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Locating-total domination critical graphs

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Abstract

A locating-total dominating set of a graph $G = (V(G), E(G))$ with no isolated vertex is a set $S \subseteq V(G)$ such that every vertex of $V(G)$ is adjacent to a vertex of S and for every pair of distinct vertices u and v in $V(G) - S$, $N(u) \cap S \neq N(v) \cap S$. Let $\gamma_t^L(G)$ be the minimum cardinality of a locating-total dominating set of G. A graph G is said to be locating-total domination vertex critical if for every vertex w that is not a support vertex, $\gamma_t^L(G-w) < \gamma_t^L(G)$. Locating-total domination edge critical graphs are defined similarly. In this paper, we study locating-total domination critical graphs.

1 Introduction

Given a simple graph G with vertex set $V(G)$ and edge set $E(G)$, the *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E(G)\}$, the *closed neighborhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$ and the *degree* of v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A subset S of vertices of $V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ is adjacent to a vertex in S. The *total domination number*, $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. For more details on domination and its variations in graphs see [6].

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A total dominating set S in a connected graph G without isolated vertices is a *locating-total dominating set*, or just LTDS, if for any two vertices x, y of $V(G)\backslash S$, $N(x) \cap S \neq N(y) \cap S$. The *locating-total domination number* $\gamma_t^L(G)$ is the minimum cardinality of a LTDS of G. Note that locating domination was introduced by Slater [7, 8] and locating-total domination was introduced by Haynes, Henning and Howard [5]. Also for recent studies on locating domination and locating-total domination we cite [1], [2] and [4].

In this paper we study the effects on decreasing locating-total domination number when a vertex or an edge is deleted. Such problems have been considered before for some domination parameters. Indeed, Brigham et al. [3] were the first to introduce vertex critical graphs for the domination number and Sumner and Blitch [9] were the first to introduce edge critical graphs for the domination number. Before presenting the main results, we need to introduce some additional definitions and notation.

A vertex with degree one in a graph G is called a *leaf*, and its neighbor is called its *support*. An edge incident to a leaf in a graph G is called a *pendant edge*. We let $S(G)$ and $L(G)$ be the set of support vertices and leaves of G, respectively. We denote by $L_x(G)$ the set of leaves adjacent to the support vertex x in G. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set A. The *diameter* $\text{diam}(G)$ of a graph G is the maximum distance over all pairs of vertices of G. The *corona, $Cor(G)$*, of a graph G is the graph obtained from G by adding a pendant edge to each vertex of G. We denote by K_n the *complete graph* of order n, and by $K_{m,n}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = m$ and $|Y| = n$. A *star* of order $n+1$ is $K_{1,n}$. The *path* and the *cycle* on n vertices are denoted by P_n and C_n, respectively.

2 γ_t^L-vertex-critical graphs

We begin with the following observation.

Observation 1 *If v is a support vertex of G, then v belongs to every $\gamma_t^L(G)$-set. Moreover, every $\gamma_t^L(G)$-set contains all leaves adjacent to v or all except one.*

Next we show that the removal of a vertex of a graph G that does not produce a graph with isolated vertices can decrease the locating-total domination number of G by at most one.

Theorem 2 *If G' is a graph obtained from G by removing a vertex $x \in V(G)\backslash S(G)$, then $\gamma_t^L(G) - 1 \leq \gamma_t^L(G')$. However the difference $\gamma_t^L(G') - \gamma_t^L(G)$ can be arbitrarily large.*

Proof. Let G' be a graph obtained from G by removing a vertex $x \in V(G)\backslash S(G)$ and let S' be a $\gamma_t^L(G')$-set. If $N_G(x) \cap S' = \emptyset$, let $S = S' \cup \{y\}$ where $y \in N(x)$. If $|N_G(x) \cap S'| \geq 1$, let $S = S' \cup \{x\}$. In both cases, S is a LTDS of G, and so $\gamma_t^L(G) \leq |S| = |S'| + 1 = \gamma_t^L(G') + 1$.

Now consider a graph G obtained by $p \geq 1$ stars $K_{1,3}$ of centers $v_1, v_2, \ldots, v_p$, and a star $K_{1,p+1}$ of center x and leaves $y_1, y_2, \ldots, y_{p+1}$ by adding edges y_iv_i, $1 \leq i \leq p$

and subdividing once the edge xy_{p+1}. Then G is a tree and $\gamma_t^L(G) = 3p+2$. Remove the vertex x from G and let G' be the resulting graph. Clearly $\gamma_t^L(G') = 4p+2$ and so $\gamma_t^L(G') - \gamma_t^L(G) = p$. ■

It follows from the previous theorem that for a vertex $x \in V(G)\backslash S(G)$, $\gamma_t^L(G-x)$ cannot be bounded above by $\gamma_t^L(G)$. But if a $\gamma_t^L(G)$-set does not contain x, then $\gamma_t^L(G') \leq \gamma_t^L(G)$. We call a graph G to be *locating-total domination vertex critical*, or just *γ_t^L-vertex-critical*, if for each vertex x of $V(G)\backslash S(G)$, $\gamma_t^L(G-x) < \gamma_t^L(G)$. If G is a γ_t^L-vertex-critical graph and $\gamma_t^L(G) = k$, we call it a *k-γ_t^L-vertex-critical graph.* It follows from Theorem 2, that in a γ_t^L-vertex-critical graph G, $\gamma_t^L(G-v) = \gamma_t^L(G) - 1$ for every $v \in V(G)\backslash S(G)$. Trivially a path P_2 is a γ_t^L-vertex-critical graph. So we consider graphs of order at least three. Also if G is a connected graph with minimum degree at least two, then $Cor(G)$ is γ_t^L-vertex-critical. On the other hand, a disconnected graph G is γ_t^L-vertex-critical if and only if each component of G is γ_t^L-vertex-critical. So only connected graphs are considered in this paper.

The next proposition summarizes for some classes of graphs those that are γ_t^L-vertex-critical.

Proposition 3 *(1) K_n is γ_t^L-vertex-critical if and only if $n \geq 4$.*
(2) No path of order at least three is γ_t^L-vertex-critical.
(3) C_n is γ_t^L-vertex-critical if and only if $n \equiv 1, 2$ (mod 4).
(4) $K_{m,n}$ is γ_t^L-vertex-critical if and only if
either $\min\{m,n\} \geq 3$ *or* $\min\{m,n\} = 1$, $\max\{m,n\} \geq 3$.

Proof. We only prove (2) and (3). Assume to the contrary that P_n, with $n \geq 3$, is a γ_t^L-vertex-critical path of order n. It is obvious that $n \geq 5$. Let v be the non-leaf neighbor of a support vertex of P_n and D any $\gamma_t^L(P_n - v)$-set. Then D is a LTDS of P_n and so $\gamma_t^L(P_n) \leq \gamma_t^L(P_n - v)$, a contradiction.

To prove (3) we just need to use the following result that can be found in [6], for $n \geq 2$, $\gamma_t^L(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$. This result is also valid for cycle C_n. ■

Now we give extremal graphs G with $\gamma_t^L(G) = 2$ that we obtain by checking all possibilities for a graph G with $\gamma_t^L(G) = 2$. We define the family $\mathcal{E} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{P_2, P_3, P_4, C_4, K_4 - \{e\}\}$, where:

- $\mathcal{F}_1$ is the class of all graphs obtained from a path P_4 by adding a new vertex x and joining x to the central vertices and to at least one leaf of the path.

- $\mathcal{F}_2$ is the class of all graphs obtained from a 4-cycle C_4 by adding a new vertex x and joining x to at least two consecutive vertices of the cycle.

- $\mathcal{F}_3$ is the class of graphs obtained from $Cor(C_3)$ be removing at least one pendant edge.

Lemma 4 *A graph G satisfies $\gamma_t^L(G) = 2$ if and only if $G \in \mathcal{E}$.*

According to Lemma 4, if for any graph $G \in \mathcal{E}$ we add a new vertex y and join y to some vertices of G, we notice whether the resulting graph is 3-γ_t^L-vertex-critical. So we have the following characterization of 3-γ_t^L-vertex-critical graphs.

Theorem 5 *A graph G is 3-γ_t^L-vertex-critical if and only if $G \in \{K_4, K_{1,3}, Cor(C_3)\}$.*

Our next result provides a descriptive characterization of γ_t^L-vertex-critical trees. Let $\mathcal{F}$ be the class of all trees T, where every vertex of T is either a leaf or a support vertex, and every support vertex has degree at least three.

Theorem 6 *A tree T is γ_t^L-vertex-critical if and only if $T \in \mathcal{F}$.*

Proof. Let T be a γ_t^L-vertex-critical tree. To prove that $T \in \mathcal{F}$ we use an induction on the order of T. It is a routine matter to check that if $\text{diam}(T) \in \{1, 2, 3\}$, then T belongs to $\mathcal{F}$. Assume that every γ_t^L-vertex-critical tree T' of order n' less than n belongs to $\mathcal{F}$. Let T be a γ_t^L-vertex-critical tree of order n and diameter at least four. We make two remarks.

i) No vertex of $V(T)\backslash S(T)$ is adjacent to two or more support vertices. Indeed, let b be a vertex not in $S(T)$ adjacent to two support vertices u, v. Then u, v belong to every $\gamma_t^L(T-b)$-set (possibly u, v are leaves in $T-b$) and clearly such a set is a LTDS of T, a contradiction.

ii) Every support vertex of T has degree at least three. Suppose to the contrary that y is a support vertex of degree two. Let x be the unique leaf adjacent to y and z the non-leaf neighbor of y. If $z \notin S(T)$, then every $\gamma_t^L(T-z)$-set contains y and x, and clearly such a set is a LTDS for T, a contradiction. So $z \in S(T)$. Let w be a leaf neighbor of z and let S' be any $\gamma_t^L(T-x)$-set. Then $\{y, w\} \cap S' \neq \emptyset$. If $y \in S'$, then S' would be a LTDS for T, which is a contradiction. Thus $y \notin S'$, and hence $w \in S$. But then $\{y\} \cup S\backslash\{w\}$ is a LTDS for T, a contradiction.

Now consider a diametrical path u_0-u_1-u_2-$\ldots$-$u_{\text{diam}(T)}$. By item (ii) u_1 is a support vertex with at least two leaves, and by item (i) u_2 is either a support vertex or a vertex of degree two. Consider the following two cases.

Case 1. u_2 has degree two. Then by item (i) u_3 is not a support vertex. Let D' be any $\gamma_t^L(T-u_3)$-set. Then, without loss of generality, $N[u_1]\backslash\{u_0\} \subset D'$ and so D' is a LTDS of T, a contradiction.

Case 2. u_2 is a support vertex. Let T' be the tree obtained from T by removing all leaves adjacent to u_1. Then u_2 remains a support vertex with degree at least three in T'. It can be seen easily that $\gamma_t^L(T) = \gamma_t^L(T') + |L_{u_1}(T)| - 1$. Now suppose that T' is not γ_t^L-vertex-critical. Then for some vertex w, $\gamma_t^L(T'-w) \geq \gamma_t^L(T')$. Note that if $w \in L_{u_2}(T')$, then since all vertices of $L_{u_2}(T')$ play the same role, we may assume that $w \neq u_1$. Hence $w \in V(T)\backslash\left(L_{u_1}(T) \cup \{u_1\}\right)$. Let S be a $\gamma_t^L(T-w)$-set. If $w \notin L_{u_2}(T')$, then S contains u_1, u_2 and all leaves of u_1 except one, say u_0. Hence

$S \backslash L_{u_1}(T)$ is a LTDS of $T' - w$ and so $\gamma_t^L(T' - w) \leq \gamma_t^L(T - w) - (|L_{u_1}(T)| - 1)$. It follows that

$$\gamma_t^L(T - w) - (|L_{u_1}(T)| - 1) \geq \gamma_t^L(T' - w) \geq \gamma_t^L(T') = \gamma_t^L(T) - (|L_{u_1}(T)| - 1)$$

and so $\gamma_t^L(T - w) \geq \gamma_t^L(T)$, contradicting the fact that T is a γ_t^L-vertex-critical tree. Thus $w \in L_{u_2}(T') \backslash \{u_1\}$. Clearly u_1 and all its leaves except u_0 are in S. If $u_2 \in S$, then $S \backslash L_{u_1}(T)$ is a LTDS of $T' - w$. If $u_2 \notin S$, then $u_3 \in S$ (because $u_0 \notin S$) and so $(\{u_2\} \cup S) \setminus (L_{u_1}(T) \cup \{u_1\})$ is a LTDS of $T' - w$. In both cases we obtain $\gamma_t^L(T' - w) \leq \gamma_t^L(T - w) - (|L_{u_1}(T)| - 1)$ implying as above $\gamma_t^L(T - w) \geq \gamma_t^L(T)$, a contradiction. We conclude that T' is a γ_t^L-vertex-critical tree. Now applying the inductive hypothesis on T', $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$.

The converse is obvious and the proof is omitted. ■

In the following theorem we produce γ_t^L-vertex-critical graphs from smaller ones.

Theorem 7 *Let G and H be k-γ_t^L-vertex-critical and k'-γ_t^L-vertex-critical graphs, respectively, each with minimum degree at least two. Let F be a graph formed by identifying a vertex of G with a vertex of H. If $\gamma_t^L(F) = k + k' - 1$, then F is γ_t^L-vertex-critical.*

Proof. Clearly $S(G) = S(H) = \emptyset$. Let x be the new vertex resulted by identifying a vertex of G with a vertex of H. Let $y \in V(F)$ and assume that $\gamma_t^L(F) = k + k' - 1$. We show that $\gamma_t^L(F - y) < k + k' - 1$. If $y = x$, then $y \in V(G) \cap V(H)$. But since both G and H are γ_t^L-vertex-critical graphs we have $\gamma_t^L(G - y) = k - 1$ and $\gamma_t^L(H - y) = k' - 1$. It follows that $\gamma_t^L(F - y) \leq k + k' - 2$. So, we suppose that $y \neq x$. Without loss of generality, let $y \in V(G)$. There is a LTDS S of size $k - 1$ in $G - y$ which also dominates x. Now in the graph H we need to find a LTDS that dominates $H - x$. This obviously follows, since $\gamma_t^L(H - x) = k' - 1$. Hence, $F - y$ has a LTDS of size $k + k' - 2$ implying that F is γ_t^L-vertex-critical. ■

3 γ_t^L-edge-critical graphs

As we will see, the removal of a non-pendant edge of a graph G can decrease the locating-total domination number by at most one.

Theorem 8 *For every non-pendant edge $e = xy$ in a graph G, $\gamma_t^L(G) - 1 \leq \gamma_t^L(G - e) \leq \gamma_t^L(G) + 2$.*

Proof. Let S_1 be a $\gamma_t^L(G - e)$-set. If either $S_1 \cap \{x, y\} = \emptyset$ or $\{x, y\} \subseteq S_1$, then S_1 is a LTDS for G implying that $\gamma_t^L(G) \leq \gamma_t^L(G - e)$. So, without loss of generality, assume that $x \in S_1$, $y \notin S_1$. Then $S_1 \cup \{y\}$ is a LTDS for G, which implies that $\gamma_t^L(G) \leq \gamma_t^L(G - e) + 1$. In both cases the lower bound follows.

Now let S_2 be a $\gamma_t^L(G)$-set. Clearly if $S_2 \cap \{x, y\} = \emptyset$, then S_2 is a LTDS for $G - e$, and hence $\gamma_t^L(G - e) \leq \gamma_t^L(G)$. Suppose that $|S_2 \cap \{x, y\}| = 1$, say $x \in S_2$. If y has two neighbors in S_2, then $S_2 \cup \{y\}$ is a LTDS of $G - e$. Thus x is the unique neighbor of y in S_2. Since e is not a pendant edge, let $z \in V - S_2$ be any neighbor of y. Then $S_2 \cup \{z\}$ is a LTDS of $G - e$. In both cases $\gamma_t^L(G - e) \leq \gamma_t^L(G) + 1$. Finally assume that $\{x, y\} \subseteq S_2$. If $\deg_{G[S_2]}(x) \geq 2$ and $\deg_{G[S_2]}(y) \geq 2$, then S_2 is a LTDS for $G - e$ and $\gamma_t^L(G - e) \leq \gamma_t^L(G)$. Thus at least one of x, y is a pendant vertex in $G[S_2]$. If $N(x) \cap N(y) \neq \emptyset$, then $S_2 \cup \{w\}$ is a LTDS for $G - e$, where $w \in N(x) \cap N(y)$. It remains to suppose that $N(x) \cap N(y) = \emptyset$. Since e is a non-pendant edge, let $x_1 \in N(x) - S_2, y_1 \in N(y) - S_2$. Then $S_2 \cup \{x_1, y_1\}$ is a LTDS for $G - e$. We obtain for both cases $\gamma_t^L(G - e) \leq \gamma_t^L(G) + 2$, and the upper bound follows. ■

Note that the right inequality in Theorem 8 is attained for a path $P_{12} = u_1$-u_2-$\ldots$-u_{12}. Then $\{u_2, u_3, u_6, u_7, u_{10}, u_{11}\}$ is $\gamma_t^L(P_{12})$-set and $\gamma_t^L(P_{12} - u_6u_7) = 8$.

We define a graph G to be a *locating-total domination edge critical graph*, or just *γ_t^L-edge-critical graph*, if for every non-pendant edge e, $\gamma_t^L(G - e) < \gamma_t^L(G)$. It follows from Theorem 8 that in a γ_t^L-edge-critical graph G, $\gamma_t^L(G - e) = \gamma_t^L(G) - 1$ for every non-pendant edge e. Also, trivially stars are γ_t^L-edge-critical graph. So we consider graphs having at least one non-pendant edge.

The following observation can easily be verified.

Observation 9 *(1) For every $n \geq 4$, the complete graph K_n is γ_t^L-edge-critical.*
(2) For $m, n \geq 3$, $K_{m,n}$ is γ_t^L-edge-critical.

Lemma 10 *Let $e = xy$ be a non-pendant edge in a γ_t^L-edge-critical graph G, and let S be any $\gamma_t^L(G - e)$-set. Then $|S \cap \{x, y\}| = 1$.*

Proof. If $|S \cap \{x, y\}| \in \{0, 2\}$, then S is a LTDS for G, a contradiction. ■

Theorem 11 *Every non-pendant edge in a γ_t^L-edge-critical graph belongs to a (not necessarily induced) cycle of length* 4.

Proof. Let $e = xy$ be a non-pendant edge of G and let S any $\gamma_t^L(G - e)$-set. By Lemma 10, we suppose that $x \in S$ and $y \notin S$. The fact that S is a $\gamma_t^L(G - e)$-set but not a LTDS for G means that there is a vertex $z \in V - S$ such that $N_G(y) \cap S = N_G(z) \cap S$. Also it is clear that $|N_G(y) \cap S| \geq 2$ and so for every vertex $w \in N_G(y) \cap S$ different to x, the subgraph induced by $\{x, y, z, w\}$ is either a cycle C_4, K_4 or K_4 minus an edge. In each case e is contained in a not necessarily induced 4-cycle C_4. ■

Theorem 11 leads immediately to the following two corollaries.

Corollary 12 *The girth of a γ_t^L-edge-critical graph is at most* 4.

Corollary 13 *A tree T is γ_t^L-edge-critical if and only if T is a star.*

We prove next that no unicycle graph is γ_t^L-edge-critical.

Theorem 14 *Unicycle graphs are not γ_t^L-edge-critical.*

Proof. Assume that G is a unicycle γ_t^L-edge-critical graph. Then by Theorem 11, G has a cycle C of length 4 and clearly that cycle is unique. Let $C = v_1\text{-}v_2\text{-}v_3\text{-}v_4\text{-}v_1$. According again to Theorem 11 each edge in $E(G) \setminus E(C)$ is a pendant edge. Since a 4-cycle C_4 is not γ_t^L-edge-critical, at least one vertex of C is a support vertex. If $V(C) = S(G)$, then for each edge e of C, every $\gamma_t^L(G-e)$-set is a LTDS for G, a contradiction. Thus suppose that $v_2 \notin S(G)$. We proceed with the following facts.

Fact 1. G contains at least two support vertices.

Assume that v_4 is the unique support vertex in G. Then every $\gamma_t^L(G - v_1v_2)$-set is a LTDS of G, a contradiction.

Fact 2. $\{v_1, v_3\} = S(G)$.

Assume that $v_1 \notin S(G)$. Then $\deg_G(v_1) = \deg_G(v_2) = 2$. By Fact 1, $v_3, v_4 \in S(G)$ and hence every $\gamma_t^L(G - v_1v_2)$-set is a LTDS of G, a contradiction. Thus $v_1 \in S(G)$ and by symmetry $v_3 \in S(G)$. Now if $v_4 \in S(G)$, then every $\gamma_t^L(G - v_1v_2)$-set is a LTDS for G, a contradiction. Hence v_1, v_3 are the unique support vertices of G.

Now consider the graph $G' = G - v_1v_2$. Then every $\gamma_t^L(G - v_1v_2)$-set D contains v_1, v_3 and not v_2 (else D would be a LTDS of G). v_3 has at least two leaves and so by Observation 1, all leaves different to v_2 are in D. But then replacing one of such leaves with v_2 in D produces a LTDS of G, a contradiction too. Therefore G is not γ_t^L-edge-critical. ■

Theorem 15 *A graph G is 3-γ_t^L-edge-critical if and only if $G \in \{K_4, K_{1,3}\}$.*

Proof. Let G be a 3-γ_t^L-edge-critical graph. Clearly G is connected and has order $n \geq 4$. We first show that $\operatorname{diam}(G) \leq 2$. Let x, y be two vertices of G such that $d(x, y) = \operatorname{diam}(G) = d$. For $i = 0, 1, 2, .., d$, let V_i be the set of all vertices of G at distance i from x. Let $e = ab$ be a non-pendant edge, where $a \in V_1, b \in V_2$, and let D be a $\gamma_t^L(G-e)$-set. Then $|D| = 2$. To dominate x, it follows that $D \subseteq V_0 \cup V_1 \cup (V_2 \setminus \{b\})$. As a result we have $d \leq 3$. If $d = 3$, then $N_{G-e}(b) \cap D = N_{G-e}(y) \cap D$. This is a contradiction. So $d \leq 2$.

If $\operatorname{diam}(G) = 1$, then it is obvious that $G = K_4$. So suppose that $\operatorname{diam}(G) = 2$. We proceed with the following two facts.

Fact 1. x is a leaf.

Suppose to the contrary that $\deg_G(x) \geq 2$. Let $e = xv_1$ be a non-pendant edge of G, and let D any $\gamma_t^L(G - e)$-set. Thus $|D| = 2$. To totally dominate x, D must contain a vertex $v_2 \in N(x) - \{v_1\}$. By Lemma 10, D contains x or v_1. If $x \in D$, then $N(y) \cap D = N(v_1) \cap D = \{v_2\}$, a contradiction. Thus $v_1 \in D$ and so v_1v_2 is not a pendant edge. By Theorem 11, the edge v_1v_2 belongs to a (not necessarily induced) cycle C of length 4. Let a, b be the two other vertices of C, with b adjacent to v_1. Since D is a $\gamma_t^L(G - e)$-set and x is dominated by v_2, it follows $\{v_1, v_2\} \subseteq N(a)$, $v_2 \notin N(b)$. Thus $N(x) = \{v_1, v_2\}$ and $V_2 = \{a, b\}$. Now by considering the edge v_2a, it can be seen that $\gamma_t^L(G - v_2a) \geq 3$, a contradiction. Hence x is a leaf.

Fact 2. V_2 is an independent set.

Indeed, if uv is an edge linking two vertices $u, v \in V_2$, then by Theorem 11, uv belongs to a (not necessarily induced) 4-cycle C, where three vertices of C are in V_2.

Let $c \in (V(C) \cap V_2) \setminus \{u, v\}$. If S is a $\gamma_t^L(G - uv)$-set, then either $N(x) \cap S = N(u) \cap S$ or $N(x) \cap S = N(v) \cap S$, a contradiction.

Therefore every edge of G is pendant and hence $G = K_{1,3}$. ■

By using a similar argument to that used in the proof of Theorem 7 we obtain the following theorem that produces γ_t^L-edge-critical graphs from smaller ones. The proof is omitted.

Theorem 16 *Let G and H be k-γ_t^L-edge-critical and k'-γ_t^L-edge-critical graphs, respectively. Let F be a graph formed by identifying a non pendant edge of G with a non pendant edge of H. If $\gamma_t^L(F) = k + k' - 1$, then F is γ_t^L-edge-critical.*

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