars. It may be interesting to determine whether there are any arithmetic progressions of three or more integers that are not the two star spectrum of a star. Also of interest is the problem of constructing an n-element spectrum using less than n prototypes.

References

[1] Peter Danziger, Alasdair Graham and Eric Mendelsohn, The chromatic spectrum of graph designs, *Bull. Inst. Combin. Appl.* 50 (2007), 71–96.

[2] Peter Danziger, Eric Mendelsohn, Gaetano Quattrocchi, On the chromatic index of path decompositions, *Discrete Math.* 284 (1-3) (2004), 107–121.

[3] Gareth A. Jones and J. Mary Jones, *Elementary Number Theory*, Springer-Verlag, London, 2008.

[4] J. J. Sylvester. Mathematical questions with their solutions. *Educational Times* 41 (1884), 21.

[5] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, 2001.

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