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Edge protection in graphs

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Abstract

We use mobile guards on the vertices of a graph to defend it against an infinite sequence of attacks on its edges. A guard on an incident vertex moves across the attacked edge to defend it; other guards may also move to neighboring vertices. We prove upper and lower bounds on the minimum number of guards needed for this eternal vertex cover problem and characterize the graphs for which the upper bound is sharp.

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices. We use mobile guards to defend G against a sequence of attacks. A number of recent papers have considered various problems associated with defending the vertices of G against a sequence of attacks; see for instance [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One version of this problem is the *eternal domination problem* (also known as the *eternal security problem*): at most one guard is located at each vertex; a guard can protect the vertex where it is located and can move to a neighboring vertex to defend an attack there. The sequence of attacks is infinitely long and requires the configuration of guards induce a dominating set before and after each attack has been defended.

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In this paper we consider infinite sequences of attacks on edges rather than on vertices. To repel an attack, a guard from an incident vertex moves across the attacked edge. We call this the *eternal vertex cover problem.* A *vertex cover* of $G = (V, E)$ is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of u and v is in C. Let $\alpha(G)$ be the *vertex covering number* of G, the minimum number of vertices required to cover all edges of G.

The eternal protection problem has two variations, depending on whether one guard or all guards are allowed to move to repel an attack. When edges are attacked, the only model that appears interesting allows each guard to move (or not) across an incident edge when an edge is attacked: one guard moves to repel the attack and others may move to better configure themselves. As a simple example, consider an even cycle C_{2n} with vertices numbered 1 to $2n$. We initially have guards on all odd numbered vertices. When an edge is attacked, the guards rotate to all even numbered vertices. To be consistent with [5], we call this the m-eternal vertex cover problem.

The m-*eternal vertex covering number*, denoted $\alpha_{\mathrm{m}}^{\infty}(G)$, is the minimum number of guards required (at most one guard per vertex) to defend G against any sequence of attacks on one edge at a time, by moving a guard along the attacked edge to its other end-vertex; any number of guards may move (to a neighboring vertex) at once. This strategy requires that the set X of vertices containing guards is a vertex cover before and after each step. To repel an attack on the edge uv, where $u, v \in X$, the guards on u and v both move along uv, crossing along the way. Since such an attack can always be repelled without changing the configuration of guards, we only consider attacks on edges with one unguarded vertex.

In Section 3 we determine $\alpha_{\mathrm{m}}^{\infty}$ for paths and cycles, and a lower bound for $\alpha_{\mathrm{m}}^{\infty}$ for trees. In Section 4 we determine a number of general upper bounds for $\alpha_{\mathrm{m}}^{\infty}$; it follows from these bounds that the lower bound for trees established in Section 3 is exact for all trees. The class of graphs with $\alpha_{\mathrm{m}}^{\infty} = 2\alpha$ is characterized in Section 5, and graphs which satisfy $\alpha_{\mathrm{m}}^{\infty} = \alpha$ are discussed in Section 6. Several open problems are also mentioned.

2 Terminology

In general we use the notation and terminology of [3]. Let $\beta_1(G)$ denote number of edges in a maximum matching. A *cyclic vertex* of G is a vertex that lies on a cycle. An *end-block* of a graph with at least two blocks is a block that contains exactly one cut-vertex. An end-vertex of a tree is referred to as a *leaf*, while the vertex adjacent to a leaf is a *support vertex*. We use $L(T)$, abbreviated L, to denote set of leaves of the tree T. The vertices of $T - L$ are sometimes referred to as the *internal vertices* of T. A *branch vertex* of a tree is a vertex of degree at least three.

Denote the open and closed neighborhoods of $X \subseteq V$ by $N(X)$ and $N[X]$, respectively, and abbreviate $N(\{x\})$ and $N[\{x\}]$ to $N(x)$ and $N[x]$. The *external private neighborhood* $\mathrm{epn}(x, X)$ *of* $x \in X$ *relative to* X is defined by $\mathrm{epn}(x, X) =$

$N(x) - N[X - \{x\}]$. Let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V$.

If the vertex v is occupied by a guard, we denote this guard by $g(v)$. Some guards themselves will have labels; the vertex containing the guard t is then denoted $g^{-1}(t)$.

Obviously, $\alpha(G) \leq \alpha_{\mathrm{m}}^{\infty}(G)$ for all graphs. For bipartite graphs, every vertex in a minimum vertex cover is contained in a maximum matching, as guaranteed by the following well-known theorem.

Theorem 1 [3, Theorem 9.13] *If G is bipartite, then $\alpha(G) = \beta_1(G)$.*

3 Paths, cycles and trees

Proposition 2 *(i) For any $n \geq 3$, $\alpha_{\mathrm{m}}^{\infty}(C_n) = \alpha(C_n) = \lceil \frac{n}{2} \rceil$.*

(ii) For any $n \geq 1$, $\alpha_{\mathrm{m}}^{\infty}(P_n) = n - 1 = \begin{cases} 2\alpha(P_n) & \text{if } n \text{ is odd} \\ 2\alpha(P_n) - 1 & \text{if } n \text{ is even.} \end{cases}$

Proof. (i) Let D be any minimum vertex cover of C_n and place a guard on each vertex of D. To protect an edge e, move a guard on e along the edge and move all other guards in the same direction along the cycle.

(ii) If $n = 1$, then no guard is required, and if $n = 2$, then the result is also trivial, so assume $n \geq 3$. The upper bound is obvious, so we only prove the lower bound. Let $P_n = v_1, \ldots, v_n$ and suppose D is a minimum m-eternal vertex covering of P_n with $|D| \leq n - 2$. Note that when guards move to protect edges, at least one guard remains on v_{n-1} or v_n.

Either before or after an attack on the edge v_1v_2, there is a guard on v_1. Let i be the smallest index such that v_i is unoccupied. To defend an attack on v_iv_{i+1}, $g(v_{i+1})$ moves to v_i, and perhaps some guards on other vertices move to their neighbors with lower indices. Let j be the smallest index now such that v_j is unoccupied and repeat the process. But $|D| \leq n - 2$, so eventually we reach a point where there is a guard on v_{n-1} and no guard on v_{n-2} and v_n. Thus $g(v_{n-1})$ cannot move to v_{n-2} to protect $v_{n-2}v_{n-1}$, and no other guard can protect this edge, a contradiction. ■

We next prove a lower bound on the m-eternal covering number of trees. It will be shown in Section 4 that this lower bound is always satisfied at equality for trees.

Proposition 3 *For any nontrivial tree T, $\alpha_{\mathrm{m}}^{\infty}(T) \geq |V - L| + 1$.*

Proof. By Proposition 2(ii) the result is true for paths, so assume T is a tree with at least one branch vertex. Let D be any set of vertices of T of cardinality $|V - L|$ and place guards on all vertices in D.

Let ℓ be any leaf of T and root T at ℓ. Either before or after an attack on the edge incident with ℓ, ℓ contains a guard. Let b_0 be the branch vertex nearest to ℓ and let Q_0 be the $\ell - b_0$ path in T. If some vertex of Q_0 is unoccupied, let u_0 be the first such vertex, let v_0 be an occupied child of u_0 and defend an attack on u_0v_0 by moving $g(v_0)$ to u_0. Other guards may move, but no guard on a predecessor of u_0 moves. Repeated attacks lead to a configuration of guards in which every vertex of Q_0 is occupied. Amongst the descendants of b_0 there are $|L| - 1$ leaves and $|L|$ unoccupied vertices. Thus there is a child c_1 of b_0 such that the subtree T_1 of T induced by c_1 and its descendants contains more unoccupied vertices than leaves. Let b_1 the branch vertex of T_1 nearest to c_1 (if it exists) and Q_1 the $c_1 - b_1$ path in T_1. If some vertex on Q_1 is unoccupied, repeat the above attack-defense sequence until each vertex of Q_1 is occupied; note that no guard in $T - T_1$ can move to T_1 during this sequence. As in the case of b_0, b_1 has a child c_2 such that the subtree T_2 of T induced by c_2 and its descendants contains more unoccupied vertices than leaves. By repeating the process we eventually obtain a subtree T_k of T that contains more unoccupied vertices than leaves, but no branch vertices, i.e., T_k is a path with at least two unoccupied vertices. It follows as in the proof of Proposition 2(ii) that the guards on T_k do not protect T_k. Since no guards on $T - T_k$ can move to protect an edge joining an unoccupied vertex of T_k to its child, D does not protect T. ■

4 General bounds

In this section we show that no more than twice the vertex covering number of guards is required to m-eternally cover G. We characterize the extremal graphs for this bound in Section 5.

Theorem 4 *Let G be a nontrivial, connected graph and let D be a vertex cover of G such that $\langle D\rangle$ is connected. Then $\alpha_{\mathrm{m}}^{\infty}(G) \leq |D| + 1$.*

Proof. Let D satisfy the hypothesis of the theorem. If $D = V$, the result is trivial, so assume $V - D \neq \varnothing$. Let $d \in V - D$ be a vertex adjacent to some vertex in D and place guards on each vertex in $D \cup \{d\}$. Initially, the guard on d is called the *shadow guard*. The shadow guard will be denoted by s throughout the proof. After each defense another guard becomes the shadow guard and we write $g(v) = s$ to indicate that the guard on v is the shadow guard, and $g^{-1}(s) = v$ to indicate the vertex occupied by s.

We repeat the following protection strategy for each attack. Suppose an edge $e = uv$ is attacked. Assume there is a guard on u and let P be a $g^{-1}(s) - u$ path in $\langle D \cup \{g^{-1}(s)\}\rangle$. Move $g(u)$ to v, and move all the guards on P along its edges in the same direction. Let $g(v) = s$ (i.e., the guard on v becomes the shadow guard). In the resulting configuration of guards, each vertex in D contains a guard, and $\langle D \cup \{g^{-1}(s)\}\rangle$ is connected. It follows that we can defend all edges of G against any sequence of attacks. ■

The exactness of the lower bound for trees obtained in Proposition 3 now follows from Theorem 4.

Corollary 5 *For any tree T, $\alpha_{\mathrm{m}}^{\infty}(T) = |V - L| + 1$.*

Proof. The subtree of T induced by its internal vertices is connected. Let D consist of these vertices and any leaf of T and note that $|D| = |V - L| + 1$. ■

The number of components of the subgraph of G induced by a vertex cover X is related to the cardinality of a minimal superset of X that induces a connected subgraph of G. Hence we have the following corollary.

Corollary 6 *Let G be a nontrivial, connected graph and let X be a vertex cover of G such that $\langle X \rangle$ has k components. Then $\alpha_{\mathrm{m}}^{\infty}(G) \leq |X| + k$.*

Proof. Since X is a vertex cover, $V - X$ is independent. Since G is connected, every component of $\langle X \rangle$ is at distance two from some other component, and can be connected to this component by the addition of one vertex. Thus the components of $\langle X \rangle$ can be connected by the addition of at most $k - 1$ vertices. Hence there exists a set D of at most $|X| + k - 1$ vertices that contains X and induces a connected subgraph of G. The result now follows from Theorem 4. ■

Finally, we bound $\alpha_{\mathrm{m}}^{\infty}(G)$ in terms of the vertex covering number $\alpha(G)$.

Corollary 7 *For any nontrivial, connected graph G, $\alpha(G) \leq \alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G)$.*

Proof. The lower bound is trivial and the upper bound follows from Corollary 6 because a minimum vertex cover of G has at most $\alpha(G)$ components. ■

5 Extremal graphs for the upper bound

It is clear from the upper bounds in Section 4 that the largest possible value of $\alpha_{\mathrm{m}}^{\infty}(G)$ is $2\alpha(G)$. We now characterize the graphs that attain this bound. We begin by describing a classes $\mathcal{T}$, $\mathcal{G}$ and $\mathcal{H}$ of graphs with $\mathcal{T} \subset \mathcal{G} \subset \mathcal{H}$; the class $\mathcal{H}$ will prove to consist of all the nontrivial extremal graphs.

The class $\mathcal{T}$: Let T' be any tree of order at least three. Subdivide each edge of T' exactly once and then delete all leaves to form the tree T. (For example, if T' is a star, then $T \cong T'$, and if $T' = P_n$, then $T = P_{2n-3}$.) Let $\mathcal{T}$ be the class of all trees formed in this way. Then any $T \in \mathcal{T}$ has the following properties.

P1. All maximal paths in T have odd order (and even length).

P2. Any two branch vertices of T are an even distance apart.

P3. Each branch vertex of T is an odd distance from any leaf.

Conversely, any tree that satisfies **P1** – **P3** is in $\mathcal{T}$. The three conditions above are equivalent to the statement that T has a unique minimum vertex cover A, which is independent and obtained by considering the unique bipartition of T and choosing

A to be the partite set of smaller cardinality. Note that A contains all branch vertices, no leaves and thus all support vertices of T.

The class $\mathcal{G}$: For any $T \in \mathcal{T}$ with minimum vertex cover A, and each pair of vertices $x, y \in A$ such that $d(x, y) = 2$, add any number (including zero) new vertices and join each new vertex to both x and y to form the graph G. Then A is also the unique minimum vertex cover of G. Let $\mathcal{G}$ be the class of all graphs thus constructed; note that $\mathcal{T} \subset \mathcal{G}$.

The class $\mathcal{H}$: Let $G \in \mathcal{G}$ with minimum vertex cover A and define $A_\varepsilon = \{x \in A : x$ is at distance two from exactly one $y_x \in A\}$. Each vertex in A_ε is a support vertex of G. For any $x \in A_\varepsilon$ such that $\deg x = r \geq 3$, join all leaf neighbors of x to y_x. In the resulting graph, $\langle \{x, y_x\} \cup (N(x) \cap N(y_x)) \rangle$ is an end-block isomorphic to $K_{2,r}$. This process may be performed for any number of vertices in A_ε, provided the new graph H has $\delta(H) = 1$; note that A is the unique minimum vertex cover of H. Let $\mathcal{H}$ be the class of all graphs constructed in this way (including the graphs in $\mathcal{G}$).

Theorem 8 *$\alpha_{\mathrm{m}}^{\infty}(G) = 2\alpha(G)$ if and only if $G \in \mathcal{H}$.*

Proof. If $G = K_2$, then obviously $\alpha_{\mathrm{m}}^{\infty}(G) = 1 < 2\alpha(G)$, so assume G has order at least three.

We first show that $\alpha_{\mathrm{m}}^{\infty}(G) = 2\alpha(G)$ implies $G \in \mathcal{H}$. Let X be a minimum vertex cover of G such that $\langle X \rangle$ has as few components as possible. Then

$$X \text{ does not contain any end-vertices of } G, \qquad (1)$$

because if $v \in X$ is an end-vertex and u is its neighbor, then $u \notin X$ by the minimality of X. Hence each neighbor of u is in X, and since $\deg u \geq 2$, $\langle (X - \{v\}) \cup \{u\} \rangle$ has fewer components than $\langle X \rangle$.

If some vertex in $V - X$ is adjacent to at least three vertices in X, then there exists a set D with at most $2\alpha(G) - 2$ vertices that contains X and such that $\langle D \rangle$ is connected. By Theorem 4, $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - 1$. If $\langle X \rangle$ contains at least one edge, then $\langle X \rangle$ has at most $\alpha(G) - 1$ components, hence by Corollary 6, $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - 1$.

Assume henceforth that $X = \{x_1, \ldots, x_\alpha\}$ is independent (thus G is bipartite) and each vertex in $V - X$ is adjacent to one or two vertices in X (thus $1 \leq \deg v \leq 2$ for each $v \in V - X$). For $i = 1, \ldots, \alpha$, let $V_i = \mathrm{epn}(x_i, X)$ and for $i, j = 1, \ldots, \alpha$, $i \neq j$, let $V_{ij} = N(x_i) \cap N(x_j)$. By (1), each end-vertex of G is in a set V_i. Also, each block B of G is a K_2 or a $K_{2,r}$, $r \geq 2$, the cut-vertices of B (or the cut-vertex x and the vertex of B at distance two from x) being in X.

Let G^* be the graph obtained by deleting all vertices but one in each set V_i and $V_{i,j}$. Then G^* contains no 4-cycles. We consider three cases, depending on whether G^* is a tree or not, and when G^* is a tree, on the structure of the end-blocks of G.

Case 1 G^* has a cycle C. Then $G \notin \mathcal{H}$. We show that $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - 1$. Since G^* has no 4-cycles, $C \cong C_{2k}$ for some $k \geq 3$. By Theorem 1, G^* has a maximum matching that saturates each vertex in X. We may choose a matching $M = \{x_i y_i : i = 1, \ldots, \alpha\}$ such that at least one vertex $w \in X \cap V(C)$ is matched to another vertex w' of C (otherwise, if $x_i \in V(C)$, $y_i \notin V(C)$, let y_i' be a neighbor of x_i on C, note that y_i' is M-unsaturated, and choose the matching $(M - x_i y_i) \cup x_i y_i'$ instead). Let x_1 be the vertex of $X - \{w\}$ on C adjacent to w' and define $D = X \cup (Y - \{y_1\})$. Then $|D| = 2\alpha(G) - 1$. Place a guard on each x_i and a shadow guard for x_i on each y_i, $i \neq 1$.

Our protection strategy in G will maintain guards on each vertex x_i and a matching from X to the occupied vertices in $V(G) - X$ that saturates all but one vertex of X. The shadow guard s_i for each $g(x_i)$ is determined by the matching. The unsaturated vertex in X will not always be the same vertex, but will remain on C and throughout the protection process its shadowless guard will be denoted $\mathbf{u}$. The guard $\mathbf{u}$ will remain adjacent to the shadow of another guard on a vertex in $X \cap V(C)$; this guard $\mathbf{a}$ is the *anchor guard* for $\mathbf{u}$ and its shadow is denoted $\mathbf{s}$. Thus we must ensure that $\mathbf{u}, \mathbf{s}, \mathbf{a}$ form a path in G at all times. Initially, $g^{-1}(\mathbf{u}) = x_1$, $g^{-1}(\mathbf{s}) = w'$, $g^{-1}(\mathbf{a}) = w$ and $g^{-1}(s_i) = y_i$. Any configuration of guards that satisfies the above-mentioned conditions will be called *equivalent to D*.

We describe our defense strategy against an attack on the edge $x_i z_i$ of G, where $z_i \in V(G) - X$, in two subcases.

Subcase 1.1 x_i lies on some cycle of G^*. Consider G.

★ If $g(x_i) \notin \{\mathbf{u}, \mathbf{a}\}$, move $g(x_i)$ and s_i to z_i and x_i, respectively; redefine $s_i = g(z_i)$, i.e. the shadow guard for the (new) guard on x_i is on z_i and we are done.

★ If $g(x_i) = \mathbf{a}$, move $g(x_i)$ and s_i to z_i and x_i (destroying the path $\mathbf{u}, \mathbf{s}, \mathbf{a}$). Let x_j be the other vertex on C at distance two from $g^{-1}(\mathbf{u})$. Since C is not a 4-cycle, $x_j \neq x_i$.

- ⋆ If s_j is adjacent to $\mathbf{u}$, then x_j becomes the anchor, i.e., $g(x_j) = \mathbf{a}$ and $s_j = \mathbf{s}$.
- ⋆ If s_j is not adjacent to $\mathbf{u}$, move $g(x_j)$ to a neighbor of $\mathbf{u}$ and the guard s_j to x_j to become the anchor; after the move, $g(x_j) = \mathbf{a}$ and $s_j = \mathbf{s}$.

★ Now suppose $g(x_i) = \mathbf{u}$. Say $g(x_j) = \mathbf{a}$ and x_k is the other vertex of C at distance two from x_j; since C is not a 4-cycle, $x_k \neq x_i$. Move $\mathbf{u}$ and $\mathbf{s}$ to z_i and x_i, respectively.

- ⋆ If z_i is a common neighbour of x_i and x_j, we simply interchange the roles of $g(z_i)$ and $\mathbf{s}$.
- ⋆ Otherwise, either z_i is adjacent to the other vertex x_m, $m \neq j$, of C at distance two from x_i, or $z_i \notin V(C)$. Let $g(z_i) = s_i$ (so $g(x_i)$ obtains a shadow) and $g(x_j) = \mathbf{u}$ (the guard on x_j becomes the shadowless guard $\mathbf{u}$).

* If $g^{-1}(s_k)$ is adjacent to x_j, let $s_k = \mathbf{s}$ and $g(x_k) = \mathbf{a}$.
* If $g^{-1}(s_k)$ is not adjacent to x_j, move $g(x_k)$ and s_k to a neighbor of x_j and to x_k, respectively. After the move, $s_k = \mathbf{s}$ and $g(x_k) = \mathbf{a}$.

After all the guard movements above, each vertex in X contains a guard, each of these guards has an adjacent shadow guard, except one vertex $g^{-1}(\mathbf{u})$ on C; for this vertex there is an anchor $\mathbf{a}$ with $g^{-1}(\mathbf{a}) \in X$ whose shadow $\mathbf{s}$ is adjacent to $\mathbf{u}$. Thus the configuration of guards is equivalent to D.

Subcase 1.2 x_i does not lie on a cycle of G^*. Then s_i exists and also does not lie on a cycle of G^* because $1 \leq \deg g^{-1}(s_i) \leq 2$. If some vertex z with $N_G(z) = N_G(z_i)$ contains a guard, move $g(x_i)$ and $g(z)$ to z_i and x_i, respectively, obtaining a configuration of guards equivalent to D. Hence we assume this is not the case.

Let P be a path $v, \ldots, y_i, x_i, z, \ldots, w$ in G^*, where $N_G(z) = N_G(z_i)$ (possibly $z = z_i$), $N_G(y_i) = N_G(g^{-1}(s_i))$, v and w are end-vertices or cyclic vertices of G^* (not necessarily both of the same type, and possibly $v = y_i$ or $z = w$) and no internal vertices of P are cyclic vertices of G^*. All vertices of P at even distance from x_i are in X, and if v or w is a cyclic vertex of G^*, then it is in X (for then it has degree at least three). By the choice of D, P initially contains at most one unoccupied vertex, since at least $\left\lfloor \frac{|V(P)|}{2} \right\rfloor$ vertices of P belong to X and each of these vertices contains a guard that also has a shadow guard. But by assumption, z is unoccupied, so all other vertices of P contain guards. Move $g(x_i)$ to z_i and s_i to x_i.

★ If v is an end-vertex of G, move all guards on the $v - g^{-1}(s_i)$ subpath of P one vertex closer to $g^{-1}(s_i)$. Then each shadow guard becomes a guard on a vertex in X, and each guard previously on X becomes a shadow guard For the resulting configuration of guards there again exists a path similar to P with only one unoccupied vertex, namely v.

★ If v is an end-vertex of G^* but not of G, then $v = x_j$ for some j such that $V_j = \varnothing$. Now each vertex in X on the $v - x_i$ subpath P' of P contains a guard that also has a shadow guard on P', which begins and ends with a vertex in X. Thus there exist vertices $x_k, x_l \in V(P')$ such that s_k and s_l occupy vertices in V_{kl}. Without loss of generality assume x_k precedes x_l on P. After moving $g(x_i)$ to z_i and s_i to x_i, move all the guards on the $s_l - g^{-1}(s_i)$ subpath of P one vertex closer to $g^{-1}(s_i)$, move $g(x_k)$ to a vertex in V_{kl} and move s_k to x_k. For the new configuration of guards there exists a path similar to P and with all vertices containing guards.

★ Finally, if v is not an end-vertex of G^*, then v is a cyclic vertex of G^*. Say v lies on a cycle C'. Let v' be the vertex adjacent to v on P. Move all guards on the $v' - g^{-1}(s_i)$ subpath of P one vertex closer to $g^{-1}(s_i)$, and then move guards on v and C' according to the strategies described in Subcase 1.1. Again there now exists a path similar to P and with all vertices containing guards.

Since each new configuration of guards is equivalent to D, this concludes the proof of Subcase 1.2. It follows that if G^* has at least one cycle, then $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - 1$.

Case 2 G^* is acyclic and $\delta(G) \geq 2$, or $\delta(G) = 1$ and G has a 4-cycle as end-block. We prove that $\alpha_{\mathrm{m}}^{\infty}(G) < 2\alpha(G)$. Since G^* is acyclic, it has $\alpha(G) - 1$ paths of length two connecting vertices in X, i.e., $V_{ij} \neq \varnothing$ for exactly $\alpha(G) - 1$ pairs i, j. Place a guard on each x_i and a shadow guard s_{ij} on a vertex v_{ij} in each nonempty V_{ij}.

★ Assume firstly that $\delta(G) \geq 2$. Then $V_i = \varnothing$ for each $i = 1, \ldots, \alpha$. If a vertex z is unoccupied, then $z \in V_{ij}$ for some i, j. To repel an attack on the edge $x_i z$, move $g(x_i)$ to z and move s_{ij} to x_i. Hence G can be protected by $2\alpha(G) - 1$ guards. Note that $G \notin \mathcal{H}$ because $\delta(G) \geq 2$.

★ Now assume that $\delta(G) = 1$ and G has a 4-cycle as end-block. Say B is such a 4-cycle. The vertex of B at distance two from the cut-vertex is in X; say this vertex is x_1. While the leaves of G do not contain guards, the protecting strategy for edges incident with vertices in the sets V_{ij} is the same as above.

- ⋆ To repel an attack on a pendant edge $x_k \ell$, assume without loss of generality that $P = x_1, v_{1,2}, x_2, \ldots, v_{k-1,k} x_k, \ell$ is the $x_1 - \ell$ path in G^*. Then each vertex of P contains a guard, with the shadow guard s_{ij} on v_{ij}. In G, move the guards on all vertices of P except x_1 one vertex closer to ℓ. If an edge $x_2 z$ is attacked, with $z \in V_{1,2}$, reverse this guard movement.
- ⋆ While a leaf of G contains a guard, if there is an attack on an edge $x_m z$, where $z \in V_{mj}$ for some j and there is a shadow guard on a vertex in V_{mj}, repel the attack as before.
- ⋆ If there is an attack on a pendant edge $x_m z$ while some leaf (say ℓ as above) is occupied, note that for some $j = 2, \ldots, k$, there is a $z - x_j$ path Q' in G^*. Let Q be the $z - \ell$ path in G^* consisting of Q' followed by the $x_j - \ell$ subpath of P. Move all guards on Q one vertex closer to z.
- ⋆ Now assume the leaf ℓ, with the path P as above, contains a guard and there is an attack on an edge $x_1 z$. Move $g(x_1)$ to z, $g(x_2)$ to the other vertex in $V_{1,2}$ and reverse all the other guard movements made to protect the edge $x_k \ell$. This gives a new configuration of guards, but the protection strategy from this configuration is virtually the same as from the initial configuration: whenever one guard on $V_{1,2}$ moves to x_2, the other one moves to x_1.

In each case G can be protected by $2\alpha(G) - 1$ guards and so $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - 1$. Moreover, no graph G considered in this case is in $\mathcal{H}$, because either G has no end-vertices, or G has an end-block that is neither an edge (thus $G \notin \mathcal{G}$) nor isomorphic to $K_{2,m}$, $m \geq 3$ (thus $G \notin \mathcal{H}$).

Case 3 G^* is acyclic, $\delta(G) = 1$ and no end-block of G is a 4-cycle. Again we have two subcases, depending on whether G has a $K_{2,r}$, $r \geq 3$, as end-block or not. We prove that $G \in \mathcal{H}$ in each case.

Subcase 3.1 each end-block of G is a K_2. Since X is independent, contains no leaves of G or of G^* and each internal vertex of G^* not in X is adjacent to exactly two vertices in X, G^* has exactly $2\alpha - 1$ internal vertices. Moreover, G^* satisfies conditions **P1** − **P3** for trees in the class $\mathcal{T}$ and so $G^* \in \mathcal{T}$. Thus G can be constructed from G^* by joining $|V_i|-1$ new leaves to each vertex x_i to obtain another tree $T \in \mathcal{T}$, and then by joining each pair of vertices $x_i, x_j \in X$ with $d(x_i, x_j) = 2$ to $|V_{ij}| - 1$ new vertices. Hence $G \in \mathcal{G} \subseteq \mathcal{H}$.

Subcase 3.2 G has a $K_{2,r}$, $r \geq 3$, as end-block. For each such end-block B_i, let $x_i \in X$ be the vertex of G of degree r and $x_j \in X$ the (unique) vertex of G at distance two from x_i. Let $z_1, \ldots, z_k \in V_{ij}$ for some $1 \leq k < |V_{ij}|$, and delete the edges $z_1x_j, \ldots, z_kx_j$. Denote the resulting graph by G'. Define the sets V_i' and V_{ij}', and the graph G'^* as before. Then each end-block of G' is a K_2, and $V_{ij}' \neq \varnothing$ whenever $V_{ij} \neq \varnothing$. As in Subcase 3.1, $G'^* \in \mathcal{T}$ and $G' \in \mathcal{G}$. By the construction of the class $\mathcal{H}$, $G \in \mathcal{H}$.

This completes the proof that $\alpha_{\mathrm{m}}^{\infty}(G) = 2\alpha(G)$ implies $G \in \mathcal{H}$.

Conversely, we prove that $\alpha_{\mathrm{m}}^{\infty}(G) = 2\alpha(G)$ for each $G \in \mathcal{H}$. Consider any graph $G \in \mathcal{H}$ with minimum vertex cover $X = \{x_1, \ldots, x_\alpha\}$, and with the sets V_i and V_{ij} defined as above. For any $x_i, x_j \in X$ such that $d(x_i, x_j) = 2$, the subgraph of G induced by $\{x_i, x_j\} \cup V_{ij}$ is isomorphic to $K_{2,r}$, where $r = |V_{ij}| \geq 1$, and the subgraph induced by $\{x_i\} \cup V_i$ is isomorphic to the star $K_{1,r}$, $r = |V_i|$. When $V_{ij}, V_i \neq \varnothing$, we call the subgraphs $\langle \{x_i, x_j\} \cup V_{ij} \rangle$ and $\langle \{x_i\} \cup V_i \rangle$ the *pieces* of G. The vertices in X are either cut-vertices of G, or vertices of degree $r \geq 3$ that belong to a piece $K_{2,r}$. Since the graph G^* (defined as before) is a tree, the pieces of G form a tree-like structure. A vertex $x \in X$ is called a *branch vertex* if it is contained in at least three pieces. An *end-piece* is a piece that contains exactly one cut-vertex, i.e., a piece isomorphic to $K_{1,r}$, or to a $K_{2,r}$ that contains a vertex x_i of degree $r \geq 3$. Because of the tree-like structure of G, we may root G at an end-vertex v by directing all edges of G away from v. The *levels* of the rooted graph G correspond to the levels of the tree G^* rooted at v.

Let D be any set of $2\alpha(G) - 1$ vertices of G and place a guard on each vertex in D. A piece has a *fair allocation of guards* if it is a star and has one guard, or it is a $K_{2,r}$ and has three guards. A piece S has *guard discrepancy* k if k equals the number of guards on S minus the number of guards in a fair allocation of S. Note that k may be positive, negative or zero. We shall consider *piece-subgraphs* of G consisting of pieces of G and of stars $K_{1,r}$ obtained by deleting one vertex of a piece $K_{2,r}$. A *path of pieces* is a connected piece-subgraph in which each cut-vertex is contained in exactly two pieces. We say a piece-subgraph H is *depleted* (respectively *neutral* or *advantaged*) if the sum of the guard discrepancies over the pieces of H is negative (respectively zero or positive). If G is rooted and H is a depleted piece (piece-subgraph consisting of one piece) in G, then one or more levels of H do not contain guards. Such a level is then called a *depleted level*.

Since G^* has exactly $\alpha(G) - 1$ vertices with two neighbors in X, and $|X| = \alpha$, $2\alpha(G) - 1$ guards can be placed on G^*, and hence on G, such that each piece has zero guard discrepancy. Since guard movements do not change the total guard discrepancy over G, we deduce that G is neutral after any guard movement.

With all these definitions in place, we show that $2\alpha(G) - 1$ guards do not protect G. Suppose to the contrary that D protects G. We follow the proof of Proposition 3. Since $\delta(G) = 1$, G has an end-vertex ℓ. Root G at ℓ. Either before or after an attack on the edge incident with ℓ, ℓ contains a guard. If G has branch vertices, let b_0 be the branch vertex nearest to ℓ. Then $b_0 \in X$. Let Q_0 be the $\ell - b_0$ path of pieces in G.

If some piece of Q_0 is depleted, let S_0 be the depleted piece nearest to ℓ and let u_0 be a vertex in a depleted level of S_0; since D is a vertex cover, u_0 has an occupied child. Since all children of u_0 are in the same orbit of the automorphism group of G, let v_0 be any occupied child of u_0. The only way to repel an attack on u_0v_0 is by moving $g(v_0)$ to u_0. Other guards may move, but no guard on a predecessor of u_0 moves to a lower level (i.e., further away from the root) than that of u_0, and thus only guards on descendants of u_0 can protect an edge on a lower level than that of u_0. Repeated attacks lead to a configuration of guards in which no piece of Q_0 is depleted and the piece containing ℓ is advantaged. Since G is neutral, there is a child c_1 of b_0 such that the piece-subgraph G_1 of G induced by the star containing c_1, and by the descendants of c_1, is depleted. Let b_1 the branch vertex of G_1 nearest to c_1 (if it exists) and Q_1 the $c_1 - b_1$ path of pieces in G_1. If some piece S_1 of Q_1 is depleted, let u_1 be a vertex in a depleted level of S_1 and repeat the above attack-defense sequence until no piece of Q_1 is depleted; note that no guard on a vertex in $G - G_1$ can move to a lower level of G_1 than that of u_1 during this sequence. As in the case of b_0, b_1 has a child c_2 such that the piece-subgraph G_2 of G induced by the star containing c_2, and by the descendants of c_2, is depleted.

By repeating the process we eventually obtain a depleted piece-subgraph G_k of G that contains no branch vertices, i.e., G_k is a path of pieces. Assume without loss of generality that the piece S of G_k that is depleted is the end-piece of G contained in G_k; otherwise, we may defend edges in preceding pieces in the same way as before. If $S \cong K_{1,r}$, then S contains no guards and so the pendant edge(s) are not covered by a guard. If $S \cong K_{2,r}$, then $r \geq 3$ and S contains at most two guards. Let x and y be the vertices of degree r in S, where x also has degree r in G, i.e. x is the vertex of G_k at maximum distance from ℓ, and let $N(x) \cap N(y) = \{z_1, \ldots, z_r\}$. If x has no guard, then at most two vertices in $\{z_1, \ldots, z_r\}$ have guards, and since $r \geq 3$ it follows that some edge xz_i is unguarded. Hence assume x contains a guard. If (say) z_1 has a guard, then y has no guard (S is depleted) and so yz_2 is not covered by a guard. Hence no z_i has a guard; the guards on S are on x and y. To repel an attack on xz_1, $g(x)$ moves to z_1. But then xz_2 and xz_3 are not covered, so guards must move to cover these edges. But only $g(y)$ can move to these edges, and only to one of xz_2 and xz_3. Hence at least one of these edges cannot be covered. We have therefore shown that $2\alpha(G) - 1$ guards cannot protect G. Hence $\alpha_{\mathrm{m}}^{\infty}(G) = 2\alpha(G)$ and the proof is complete. ■

Question 1 *If G^* has k cycles, is $\alpha_{\mathrm{m}}^{\infty}(G) \leq 2\alpha(G) - k$?*

6 Lower bound

Our main question in this section is the following.

Question 2 *For which graphs is $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$?*

An elegant characterization resolving this question seems difficult, but a sufficient condition is given in the next result.

Proposition 9 *If G has two disjoint minimum vertex covers and each edge of G is contained in a maximum matching, then $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$.*

Proof. If G has two disjoint minimum vertex covers A_1 and A_2, then both A_1 and A_2 are independent and each edge joins a vertex in A_1 to a vertex in A_2. Thus G is bipartite and so by Theorem 1 there is a perfect matching between A_1 and A_2. Place a guard on each vertex in A_1. To defend an edge uv, where $u \in A_1$, let M be a perfect matching containing uv. Then all guards move from A_1 to A_2 along the edges of M. This strategy can be repeated indefinitely and so $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$. ■

Both the existence of disjoint minimum vertex covers and the property that each edge is contained in a maximum matching are required to ensure that $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$. The paths P_{2n} have disjoint minimum vertex covers but a unique maximum matching, i.e., not every edge is contained in a maximum matching, and by Corollary 5, $\alpha_{\mathrm{m}}^{\infty}(P_{2n}) = 2n - 1 > \alpha(P_{2n})$ for $n \geq 2$. On the other hand, if G is the graph consisting of two triangles which share one vertex v, then every edge of G is contained in a maximum matching ($\beta_1(G) = 2$), G has distinct but not disjoint minimum vertex covers consisting of v and one other vertex of each triangle, so $\alpha(G) = 3$, but an attack on an edge incident with v cannot be defended by the guards in any minimum vertex cover. Odd cycles show that the conditions are not necessary for G to satisfy $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$; they do not have disjoint minimum vertex covers but $\alpha_{\mathrm{m}}^{\infty}(C_n) = \alpha(C_n)$ for all n.

Question 3 *Is it true that if $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$, then each edge of G is contained in a maximum matching?*

An interesting class of graphs to examine is vertex transitive graphs. Let $G \times H$ denote the Cartesian product of G and H. The following is easy to verify.

Fact 10 *Each graph in the following classes satisfies $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$.*

(*i*) *K_n*

(*ii*) *Petersen graph*

(*iii*) $K_m \times K_n$

(*iv*) $C_m \times C_n$

(*v*) *Circulant graphs (to repel an attack along the edge uv, move (say) $g(u)$ to v and move each other guard along its incident edge that corresponds to uv in the same orientation of the cycle).*

A natural question then is the following.

Question 4 *Do all vertex transitive graphs G have $\alpha_{\rm m}^{\infty}(G) = \alpha(G)$?*

We now exhibit a family of graphs that are not vertex transitive, but which satisfy $\alpha_{\rm m}^{\infty}(G) = \alpha(G)$ (see Theorem 11(*ii*)). We sometimes call $P_n \times P_m$ the *$n \times m$ grid graph.*

Theorem 11 (*i*) $\alpha_{\rm m}^{\infty}(P_1 \times P_n) = n - 1$.

(*ii*) *If n is even, then $\alpha_{\rm m}^{\infty}(P_n \times P_m) = \frac{nm}{2} = \alpha(P_n \times P_m)$.*

(*iii*) *If $n, m > 1$ are odd, $n \geq m$, then $\alpha_{\rm m}^{\infty}(P_n \times P_m) = \lceil \frac{nm}{2} \rceil = \alpha(P_n \times P_m) + 1$.*

Proof. Statement (*i*) is Proposition 2.

(*ii*) If n is even, then $G = P_n \times P_m$ is Hamiltonian. By choosing alternative vertices on a Hamilton cycle we obtain a minimum vertex cover; the remaining $\frac{nm}{2}$ vertices are also a minimum vertex cover. By placing the nm vertices of G in the form of an $n \times m$ grid and rotating and reflecting Hamilton cycles (the number of rotations and reflections depend on the parity of m), we note that every edge of G, $n \geq 4$, lies on a Hamilton cycle and is thus contained in a perfect matching. Hence by Proposition 9, $\alpha_{\rm m}^{\infty}(G) = \alpha(G) = \frac{nm}{2}$.

(*iii*) Let $G = P_n \times P_m$ and note that G has a unique minimum vertex cover A of cardinality $\lfloor \frac{nm}{2} \rfloor$ which consists of the vertices at odd distance from the vertices of degree two, while $B = V - A$ is a vertex cover of cardinality $\lceil \frac{nm}{2} \rceil$. Call the vertices in A the *A-vertices*, and the vertices in B the *B-vertices.* Label the vertices in row i of the grid by $v_{i1}, \ldots, v_{im}$, $i = 1, \ldots, n$; the vertices whose indices sum to odd (respectively even) numbers are A-vertices (respectively B-vertices). Let b_0 be any B-vertex and $G_0 = G - b_0$. The sets $A \subseteq V(G_0)$ and $B_0 = V(G_0) - A = B - \{b_0\}$ are minimum vertex covers of G_0. Then G_0 is Hamiltonian and by considering various Hamilton cycles of G_0 we note that every edge of G_0 lies on a Hamilton cycle and hence is contained in a maximum matching of G_0. [Two Hamilton cycles of $(P_5 \times P_5) - v_{2,2}$ are illustrated in Figure 1 (a) and (b). Superimposing the two cycles and then reflecting in the southeast – northwest diagonal covers all edges of $(P_5 \times P_5) - v_{2,2}$.]

In G, initially place guards on all the A-vertices and on b_0. By Proposition 9 the edges in G_0 can be protected without using $g(b_0)$. Now consider an attack on an edge ub_0. While there is a guard on b_0 we only need to consider an attack when the other guards in G are on B-vertices. [See Figure 1 (c).] Let $b_1 \neq b_0$ be a B-vertex adjacent to u and let M_0 be a perfect matching of G_0 that contains ub_1. Move $g(b_0)$ to u and move all guards on B-vertices other than b_1 to A-vertices according to M_0. (Hence we use all edges of M_0 except the edge ub_1.) Now all guards in G are on A-vertices, except that there is a guard on the B-vertex b_1. [See Figure 1 (d).] Define $G_1 = G - b_1$ and $B_1 = V(G_1) - A = B - \{b_1\}$ and repeat the above-mentioned strategy. It is clear that the $\lceil \frac{nm}{2} \rceil$ guards can protect G against any sequence of attacks. Since A is the unique minimum vertex cover of G and any movement of guards to defend an edge destroys this vertex cover, $\lfloor \frac{nm}{2} \rfloor$ guards cannot protect G and the result follows. ∎

We saw graphs G, H in Theorem 11(ii) that require more guards than their vertex cover number, yet $\alpha_{\mathrm{m}}^{\infty}(G \times H) = \alpha(G \times H)$. The next question is whether $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$ implies that $\alpha_{\mathrm{m}}^{\infty}(G \times H) = \alpha(G \times H)$.

Proposition 12 *For $n \geq 2$, $\alpha(K_n \times G) = \alpha_{\mathrm{m}}^{\infty}(K_n \times G)$.*

Proof. View $K_n \times G$ as consisting of n copies $G_1, \ldots, G_n$ of G, where corresponding vertices in any two copies are joined by a perfect matching. Denote the copy of $v \in V(G)$ in G_i by v_i. Place guards on a minimum vertex cover of $K_n \times G$. An attack on an edge between G_i and G_j is handled by moving all guards from G_i to their corresponding vertices in G_j and vice versa (some guards may cross in the process). Consider an attack on an edge $u_i v_i$ of G_i; assume without loss of generality that $i = 1$ (attacks on G_i, $i \geq 2$, are treated identically) and that there is a guard on u_1. Since v_1 has no guard, v_i has a guard for each $i \geq 2$. Consider the 4-cycle u_1, v_1, v_2, u_2, u_1 and note that u_2 may or may not contain a guard. Move $g(u_1)$, $g(u_2)$ (if it exists) and $g(v_2)$ to v_1, u_1 and u_2 respectively, and switch all other guards between G_1 and G_2. The new guard configuration on G_1 is the same as the former configuration on G_2 and vice versa, thus the guards on $K_n \times G$ form a vertex cover. ∎

Using the same ideas (but involving all copies of G in each movement of guards instead of just two), it is simple to prove the following.

Proposition 13 *$\alpha(C_{2n} \times G) = \alpha_{\mathrm{m}}^{\infty}(C_{2n} \times G)$.*

Question 5 *Let G and H be graphs such that $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$. Is it true that $\alpha_{\mathrm{m}}^{\infty}(G \times H) = \alpha(G \times H)$?*

We conjecture that the following weaker result is true.

Conjecture 6 *Let G and H be graphs such that $\alpha_{\mathrm{m}}^{\infty}(G) = \alpha(G)$ and $\alpha_{\mathrm{m}}^{\infty}(H) = \alpha(H)$. Then $\alpha_{\mathrm{m}}^{\infty}(G \times H) = \alpha(G \times H)$.*

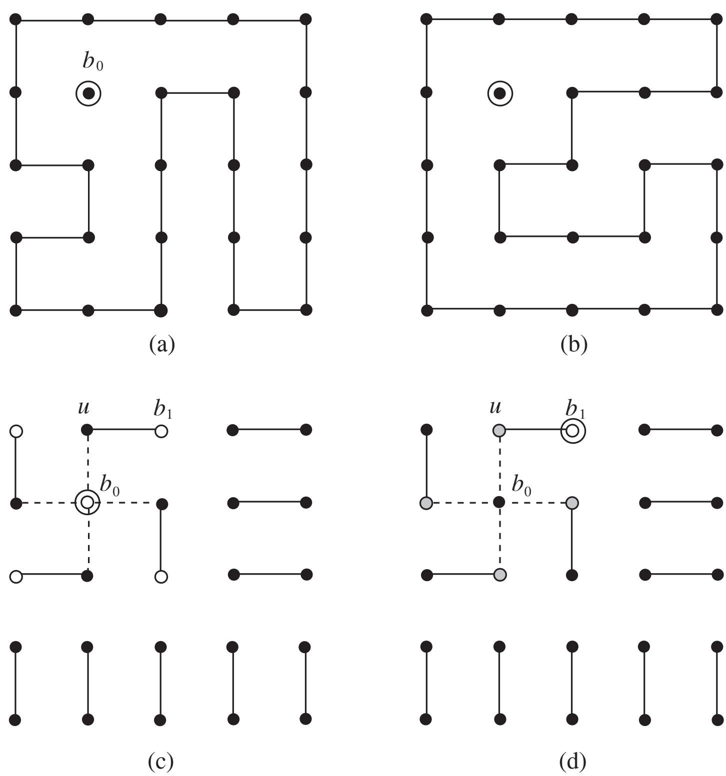

Figure 1: Hamilton cycles, matchings and guards in $(P_5 \times P_5) - v_{2,2}$

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