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A class of full $(d, 1)$-colorable trees

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Abstract

Let j and k be nonnegative integers. An $L(j, k)$-labeling of a graph G, where $j \geq k$, is a function $f : V(G) \rightarrow \mathbf{Z}^+ \cup \{0\}$ such that if u and v are adjacent vertices in G, then $|f(u) - f(v)| \geq j$, while if u and v are vertices such that $d(u, v) = 2$, then $|f(u) - f(v)| \geq k$. The largest label used by f is the span of f. The smallest span among all $L(j, k)$-labelings f of G, denoted $\lambda_{j,k}(G)$, is called the span of G. An $L(j, k)$-labeling of G that has a span of $\lambda_{j,k}(G)$ is called a span labeling of G. We say that G is (j, k)-full colorable, denoted (j, k)-FC, if there exists a span labeling f of G such that the set $\{i \,|\, f^{-1}(\{i\}) = \emptyset$, where $1 \leq\ i \leq \text{span}(f) - 1\} = \emptyset$. Fishburn and Roberts showed (in *SIAM J. Discrete Math.* **20** (2006), 428–443) that if T is a tree of order $n \geq \Delta(T) + 2$, then T is $(2, 1)$-FC. In this paper, we show that there exists a class of $(d, 1)$-FC trees where $d \geq 3$.

1 Introduction

The frequency of a transmitting radio station may interfere with frequencies of neighboring radio stations. Similar interference may occur in cell phone networks and other radio transmitting networks. Finding the most efficient way to assign non-interfering frequencies to transmitters within a given region is called the channel assignment problem.

Roberts [5] modeled the channel assignment problem by representing each transmitter by a vertex of a graph and joining two vertices by an edge if and only if their corresponding transmitters are within some specified distance of each other. Frequencies (or channels) are represented by non-negative integers. Adjacent vertices must be assigned frequencies with absolute difference of at least j, while vertices at distance two are assigned frequencies with absolute difference of at least k (where $j \geq k$).

More formally, an $L(j,k)$*-labeling* of a graph G, where $j \geq k$, is defined (in [3]) as a function $f : V(G) \to \mathbf{Z}^+ \cup \{0\}$ such that if u and v are adjacent vertices in G, then $|f(u) - f(v)| \geq j$, while if u and v are vertices such that $d(u,v) = 2$, then $|f(u) - f(v)| \geq k$. Here, $d(u,v)$ denotes the distance between u and v, i.e. the length of a shortest $u-v$-path in G if such a path exists. (The concept of $L(j,k)$-labelings of graphs is a generalization of $L(2,1)$-labelings of graphs, as introduced in [4].) We assume throughout the paper that $f^{-1}(\{0\}) \neq \emptyset$. The largest label used by f is the *span of* f, denoted span(f). The smallest span among all $L(j,k)$-labelings of G, denoted $\lambda_{j,k}(G)$, is called the *span of* G. An $L(j,k)$-labeling of G that has a span of $\lambda_{j,k}(G)$ is called a *span labeling of* G.

We say that a span labeling f has ℓ *holes* if the set $\{i \,|\, f^{-1}(\{i\}) = \emptyset$, where $1 \leq i \leq \text{span}(f) - 1\}$ has cardinality ℓ. The $L(j,k)$*-hole index of* G, denoted $\rho_{j,k}(G)$, is defined as the minimum number of holes over all span $L(j,k)$-labelings of G. If $\rho_{j,k}(G) = 0$, then we say that G is (j,k)*-full colorable*, denoted (j,k)-FC (or just FC if the context is clear).

Fishburn and Roberts showed (in [2]) that if T is a tree of order $n \geq \Delta(T) + 2$, then T is $(2,1)$-FC. In this paper, we show that if $d \geq 3$, then there is a class of trees where each member is $(d,1)$-FC.

We will use Δ to denote the maximum degree of the tree in question, and assume that $3 \leq d < \Delta$.

The following result is due to Chang et al.

Theorem 1 *([1]) For any tree* T, $\Delta + d - 1 \leq \lambda_{d,1}(T) \leq \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$; *moreover, the lower and upper bounds are both attainable.*

Throughout we let $b_0 = \Delta + d - 1$, and $b_1 = \min\{\Delta + 2d - 2, 2\Delta + d - 2\}$. Since $d < \Delta$, we have $b_1 = \Delta + 2d - 2$. A vertex of degree one will be called a *leaf*. The *eccentricity* of a vertex u of a connected graph G, denoted $e(u)$, is $\max\{d(u,v) \,|\, v \in V(G)\}$.

Let $u_0 \in V(T)$. The set of vertices at distance i from u_0, $0 \leq i \leq e(u_0)$, will be denoted U_i, and the sets $U_0, \ldots, U_{e(u_0)}$ will then be called the *level decomposition of* T *with respect to* u_0. The vertices of $\cup_{i=1}^{\lfloor \frac{e(u_0)}{2} \rfloor} U_{2i}$ will be called the *even level vertices* of T, and denoted $U(T)_{\text{even}}$, or just U_{even} when the context is clear. The remaining vertices will be called the *odd level vertices* of T.

Let $\mathcal{T}$ consists of all trees T of order at most b_1 such that the level decomposition $U_0, \ldots, U_{e(u_0)}$ with respect to a maximum degree vertex u_0 has the property $|U_{\text{even}}| \geq$

$d-1$.

The aim of this paper is to prove the following result.

Theorem 2 *If $T \in \mathcal{T}$, then T is $(d,1)$-FC with span b_0.*

2 Preliminaries

We begin with some preliminary results and terminology.

Suppose $U_0,\ldots,U_{e(u_0)}$ is the level decomposition of a tree T of order at most b_1 with respect to a maximum degree vertex u_0. Moreover, suppose $|U_{\text{even}}| \geq d-1$.

The *open neighborhood* of a vertex v, denoted $N(v)$, is the set $\{u|u$ and v are adjacent$\}$. Let $n_i = |U_i|$ for $i = 0,\ldots,e(u_0)$, and suppose $U_i = \{u_{i,1},\ldots,u_{i,n_i}\}$. Moreover, let $N_{i,j} = N(u_{i,j}) \cap U_{i+1}$ for $i = 0,\ldots,e(u_0)-1$ and $j = 1,\ldots n_i$. The vertex $u_{i,j}$ is called the *parent* of $N_{i,j}$, while each vertex of $N_{i,j}$ is called a *child* of $u_{i,j}$. If $N(u_{i,j}) \cap U_{i-1} \neq \emptyset$, then $|N(u_{i,j}) \cap U_{i-1}| = 1$, and this vertex will be called the parent of $u_{i,j}$ and denoted $p_{i,j}$. Note that $u_{0,1} = u_0$.

The *sublevel vertices* of T are the vertices of the set $V(T) - (U_0 \cup U_1)$, which has cardinality at most $b_1 - \Delta - 1 = 2d-3$. If $v \in N_{i,j}$, then the cardinality of $N_{i,j}$ is called the *sibling number* of v, and is denoted $\mathrm{sib}(v)$.

If $u \in U_i$ and $v \in U_k$, where $i < k$, and u is joined to v by a path $P:\ (u=)w_1,w_2,\ldots,w_m(=v)$, where w_{i-1} is the parent of w_i for $i = 2,\ldots,m$, then u is called a *predecessor* of v, while v is called a *descendant* of u.

Lemma 3 *There is at most one more vertex besides u_0 with degree greater than d.*

Proof. Omitted. □

3 The Algorithm

The following procedure produces an FC labeling f of T with $\mathrm{span}(f) = b_0$ for the following case.

Case 1. The only vertex with degree larger than d, is u_0.

Among all vertices in U_1, choose one with the minimum number of descendants, say u. If u has at least two descendants, then T has at least $2|U_1| = 2\Delta \geq 2(d+1) = 2d+2$ sublevel vertices, which is a contradiction as there are $2d-3$ of these vertices. Thus u has zero or one descendant(s). The child of u (if it exists) is denoted u^*, is a leaf and is assigned the label $2d$ ($\leq d+\Delta-1 = b_0$) by f. Note that if u^* exists, then $|U_{\text{even}} - \{u^*\}| \geq |U_2 - \{u^*\}| \geq |U_1 - \{u\}| = \Delta - 1 \geq d$. If u^* does not exist, then, by assumption, $|U_{\text{even}} - \{u^*\}| = |U_{\text{even}}| \geq d-1$.

Step 1: Label u_0 and the principal vertices.

```
Let f(u_0) = 0.
Let E = U_even.
for i = 1 to d − 1 do
    begin
        Let v_i ∈ E such that
            (1) deg(v_i) is maximized, and
            (2) subject to (1) sib(v_i) is maximized.
        Let E = E − {v_i}.
        Let f(v_i) = i.
    end
```

We call the even level vertices that have been labeled so far the *principal vertices.*

Step 2: Label the remaining even level vertices.

```
If e(u_0) is odd, then let M = e(u_0) − 2; otherwise let M = e(u_0) − 1.

for i = 1 to M step 2 do
    for j = 1 to n_i do
    begin
        Let x_1, ..., x_s be the principal vertices of N_{i,j}.
        Arrange the remaining vertices of N_{i,j}, say y_1, ..., y_t,
         such that deg(y_1) ≥ ... ≥ deg(y_t).
        Label these vertices greedily in the order y_1, ..., y_t by
         using labels from the set{0, ..., s + t} − {f(x_1), ..., f(x_s), f(p_{i,j})}.
    end
```

For $i \in \{1, \ldots, e(u_0)\}$ and $j \in \{1, \ldots, n_i\}$, define the *maximum child label of* $u_{i,j}$, denoted $\text{mlab}(u_{i,j})$, as $\begin{cases} \max\{f(x) \mid x \in N_{i,j}\} & \text{if } u_{i,j} \text{ is not a leaf,} \\ -1 & \text{otherwise.} \end{cases}$

Step 3: Label the vertices of U_1.

Without loss of generality, arrange the vertices of U_1 such that $u_{1,\Delta} = u$ and $\text{mlab}(u_{1,1}) \geq \text{mlab}(u_{1,2}) \geq \ldots \geq \text{mlab}(u_{1,\Delta-1})$. We then let $f(u_{1,j}) = \Delta + d - j$ for $j = 1, \ldots, \Delta$.

Step 4: Label the odd level vertices of T.

```
If e(u_0) is odd, then let M = e(u_0) − 1; otherwise let M = e(u_0) − 2.

for i = 2 to M step 2 do
    for j = 1 to n_i do
    begin
        Arrange the vertices of N_{i,j}, say x_1, ..., x_t
         such that mlab(x_1) ≥ ... ≥ mlab(x_t).
```

```
    Label these vertices greedily in the order x_t, ..., x_1 by
     using labels from the set {b_0 - t, ..., b_0} - {f(p_{i,j})}.
end
```

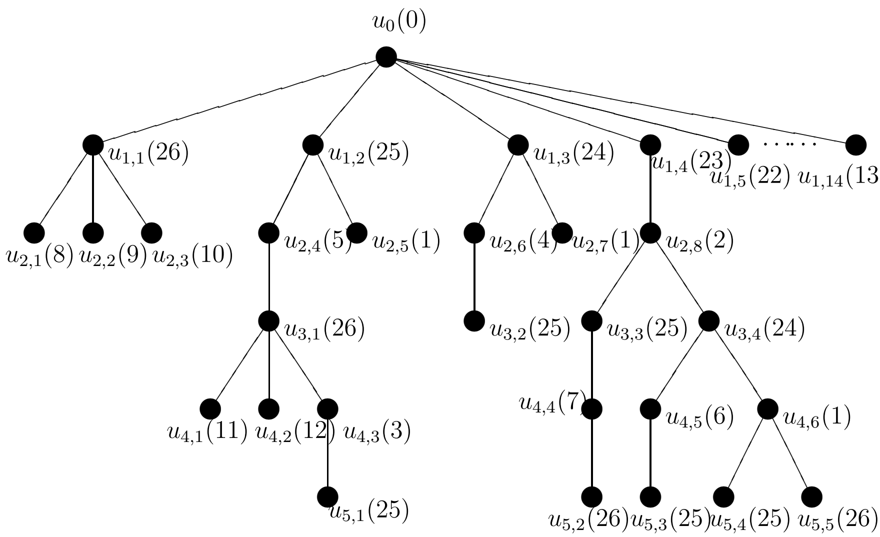

Figure 1: An example of a full $L(13, 1)$-colorable tree T with span $b_0 = 26$, order $b_1 = 38$, $\Delta = 14$, and with 23 sublevel vertices. The labels assigned by the algorithm appear in parentheses.

For the example tree T, shown in Figure 1, we choose $d = 13$ and $\Delta = 14$ so that $b_0 = 26$ and $b_1 = 38$. Note that T has order b_1.

We label u_0 by 0. The principal vertices are (in non-decreasing order of degree and sibling number) $u_{4,6}, u_{2,8}, u_{4,3}, u_{2,6}, u_{2,4}, u_{4,5}, u_{4,4}, u_{2,1}, u_{2,2}, u_{2,3}, u_{4,1}$ and $u_{4,2}$.

The remaining even level vertices are $u_{2,5}$ and $u_{2,7}$. For both vertices, the available labels are $0, 1$ and 2. Since both these vertices have a grandparent labeled 0, we assign the next available label 1 to each of them.

Next we label the vertices of U_1. The maximum child labels of vertices in U_1 are: $\text{mlab}(u_{1,1}) = 10$, $\text{mlab}(u_{1,2}) = 5$, $\text{mlab}(u_{1,3}) = 4$, $\text{mlab}(u_{1,4}) = 2$, $\text{mlab}(u_{1,5}) = \ldots = \text{mlab}(u_{1,14}) = -1$. We arrange the vertices of U_1 in order of non-increasing maximum child label. This gives us the order $u_{1,1}, u_{1,2}, \ldots, u_{1,14}$. We then assign $b_0, b_0 - 1, \ldots, d = 26, 25, \ldots, 13$ in this order to these vertices.

The labeling of the remaining vertices now takes place. We illustrate this labeling by looking at $u_{3,3}$ and $u_{3,4}$. Here $\text{mlab}(u_{3,3}) = 7$ and $\text{mlab}(u_{3,4}) = 6$. The labels available for $u_{3,3}$ and $u_{3,4}$ are $\{b_0 - 2, \ldots, b_0\} - \{f(u_{1,4})\} = \{24, 25, 26\} - \{23\} = \{24, 25, 26\}$. Thus, $u_{3,4}$ is assigned 24, while $u_{3,3}$ is assigned 25. As another illustration look at

$u_{5,4}$ and $u_{5,5}$. Here $\text{mlab}(u_{5,4}) = \text{mlab}(u_{5,5}) = -1$. The labels available for $u_{5,4}$ and $u_{5,5}$ are $\{b_0 - 2, \ldots, b_0\} - \{f(u_{3,4})\} = \{24, 25, 26\} - \{24\} = \{25, 26\}$. Thus, $u_{5,4}$ is assigned 25, while $u_{5,5}$ is assigned 26.

4 Correctness

We now prove that f is indeed a full-coloring of span b_0 for Case 1.

Observation 1 *If x and y are vertices in T such that $d(x, y) = 2$, then $f(x) \neq f(y)$.*

Observation 2 *The largest label assigned to any $v \in U_{\text{even}} - \{u^*\}$ is $d - 1$.*

Proof. Since $\deg(v) \leq d$ for any vertex $v \neq u_0$, the largest label assigned by Steps 1 and 2 will be $d - 1$. □

Observation 3 *Let $i \geq 3$ be odd. The smallest label assigned by f to $v \in U_i$ is Δ.*

Proof. Since $\deg(v) \leq d$ for any vertex $v \neq u_0$, the smallest label assigned by Step 4 will be Δ. □

Proposition 4 *Let i be a nonnegative even integer, let $x \in N_{i,j}$ and suppose vertex y is a child of x. Then $|f(x) - f(y)| \geq d$.*

Proof. Note that if $x \in N_{i,j}$ and vertex y is a child of x, then x is an odd level vertex, while y is an even level vertex of T. If $x = u$, and u^* exists, then $|f(x) - f(y)| = |f(u) - f(u^*)| = |d - 2d| = |-d| = d$. Let $N^*_{i,j} := N_{i,j} - \{u\}$.

Before proceeding with the proof, we establish five claims.

Claim 1 *Let $v \in N^*_{i,j}$ and suppose y'' is a child of v such that $f(y'') = \text{mlab}(v)$. If y'' is not a principal vertex, then $\deg(v) \geq \text{mlab}(v) + 1$.*

Proof. Partition $N(v) \cap U_{i+2}$ into s principal vertices and t non-principal vertices. According to Step 2, the largest label that could be assigned to a non-principal vertex is $s + t$. Thus, $\deg(v) = s + t + 1 \geq f(y'') + 1 = \text{mlab}(v) + 1$. ◇

Claim 2 *If $x \in N^*_{i,j}$, then $f(x) = b_0 - k$ where k is a nonnegative integer.*

Proof. As x is an odd level vertex, by Steps 3 and 4, the largest label that x could receive is b_0. Thus $f(x) = b_0 - k$ for some nonnegative integer. ◇

Claim 3 *Let $x \in N^*_{i,j}$ such that $f(x) = b_0 - k$ for some nonnegative integer k, and suppose y is a child of x. Then $k \leq 1$ and $f(x) - f(y) \geq d$ or $2 \leq k \leq \Delta - 2$ and $d + 1 \leq f(x) \leq b_0 - 2$.*

Proof. If $k \le 1$, then $f(x) \ge b_0 - 1$, and, by Observation 2, $f(x) - f(y) \ge b_0 - 1 - (d-1) = \Delta + d - 2 - (d-1) = \Delta - 1 \ge d$. For $k \ge 2$, if $i = 0$, then, by Step 3, $f(x) \ge d+1$, and if $i \ge 2$, then, by applying Observation 3 to $x \in U_{i+1}$, we have $f(x) \ge \Delta \ge d+1$. Thus, $d+1 \le f(x) \le b_0 - 2$ if $2 \le k \le \Delta - 2$. $\diamond$

Claim 4 *Let $x \in N^*_{i,j}$ such that $f(x) = b_0 - k$ for some integer $k \ge 2$. Then there exist either k or $k-1$ vertices in $N^*_{i,j}$ that are labeled greater than $f(x)$.*

Proof. Suppose $i = 0$. The labels used, according to Step 3, for vertices in $N^*_{i,j}$ are $d+1, \ldots, b_0$. By Claim 3, $f(x) = b_0 - k \ge d+1$, and so $f(x)$ appears on the aforementioned list of labels. The vertices with labels greater than $b_0 - k$ are the vertices with labels $b_0 - k + 1, \ldots, b_0$, and there are k of these vertices.

Thus, suppose $i \ge 2$. Let $t = |N^*_{i,j}|$. The possible labels for vertices in $N^*_{i,j}$, according to Step 4, are $b_0 - t, \ldots, b_0$, accounting for $t+1$ labels. Since $|\{f(v) \,|\, v \in N^*_{i,j}\}| = t$ and $\{f(v) \,|\, v \in N^*_{i,j}\} \subseteq \{b_0 - t, \ldots, b_0\} - \{f(p_{i,j})\}$, at most one of the numbers in $\{b_0 - t, \ldots, b_0\}$ is not used to label vertices in $N^*_{i,j}$. Hence, $|\{f(v) \,|\, v \in N^*_{i,j}, f(v) > b_0 - k\}| \ge (b_0 - (b_0 - k + 1) + 1) - 1 = k - 1$. $\diamond$

Let $N^{**}_{i,j} = \{x \in N^*_{i,j} \,|\, f(x) \le b_0 - 2\}$.

Claim 5 *Let $x \in N^{**}_{i,j}$ and suppose y is a child of x. Then $f(x) - f(y) \ge d$.*

Proof. Suppose, to the contrary, that there exists $x' \in N^{**}_{i,j}$ and a child y' of x' such that $f(x') - f(y') < d$. Since $\mathrm{mlab}(x') \ge f(y')$, it follows that $f(x') - \mathrm{mlab}(x') \le f(x') - f(y') < d$. Among all vertices $x' \in N^{**}_{i,j}$ with the property that $f(x') - \mathrm{mlab}(x') < d$, let x be one for which $f(x)$ is as large as possible.

Since $x \in N^{**}_{i,j}$, it follows that $f(x) = b_0 - k$ for some integer $k \ge 2$. By Claim 3, we have $2 \le k \le \Delta - 2$ and $d + 1 \le f(x) \le b_0 - 2$. Moreover, $f(x) - \mathrm{mlab}(x) \le d - 1$, and so $\mathrm{mlab}(x) \ge f(x) - d + 1 = b_0 - k - d + 1 = \Delta - k \ge 2$.

Let $V_{\ge f(x)} = \{v \in N^*_{i,j} \,|\, f(x) \le f(v)\}$. Then, by Claim 4, $|V_{\ge f(x)}| \in \{k, k+1\}$.

We now show that there exist distinct vertices v and w in $V_{\ge f(x)}$ such that $\mathrm{mlab}(v) = \mathrm{mlab}(w)$: Suppose $x = v_1, \ldots, v_K$ are the vertices of $V_{\ge f(x)}$, where $K \in \{k, k+1\}$. Suppose $f(v_1) < \ldots < f(v_K)$, and suppose, to the contrary, that $\mathrm{mlab}(x) = \mathrm{mlab}(v_1) < \ldots < \mathrm{mlab}(v_K)$. By Observation 2, the largest label assigned to any even level vertex, except for u^*, is $d-1$, and so $\Delta - k \le \mathrm{mlab}(x) \le \mathrm{mlab}(v_K) - K + 1 \le d - 1 - K + 1 \le d - 1 - k + 1 = d - k$, whence $d < \Delta \le d$, which is a contradiction.

Each $v \in V_{\ge f(x)}$ has a child, since $\mathrm{mlab}(v) \ge \mathrm{mlab}(x) \ge 2$. For each $v \in V_{\ge f(x)}$, let v' denote a child of v for which $f(v') = \mathrm{mlab}(v)$.

Let v and w be distinct vertices in $V_{\ge f(x)}$ such that $f(v') = f(w')$. Then v' and w' cannot both be principal vertices, since by Step 1 distinct principal vertices receive different labels. Thus, either both v' and w' are not principal vertices, or one, say v', is a principal vertex, while the other vertex, say w', is a non-principal vertex.

Note that $f(v') = f(w') \ge \mathrm{mlab}(x) = \Delta - k \ge 2$.

Case i. Both v' and w' are non-principal vertices. Therefore, by Claim 1, $\deg(v) \geq \text{mlab}(v)+1 \geq \text{mlab}(x)+1$, and so v has at least $\text{mlab}(x)$ descendants. Similarly, we conclude that w also has at least $\text{mlab}(x)$ descendants.

Let $\bar{P} = \{z \in V_{\geq f(x)} - \{v, w\} \mid z' \text{ is not a principal vertex }\}$, and let $P = V_{\geq f(x)} - \bar{P} - \{v, w\}$.

For every $z \in \bar{P}$, z' is not a principal vertex, and so, by Claim 1, $\deg(z) \geq \text{mlab}(z) + 1 \geq \text{mlab}(x) + 1 \geq 3$, and so z has at least two descendants.

Next, suppose $z \in P$, and so z' is a principal vertex. Then $1 \leq \deg(v') \leq \deg(z')$. If $\deg(z') > \deg(v') \geq 1$, then z' has at least one child, and so z has at least two descendants. Thus, $\deg(z') = \deg(v')$ and so $\text{sib}(z') \geq \text{sib}(v') \geq 2$ and again z has at least two descendants.

Let D be the set of all descendants of vertices of $V_{\geq f(x)}$. The vertices v and w each contributes at least $\text{mlab}(x)$ descendants to D, while each vertex of $V_{\geq f(x)} - \{v, w\}$ contributes at least two vertices to D. Thus, $|D| \geq 2\text{mlab}(x) + 2(K-2) \geq 2(\Delta - k) + 2(k-2) = 2\Delta - 4 \geq 2(d+1) - 4 = 2d-2$. But there are at most $2d-3$ sublevel vertices, which is a contradiction.

Case ii. v' is principal vertex, while w' is a non-principal vertex.

As w' is not a principal vertex, by Claim 1, $\deg(w) \geq \text{mlab}(w) + 1 \geq \text{mlab}(x) + 1$, and so w has at least $\text{mlab}(x) \geq 2$ children (and therefore descendants). Moreover, $\deg(v') \geq \deg(w')$.

Suppose $\deg(v') = \deg(w')$. Then $\text{sib}(v') \geq \text{sib}(w')$ and so $\deg(v) \geq \deg(w)$. As w has at least $\text{mlab}(x)$ children, vertex v will also have at least $\text{mlab}(x)$ descendants, and the result follows as before.

Thus, $\deg(v') > \deg(w') \geq 1$, and so v has at least two descendants. As before, each vertex in $V_{\geq f(x)}$ has at least two descendants.

Let $\bar{P}$, P and D be as before. Suppose $z \in \bar{P}$. Then z' is not a principal vertex, and by Claim 1, $\deg(z) \geq \text{mlab}(z) + 1 \geq \text{mlab}(x) + 1$, and so z has at least $\text{mlab}(x)$ descendants. Each vertex in $\{w, z\}$ has at least $\text{mlab}(x)$ descendants, while each vertex in $V_{\geq f(x)} - \{w, z\}$ has at least two descendants, and the result follows as before.

Thus, $\bar{P} = \emptyset$, whence $P = V_{\geq f(x)} - \{v, w\}$. Let $z \in P$. Then z' is a principal vertex of T, and so $\deg(z') \geq \deg(w')$.

If $\deg(z') = \deg(w')$, then $\text{sib}(z') \geq \text{sib}(w')$ and so $\deg(z) \geq \deg(w)$. As w has at least $\text{mlab}(x)$ children, vertex z will also have at least $\text{mlab}(x)$ descendants, and the result follows as before.

Thus, $\deg(z') > \deg(w') \geq 1$, and so $\deg(z') \geq 2$. Each vertex of $V_{\geq f(x)} - \{w\}$ contributes at least two vertices to D, while w contributes at least $\text{mlab}(x)$ vertices to D. Note that if $z \in V_{\geq f(x)} - \{w\}$, then z' is a principal vertex such that $f(z') = \text{mlab}(z) \geq \text{mlab}(x)$. Thus, for each principal vertex z' considered in the previous paragraph, we have $f(z') = \text{mlab}(z) \geq \text{mlab}(x)$. Since $\text{mlab}(x) \leq d-1$, we have

$\{1,\ldots,\text{mlab}(x)-1\} \subseteq \{1,\ldots,d-2\}$. Let P' be the set of principal vertices labeled from 1 to $\text{mlab}(x)-1$. Every vertex in P' contributes at least one more vertex to the total sublevel vertices. Therefore we have at least $2(K-1)+\text{mlab}(x)+\text{mlab}(x)-1 = 2K+2\text{mlab}(x)-3 \geq 2k+2(\Delta-k)-3 = 2\Delta-3 \geq 2(d+1)-3 = 2d-1$ sublevel vertices, which is a contradiction. Our claim follows. $\diamond$

As an immediate consequence of Claims 3 and 5, the proof of Proposition 4 is complete. $\square$

Proposition 5 *Let $y \in N_{i,j}$ with i an even nonnegative integer. Then $f(y) - f(u_{i,j}) \geq d$.*

Proof. Note that $f(y)-f(u_0) \geq d$ for any $y \in U_1$. Therefore we assume $i \geq 2$. Also note that $u_{i,j} \neq u^*$ since u^* has no children. Let $x = u_{i,j}$. Suppose, to the contrary, there exists $y \in N_{i,j}$ such that $f(y) - f(x) < d$. Before proceeding further, we prove two claims.

Claim 6 $2 \leq f(x) \leq d-1$ *and* $f(y) \leq f(x)+d-1$.

Proof. The upper bound on $f(x)$ follows from Observation 2. Since y is an odd level vertex, by Observation 3, $f(y) \geq \Delta \geq d+1$. If $f(x) \leq 1$, then certainly $f(y)-f(x) \geq d$. Since $f(y)-f(x) < d$, it follows that $f(y) \leq f(x)+d-1$. $\diamond$

Claim 7 $\deg(x) \geq \Delta - f(x) + 1$.

Proof. Consider the vertices of $N_{i,j}$ labeled at least $f(y)$. The possible labels of these vertices can be listed as $f(y), f(y)+1, \ldots, b_0$ with at most one of these labels not being assigned. Thus, x has at least $b_0 - f(y) \geq \Delta+d-1-(f(x)+d-1) = \Delta - f(x)$ children. Since x also has one parent, the result follows. $\diamond$

We now consider two cases:

Case i. x is a principal vertex. Then there exists at least $f(x)$ principal vertices, and since the principal vertices are labeled in non-increasing order with regard to their degrees, each of these $f(x)$ principal vertices has, according to Claim 7, at least degrees $\Delta - f(x)$ children. The principal vertices are also amongst the sublevel vertices. In total we have at least $f(x)(\Delta - f(x)) + (d-1)$ sublevel vertices. This gives us a total of $\Delta f(x) - f(x)^2 + d - 1 \geq (d+1)f(x) - f(x)^2 + d - 1 = f(x)d + f(x) - f(x)^2 + d - 1$ vertices. Define the function g by $g(z) = zd + z - z^2 + d - 1$, where, according to Claim 6, $2 \leq z \leq d-1$. Then $f'(z) = d+1-2z$, $f''(z) = -2$, while $f'(z) = 0$ if and only if $z = \frac{d+1}{2}$. As $d \geq 3$, the minimum value of $g(z)$ will occur at $z = 2$ or $z = d-1$. But $g(2) = 3d-3 > 2d-3$ and $g(d-1) = 3d-3 > 2d-3$. Therefore we have a contradiction with the fact that there are at most $2d-3$ sublevel vertices.

Case ii. x is not a principal vertex.

Then all the principal vertices must have at least $\Delta - f(x)$ children. Therefore the sublevel vertices number at least $(d-1)(\Delta - f(x)) + \Delta - f(x) = d(\Delta - f(x)) \geq d(d+1) - df(x) = d^2 + d(1 - f(x))$. Define the function g by $g(z) = d^2 + d(1-z)$, where $2 \leq z \leq d-1$. The minimum value of $g(z)$ will occur where z is a maximum, which is $g(d-1) = 2d > 2d-3$. Once again we have a contradiction. □

Label 0 is assigned to u_0, while labels $1, \ldots, d-1$ are assigned in Step 1. Step 3 assigns the labels $d, \ldots, b_0$.

The function f is thus a valid $(d,1)$-FC labeling of T with span b_0.

5 Modification of Algorithm and Correctness

We now modify the algorithm of Section 3 to produce a $(d,1)$-FC labeling of T with span b_0 for the following case.

Case 2. Assume that there exists $w \in V(T)$ where $w \neq u_0$ and $\deg(w) > d$. Recall that w is the only such vertex and $d+1 \leq \deg(w) \leq \Delta$.

Suppose $w \in N_{i,j}$ and suppose that at least $d-2$ of the children of w are not leaves. Then there are at least $2(d-2)+2 = 2d-2$ sublevel vertices, which is a contradiction. Thus, at most $d-3$ of the children of w are not leaves, while at least $\deg(w) - 1 - (d-3) = \deg(w) - d + 2$ of the children of w are leaves. Let C be any set of $\deg(w) - d$ leaves of w, and let $T^* = T - C$. Then T^* is a tree such that $\deg_{T^*}(w) = d$, and has order at most b_1. Note that u_0 is the only vertex in T^* with degree larger than d.

Suppose i is odd. Then w is an even level vertex, and so $C \cap U_{\text{even}}(T) = \emptyset$, whence $|U(T^*)_{\text{even}}| = |U(T)_{\text{even}}| \geq d-1$. We conclude that $T^* \in \mathcal{T}$. Besides w, there are at least $d-2$ other even level vertices. The even level vertices, as well as the children of w, are all contained in the sublevel vertices, and so we have at least $d-1+d = 2d-1$ sublevel vertices, which is a contradiction.

Hence, i is even, and so w is an odd level vertex. Then $U_{i+2} \cap N_{T^*}(w) \subseteq U(T^*)_{\text{even}}$, and so $|U(T^*)_{\text{even}}| \geq |U_{i+2} \cap N_{T^*}(w)| = \deg_{T^*}(w) - 1 = d-1$. We conclude that $T^* \in \mathcal{T}$. Let f^* be the labeling function on T^* as determined in Case (1). Suppose $C = \{c_1, \ldots, c_{|C|}\}$. Extend f^* on $V(T^*)$ to the function f on $V(T) = V(T^*) \cup C$ by letting $f(c_k) = d-1+k$ for $k = 1, \ldots, |C|$.

From now on $\text{mlab}_{f^*}(v)$ will refer to the maximum child label of v under f^* for any $v \in V(T^*)$. Note that $d-2 \leq \text{mlab}_{f^*}(w) \leq d-1$, since w has $d-1$ children in T^*.

Suppose w' is a sibling of w. We show that $\text{mlab}_{f^*}(w) > \text{mlab}_{f^*}(w')$: Suppose $\text{mlab}_{f^*}(w) = d-2$. Then the parent of w must have been assigned label $d-1$ by f^*. Since no child of w' is assigned the label $d-1$ by f^*, we must have $\text{mlab}_{f^*}(w') \leq d-2$.

Suppose $\text{mlab}_{f^*}(w') = d-2$. Let w_1' be the child of w' labeled $\text{mlab}_{f^*}(w')$ and let w_1 be the child of w labeled $\text{mlab}_{f^*}(w)$. If w_1' is a principal vertex, then w_1 is not a principal vertex so either $\deg(w_1') > \deg(w_1) \geq 1$ or $\deg(w_1') = \deg(w_1)$

and $\text{sib}(w_1') \geq \text{sib}(w_1)$. If $\deg(w_1') > \deg(w_1) \geq 1$, then the degree of any principal vertex labeled less than $d-2$ must also be at least $\deg(w_1') \geq 2$, and so we will have at least $2(d-2)+1+1 = 2d-2$ sublevel vertices, which is a contradiction. If $\text{sib}(w_1') \geq \text{sib}(w_1) = d-1$ then the children of w and w' will contribute at least $2d-2$ vertices to the sublevel vertices of T^*, which is a contradiction.

Therefore w_1' is not a principal vertex. Now, since the common parent of w and w' has been assigned $d-1$ by f^*, every label on the list $0,\ldots,d-2$ must have been assigned to the children of w', implying that $\deg_{T^*}(w') = d$ which is again a contradiction as before.

Next suppose $\text{mlab}_{f^*}(w) = d-1 = \text{mlab}_{f^*}(w')$. Then, by similar reasoning as before, every label on the list $0,\ldots,d-1$, except for the label assigned to the common parent of w and w', must have been assigned to the children of w', implying that $\deg_{T^*}(w') = d$, and once more we obtain a contradiction.

Since $\text{mlab}_{f^*}(w) > \text{mlab}_{f^*}(w')$ for every sibling w' of w, by Steps 3 and 4, $f^*(w) \in \{b_0-1, b_0\}$.

Define the function g on $V(T)$ by $g(v) = \begin{cases} b_0-1 & \text{if } f(v) = b_0, \\ b_0 & \text{if } f(v) = b_0-1, \\ f(v) & \text{otherwise.} \end{cases}$

Define the function h on $V(T)$ by $h = \begin{cases} f & \text{if } f(w) = b_0, \\ g & \text{if } f(w) = b_0-1. \end{cases}$

As an example, consider the tree T of Figure 2. We choose $d = 13$ and $\Delta = 14$ so that $b_0 = 26$. Note that $n(T) \leq b_1 = 38$. Assign reference labels $u_{i,j}$ to the vertices of T as shown. The figures below show the labelings of T^* and of T. The vertex $u_{3,1}$ is w, and $C = \{u_{4,13}\}$.

Since $f(w) = 25 = b_0-1$, labels b_0 and b_0-1 are swapped, while all other labels are left untouched. The tree T labeled by h appears in Figure 3.

We now prove that h is a valid $(d,1)$-FC labeling of T.

We do this by proving the following three lemmas.

Lemma 6 *The function h restricted to T^* is a valid $(d,1)$-FC labeling.*

Proof. The only vertices impacted by a label swap are those odd level vertices which received one of the labels b_0-1 or b_0, and so the result follows directly from Claim 3 and Claim 6. □

Lemma 7 *For any $v \in C$, $h(w) - h(v) \geq d$.*

Proof. Note that $h(w) - h(v) \geq b_0 - (\Delta - 1) = d$. □

Lemma 8 *No vertex $v \in C$ shares a label with any of its siblings or with the parent of w.*

Proof. The largest label received by any $v \in U_{\text{even}} - \{u^*\} - C$ is $d-1$. □

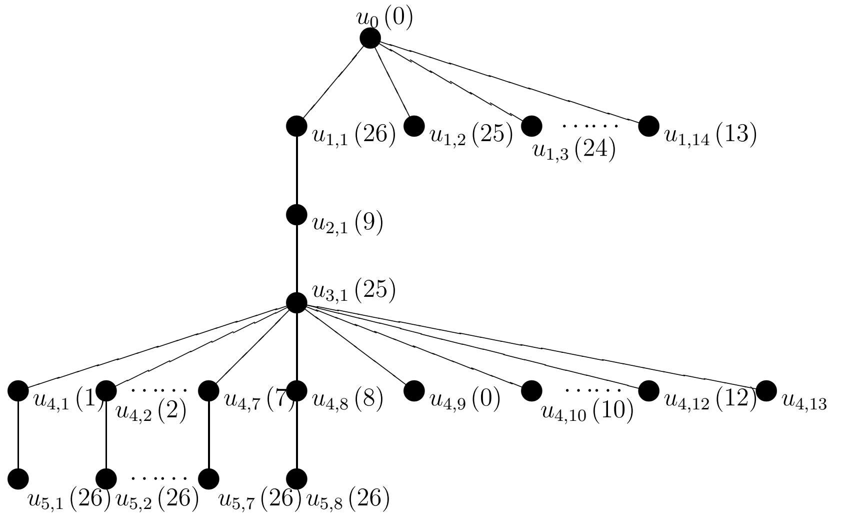

Figure 2: The labeling f^* of T^*. The vertex $u_{4,13}$ is the only vertex in C and remains unlabeled for now.

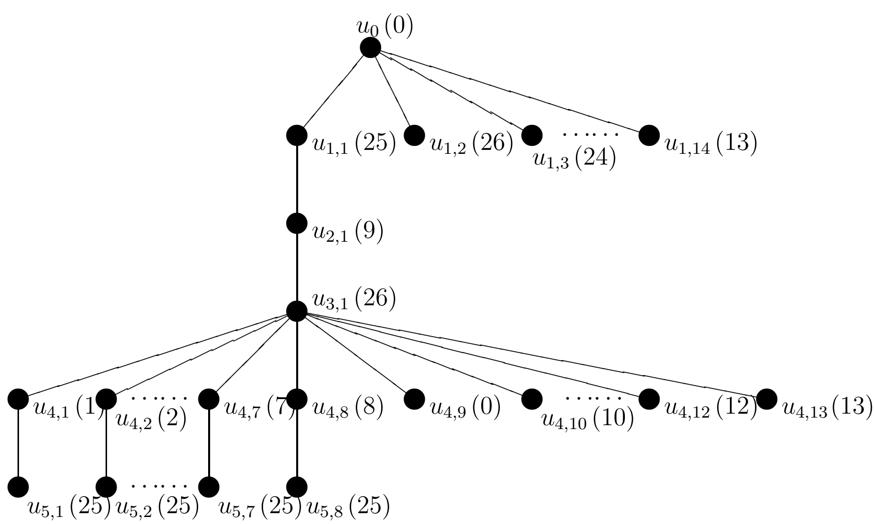

Figure 3: The labeling h of T.

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