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Domination with respect to nondegenerate properties: bondage number

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Abstract

For a graphical property $\mathcal{P}$ and a graph G, a subset S of vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the minimum cardinality of a dominating $\mathcal{P}$-set. The bondage number with respect to the property $\mathcal{P}$ of a nonempty graph G, denoted $b_{\mathcal{P}}(G)$, is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number with respect to $\mathcal{P}$ unequal to $\gamma_{\mathcal{P}}(G)$.

In this paper we show that some known sharp upper bounds for the ordinary bondage number are sharp upper bounds for $b_{\mathcal{P}}(G)$ as well.

1 Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes, et al. [14]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x, G)$ denotes the set of all neighbors of x in G, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of x is $\deg(x, G) = |N(x, G)|$. The maximum and minimum degrees of vertices in the graph G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. By $d_G(x, y)$ we denote the distance of the vertices x and y in the graph G. If X and Y are nonempty subsets of $V(G)$, we let $E(X, Y)$ represents the set of edges of the form xy where $x \in X$ and $y \in Y$, and let $e(X, Y)$ denote the cardinality of $E(X, Y)$.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any non-empty subset of $\mathcal{G}$. We say that a *graph G has property* $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example we list some graph properties:

- $\mathcal{I} = \{H \in \mathcal{G} : H \text{ is totally disconnected}\}$;
- $\mathcal{C} = \{H \in \mathcal{G} : H \text{ is connected}\}$;

- $\mathcal{F} = \{H \in \mathcal{G} : H$ is a forest$\}$;

A graph property $\mathcal{P}$ is called: (a) *hereditary (induced-hereditary)*, if from the fact that a graph G has property $\mathcal{P}$, it follows that all subgraphs (induced subgraphs) of G also belong to $\mathcal{P}$; (b) *nondegenerate* if $\mathcal{I} \subseteq \mathcal{P}$, and (c) *additive* if it is closed under taking disjoint union of graphs. Note that $\mathcal{I}$ and $\mathcal{F}$ are nondegenerate, hereditary and additive whereas $\mathcal{C}$ is neither nondegenerate nor induced-hereditary nor additive.

A set $D \subseteq V(G)$ *dominates* a vertex $v \in V(G)$ if either $v \in D$ or $N(v, G) \cap D \neq \emptyset$. If D dominates all vertices in a subset T of $V(G)$ we say that D *dominates* T. When D dominates $V(G)$, D is called a *dominating set* of the graph G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Any set $S \subseteq V(G)$ such that the subgraph $\langle S, G\rangle$ satisfies property $\mathcal{P}$ is called a $\mathcal{P}$-set. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [9]. The *domination number with respect to the property* $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of G. Any dominating set of cardinality $\gamma_{\mathcal{P}}(G)$ is called a $\gamma_{\mathcal{P}}(G)$*-set*, or just $\gamma_{\mathcal{P}}$-set when the graph G is clear from the context. Note that there may be no dominating $\mathcal{P}$-set of G at all. For example, all graphs having at least two components are without dominating $\mathcal{C}$-sets. On the other hand, if a property $\mathcal{P}$ is nondegenerate then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. This fact will be used in the sequel, without specific reference. Michalak [18] has considered these parameters where the property is additive and induced-hereditary.

Note that:

(a) in the case $\mathcal{P} = \mathcal{G}$ we have $\gamma_{\mathcal{G}}(G) = \gamma(G)$;

(b) in the case $\mathcal{P} = \mathcal{I}$, $\gamma_{\mathcal{I}}(G)$ is the well known as the *independent domination number* $i(G)$;

(c) in the case $\mathcal{P} = \mathcal{C}$, $\gamma_{\mathcal{C}}(G)$ is denoted by $\gamma_c(G)$ and is called the *connected domination number*;

(d) in the case $\mathcal{P} = \mathcal{F}$, $\gamma_{\mathcal{F}}(G)$ is denoted by $\gamma_a(G)$ and is called the *acyclic domination number* [16].

Observe that if $\mathcal{I} \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{G}$ then [9] $i(G) \geq \gamma_{\mathcal{P}_1}(G) \geq \gamma_{\mathcal{P}_2}(G) \geq \gamma(G)$.

The study of effects on domination related parameters when a graph is modified by deleting a vertex or adding or deleting an edge is classical; see for instance [2, 4, 8, 10, 13, 18, 19, 20, 22] and for surveys [14, Chapter 5] and [15, Chapter 16].

When we remove an edge from a graph G, the domination number with respect to the property $\mathcal{P}$ can increase or decrease. For instance, if G is a star $K_{1,p}$, $p \geq 2$, and $\{K_1, 2K_1\} \subseteq \mathcal{P} \subseteq \mathcal{G}$ then $\gamma_{\mathcal{P}}(G) = 1$ and $\gamma_{\mathcal{P}}(G - e) = 2$ for all e. If a graph G is obtained by three stars $K_{1,p}$ and three edges e_1, e_2, e_3 joining their centers then $\gamma_a(G) = 2 + p$ and $\gamma_a(G - e_i) = 3$, $i = 1, 2, 3$.

One measure of the stability of the domination number of G under edge removal is the bondage number defined in [6] (previously called the *domination line-stability* in [2]). The *bondage number* $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater then $\gamma(G)$. The concept of the bondage number has been topic of several publications, see [2, 3, 5, 6, 11, 12, 17, 23, 24, 25, 26, 27].

For every graph G with at least one edge and every nondegenerate property $\mathcal{P}$, we define the *bondage (minus bondage, plus bondage,* respectively*) number with respect to the property* $\mathcal{P}$, denoted $b_{\mathcal{P}}(G)$ ($b^-_{\mathcal{P}}(G)$, $b^+_{\mathcal{P}}(G)$, respectively) to be the cardinality of a smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}}(G-U) \neq \gamma_{\mathcal{P}}(G)$ ($\gamma_{\mathcal{P}}(G-U) < \gamma_{\mathcal{P}}(G)$, $\gamma_{\mathcal{P}}(G-U) > \gamma_{\mathcal{P}}(G)$, respectively).

Since $\gamma_{\mathcal{P}}(G - E(G)) = |V(G)| > \gamma_{\mathcal{P}}(G)$ for every nonempty graph G and every nondegenerate property $\mathcal{P}$, it follows that $b^+_{\mathcal{P}}(G)$ always exists. If $\gamma_{\mathcal{P}}(G-U) \geq \gamma_{\mathcal{P}}(G)$ for all $U \subseteq E(G)$, we write $b^-_{\mathcal{P}}(G) = \infty$. Hence $b_{\mathcal{P}}(G) = min\{b^+_{\mathcal{P}}(G), b^-_{\mathcal{P}}(G)\}$.

Necessary and sufficient conditions for $b^+_{\mathcal{P}}(G) = 1$ ($b^-_{\mathcal{P}}(G) = 1$, respectively) may be found in [19] Theorem 3.1 (Theorem 3.4, respectively).

We need the following definitions and results.

Observation 1.1. *Let* $\mathcal{H} \subseteq \mathcal{G}$ *be nondegenerate and let* G *be a nonempty graph.*

(i) $b^-_{\mathcal{G}}(G) = \infty$ *and* $b^+_{\mathcal{G}}(G) = b_{\mathcal{G}}(G) = b(G)$.

(ii) Let $\Delta(G) \leq 2$. *Then* $b^-_{\mathcal{H}}(G) = \infty$ *and* $b^+_{\mathcal{H}}(G) = b^+_{\mathcal{G}}(G) = b(G)$.

(iii) For the cycle of order n,

$$b^-_{\mathcal{H}}(C_n) = \infty \text{ and } b^+_{\mathcal{H}}(C_n) = b_{\mathcal{H}}(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \ (mod\ 3), \\ 2 & \text{otherwise.} \end{cases}$$

(iv) For the path of order $n \geq 2$,

$$b^-_{\mathcal{H}}(P_n) = \infty \text{ and } b^+_{\mathcal{H}}(P_n) = b_{\mathcal{H}}(P_n) = \begin{cases} 2 & \text{if } n \equiv 1 \ (mod\ 3), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. (i) $\gamma_{\mathcal{G}}(G - e) \geq \gamma_{\mathcal{G}}(G)$ for every edge $e \in E(G)$ ([14]).

(ii) If T is a graph with $\Delta(T) \leq 2$ then $\gamma(T) = i(T)$ ([1]) which implies $\gamma(T) = \gamma_{\mathcal{H}}(T)$. Hence $b^+_{\mathcal{H}}(G) = b^+_{\mathcal{G}}(G) = b(G)$ and $b^-_{\mathcal{H}}(G) = b^-_{\mathcal{G}}(G) = \infty$ (by (i)).

(iii)–(iv) $b^-_{\mathcal{H}}(C_n) = b^-_{\mathcal{H}}(P_n) = \infty$ because of (ii). The required results for $b^+_{\mathcal{H}}(C_n)$ and $b^+_{\mathcal{H}}(P_n)$ provided $\mathcal{H} = \mathcal{G}$ due to Fink et al. [6]. The rest follows immediately by (ii). ■

Let $\mathcal{P} \subseteq \mathcal{G}$ be nondegenerate and let G be a graph. Fricke et al. [8] defined a vertex $v \in V(G)$ to be:

(a) $\gamma_{\mathcal{P}}$-*good*, if v belongs to some $\gamma_{\mathcal{P}}$-set of G;

(b) $\gamma_{\mathcal{P}}$-*bad*, if v belongs to no $\gamma_{\mathcal{P}}$-set.

We denote:

$\mathbf{G}_{\mathcal{P}}(G) = \{x \in V(G) : x \text{ is } \gamma_{\mathcal{P}}\text{-good }\}$;

$\mathbf{B}_{\mathcal{P}}(G) = \{x \in V(G) : x \text{ is } \gamma_{\mathcal{P}}\text{-bad }\}$;

$V^-_{\mathcal{P}}(G) = \{x \in V(G) : \gamma_{\mathcal{P}}(G - x) < \gamma_{\mathcal{P}}(G)\}$.

Clearly, $\{\mathbf{G}_{\mathcal{P}}(G), \mathbf{B}_{\mathcal{P}}(G)\}$ is a partition of $V(G)$.

Observation 1.2. *Let $G_1, G_2, .., G_k$ be mutually vertex disjoint graphs and $G = \cup_{i=1}^k G_i$, $k \geq 2$.*

(a) If $\mathcal{H}$ is nondegenerate and additive then $\gamma_{\mathcal{H}}(G) \leq \sum_{i=1}^k \gamma_{\mathcal{H}}(G_i)$.

(b) If $\mathcal{H}$ is nondegenerate and induced-hereditary then $\gamma_{\mathcal{H}}(G) \geq \sum_{i=1}^k \gamma_{\mathcal{H}}(G_i)$.

(c) If $\mathcal{H}$ is additive and induced-hereditary then $\gamma_{\mathcal{H}}(G) = \sum_{i=1}^k \gamma_{\mathcal{H}}(G_i)$, $\mathbf{B}_{\mathcal{H}}(G) = \cup_{i=1}^k \mathbf{B}_{\mathcal{H}}(G_i)$, $\mathbf{G}_{\mathcal{H}}(G) = \cup_{i=1}^k \mathbf{G}_{\mathcal{H}}(G_i)$ and $V_{\mathcal{H}}^-(G) = \cup_{i=1}^k V_{\mathcal{H}}^-(G_i)$.

Proof. (a) Let M_i be a $\gamma_{\mathcal{H}}(G_i)$-set, $i = 1, 2, \dots, k$. Since $\mathcal{H}$ is additive, $M = \cup_{i=1}^k M_i$ is a dominating $\mathcal{H}$-set of G and $\gamma_{\mathcal{H}}(G) \leq |M| = \sum_{i=1}^k |M_i|$.

(b) Let M be a $\gamma_{\mathcal{H}}(G)$-set and let $M_i = M \cap V(G_i)$, $i = 1, 2, \dots, k$. Since $\mathcal{H}$ is induced-hereditary, M_i is a dominating $\mathcal{H}$-set of G_i, $i = 1, 2, \dots, k$. This implies $\gamma_{\mathcal{H}}(G) = |M| = \sum_{i=1}^k |M_i| \geq \sum_{i=1}^k \gamma_{\mathcal{H}}(G_i)$.

(c) Any additive and induced-hereditary property is clearly nondegenerate. It immediately follows by (a) and (b) that $\gamma_{\mathcal{H}}(G) = \sum_{i=1}^k \gamma_{\mathcal{H}}(G_i)$. From this we have:

(i) for each $\gamma_{\mathcal{H}}(G)$-set M, $M \cap V(G_i)$ is a $\gamma_{\mathcal{H}}(G_i)$-set, $i = 1, 2, \dots, k$;

(ii) if M_i is a $\gamma_{\mathcal{H}}(G_i)$-set, $i = 1, 2, \dots, k$, then $\cup_{i=1}^k M_i$ is a $\gamma_{\mathcal{H}}(G)$-set.

First let $x \in V(G_j) \cap \mathbf{G}_{\mathcal{H}}(G)$, $j \in \{1, 2, .., k\}$. Then x is in a $\gamma_{\mathcal{H}}(G)$-set, say M. Now by (i), x is in the $\gamma_{\mathcal{H}}(G_j)$-set $M \cap V(G_j)$ which implies $x \in \mathbf{G}_{\mathcal{H}}(G_j)$. Hence $\mathbf{G}_{\mathcal{H}}(G) \subseteq \cup_{i=1}^k \mathbf{G}_{\mathcal{H}}(G_i)$.

On the other hand, let x_j be a $\gamma_{\mathcal{H}}$-good vertex of G_j, $j \in \{1, 2, .., k\}$. Then x is in a $\gamma_{\mathcal{H}}(G_j)$-set, say M_j. Now by (ii), $M = \cup_{i=1}^k M_i$ is a $\gamma_{\mathcal{H}}(G)$-set, where M_i is a $\gamma_{\mathcal{H}}(G_i)$-set, $i = 1, .., k$. Since $x \in M$, it follows that $x \in \mathbf{G}_{\mathcal{H}}(G)$ which leads to $\mathbf{G}_{\mathcal{H}}(G_j) \subseteq \mathbf{G}_{\mathcal{H}}(G)$, $j = 1, 2, .., k$.

Thus $\mathbf{G}_{\mathcal{H}}(G) = \cup_{i=1}^k \mathbf{G}_{\mathcal{H}}(G_i)$ and since for any graph T, $\{\mathbf{G}_{\mathcal{H}}(T), \mathbf{B}_{\mathcal{H}}(T)\}$ is a partition of $V(T)$ it follows that $\mathbf{B}_{\mathcal{H}}(G) = \cup_{i=1}^k \mathbf{B}_{\mathcal{H}}(G_i)$.

Finally $V_{\mathcal{H}}^-(G) = \cup_{i=1}^k V_{\mathcal{H}}^-(G_i)$ because of $\gamma_{\mathcal{H}}(G) - \gamma_{\mathcal{H}}(G-x) = \gamma_{\mathcal{H}}(G_s) - \gamma_{\mathcal{H}}(G_s - x)$ for each $x \in V(G_s)$, $s = 1, 2, \dots, k$. ■

Lemma 1.3. *[19] Let G be a graph of order at least two and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with K_1.*

(i) Let $v \in V_{\mathcal{H}}^-(G)$. Then:

(i.1) $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$;

(i.2) if M is a $\gamma_{\mathcal{H}}(G - v)$-set then $M \cup \{v\}$ is a $\gamma_{\mathcal{H}}(G)$-set;

(i.3) $N(v, G) \subseteq \mathbf{B}_{\mathcal{H}}(G - v)$.

(ii) If $u \in \mathbf{B}_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G - u) = \gamma_{\mathcal{H}}(G)$.

Lemma 1.4. *[19] Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with K_1. Let x and y be two different and nonadjacent vertices in a graph G. If $x \in V_{\mathcal{H}}^-(G)$ and $y \in \mathbf{B}_{\mathcal{H}}(G-x) - V_{\mathcal{H}}^-(G)$ then $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)$. If $x \in V_{\mathcal{H}}^-(G)$ and $y \in \mathbf{G}_{\mathcal{H}}(G-x)$ then $\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G) - 1$.*

In this paper we show that some known sharp upper bounds for the ordinary bondage number are sharp upper bounds for $b_{\mathcal{P}}(G)$ as well.

2 Upper Bounds

Proposition 2.1. *Let G be a graph with $V(G) \neq V^-_{\mathcal{H}}(G)$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. Then:*

(i) if $v \in V(G) - V^-_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G - E(\{v\}, N(v,G))) > \gamma_{\mathcal{H}}(G)$;

(ii) (Bauer et al. [2] when $\mathcal{H} = \mathcal{G}$) $b^+_{\mathcal{H}}(G) \leq min\{\deg(x,G) : x \in V(G) - V^-_{\mathcal{H}}(G)\}$;

(iii) if $V^-_{\mathcal{H}}(G) = \emptyset$ then $b^+_{\mathcal{H}}(G) \leq \delta(G)$.

Proof. (i) By Observation 1.2(b), for any vertex $v \in V(G) - V^-_{\mathcal{H}}(G)$ we have $\gamma_{\mathcal{H}}(G - E(\{v\}, N(v,G))) \geq \gamma_{\mathcal{H}}(G - v) + 1 > \gamma_{\mathcal{H}}(G)$.
(ii) and (iii): The results follow immediately by (i). ■

Clearly, $V^-_{\mathcal{H}}(C_{3k}) = \emptyset$. Hence, by Observation 1.1(iii) it follows that the bound stated in Proposition 2.1 (iii) is sharp.

A vertex v of a graph G is *$\gamma_{\mathcal{P}}$-critical* if $\gamma_{\mathcal{P}}(G - v) \neq \gamma_{\mathcal{P}}(G)$. The graph G is *vertex-$\gamma_{\mathcal{P}}$-critical* if all its vertices are $\gamma_{\mathcal{P}}$-critical. By Proposition 2.1 it immediately follows:

Corollary 2.2. *(Teschner [23] when $\mathcal{H} = \mathcal{G}$) Let G be a graph and $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. If $b^+_{\mathcal{H}}(G) > \Delta(G)$ then G is a vertex-$\gamma_{\mathcal{H}}$-critical graph.*

The bondage number of vertex-$\gamma_{\mathcal{G}}$-critical graphs is examined in [24], [21].

Proposition 2.3. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. Let G be a graph and $G_{u,v} = G - E(\{u\}, N(u,G)) - E(\{v\}, V(G) - N[u,G])$ where u and v are two adjacent vertices of G. Then:*

(i) $\gamma_{\mathcal{H}}(G_{u,v}) > \gamma_{\mathcal{H}}(G)$;

(ii) (Hartnell and Rall [11] when $\mathcal{H} = \mathcal{G}$) $b^+_{\mathcal{H}}(G) \leq \deg(u,G) + e(\{v\}, V(G) - N[u,G]) = \deg(u,G) + \deg(v,G) - 1 - |N(u,G) \cap N(v,G)|$.

Proof. Let M be a $\gamma_{\mathcal{H}}$-set of $G_{u,v}$. Then $u \in M$ and there is a vertex $w \in N[v, G_{u,v}] \cap M$. Since $N[v, G_{u,v}] \subseteq N(u,G)$ and $\mathcal{H}$ is induced-hereditary, $M - \{u\}$ is a dominating-$\mathcal{H}$-set of G. Therefore $b^+_{\mathcal{H}}(G) \leq |E(G) - E(G_{u,v})| = \deg(u,G) + e(\{v\}, V(G) - N[u,G]) = \deg(u,G) + \deg(v,G) - 1 - |N(u,G) \cap N(v,G)|$. ■

For any graph G without isolates we define:
$\delta_\delta(G) = \min\{\deg(x,G) : x \in V(G)$ is adjacent to a vertex having minimum degree$\}$.

Corollary 2.4. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. If G is a graph without isolates, then $b^+_{\mathcal{H}}(G) \leq \delta(G) + \delta_\delta(G) - 1$.*

Another upper bound is the following result, in terms of maximum degree and the edge connectivity of a graph.

Theorem 2.5. *(Hartnell and Rall [12] and Teschner [26] when $\mathcal{H} = \mathcal{G}$) Let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and additive. If a connected graph G has edge-connectivity $\lambda(G) \geq 1$, then $b_{\mathcal{H}}(G) \leq \Delta(G) + \lambda(G) - 1$.*

Proof. Clearly, $\mathcal{H}$ is nondegenerate and closed under union with K_1. Let $E_1 \subseteq E(G)$ be such that $G' = G - E_1$ is not connected and $|E_1| = \lambda(G)$. Assume that for every subset $S \subseteq E_1$, $\gamma_{\mathcal{H}}(G - S) = \gamma_{\mathcal{H}}(G)$, otherwise $b_{\mathcal{H}}(G) \leq \lambda(G)$ and we are done. Let $V_1 \subseteq V(G)$ be the set of all vertices that are incident in G to an edge of E_1. If $x \in V_1$ and $\gamma_{\mathcal{H}}(G' - x) \geq \gamma_{\mathcal{H}}(G')$ then by Proposition 2.1, $b^+_{\mathcal{H}}(G') \leq \deg(x, G')$ which implies $b_{\mathcal{H}}(G) \leq \lambda(G) + b^+_{\mathcal{H}}(G') \leq \lambda(G) + \deg(x, G) - 1$. So let $V_1 \subseteq V^-_{\mathcal{H}}(G')$. Denote by G_1 and G_2 the components of $G'$and let $x_i \in V(G_i)$, $i = 1, 2$ be adjacent in G. By Observation 1.2(c) we have $\gamma_{\mathcal{H}}(G') = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2)$, $\gamma_{\mathcal{H}}(G' - x_1) = \gamma_{\mathcal{H}}(G_1 - x_1) + \gamma_{\mathcal{H}}(G_2)$, $\gamma_{\mathcal{H}}(G' - x_2) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2 - x_2)$. From this and since $x_1, x_2 \in V^-_{\mathcal{H}}(G')$, Lemma 1.3 implies that $x_i \in V^-_{\mathcal{H}}(G_i)$, $i = 1, 2$. Now again by Lemma 1.3, there is a $\gamma_{\mathcal{H}}(G_i - x_i)$-set M_i such that $M_i \cup \{x_i\}$ is a $\gamma_{\mathcal{H}}(G_i)$-set and $N[x_i, G_i] \cap M_i = \emptyset$, $i = 1, 2$ provided G_i has order at least two; otherwise let $M_i = \emptyset$. Hence the set $M = M_1 \cup M_2 \cup \{x_1, x_2\}$ is a dominating set of G' with $|M| = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) = \gamma_{\mathcal{H}}(G')$. Since $\mathcal{H}$ is additive, M is an $\mathcal{H}$-set of G'. Thus M is a $\gamma_{\mathcal{H}}(G')$-set. Clearly, $M - \{x_2\}$ is a dominating set of $G' + x_1x_2$. Since $\mathcal{H}$ is induced-hereditary, $M - \{x_2\}$ is an $\mathcal{H}$-set of $G' + x_1x_2$. Hence $\gamma_{\mathcal{H}}(G') > |M - \{x_2\}| \geq \gamma_{\mathcal{H}}(G' + x_1x_2)$, a contradiction. ∎

Notice that the bound stated in the above theorem is attainable, for example when $G = C_{3k+1}$ (because of Observation 1.1 (iii)).

Theorem 2.6. *Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and additive. Let G be a graph, $u, v, w \in V(G)$, $d_G(u, v) = 2$ and $uw, vw \in E(G)$.*

(a) Whenever $u, v \in V^-_{\mathcal{H}}(G)$, let at least one of $u \notin V^-_{\mathcal{H}}(G - v)$ and $v \notin V^-_{\mathcal{H}}(G - u)$ hold. Then $b_{\mathcal{H}}(G) \leq \deg(u, G) + \deg(v, G) - 1$.

(b) Let $u, v \in V^-_{\mathcal{H}}(G)$, $u \in V^-_{\mathcal{H}}(G - v)$ and $v \in V^-_{\mathcal{H}}(G - u)$. Then $b^-_{\mathcal{H}}(G) \leq \deg(w, G) - 2$.

Proof. Let $G_u = G - E(\{u\}, N(u, G))$, $G_{u,v} = G_u - E(\{v\}, N(v, G_u))$, $G_{u,v,w} = G_{u,v} - E(\{w\}, N(w, G_{u,v}))$, $G_1 = G_{u,v,w} + \{uw, vw\}$ and $G_2 = G - E(\{w\}, N(w, G) - \{u, v\})$.

(a) If $u \notin V^-_{\mathcal{H}}(G)$, then Proposition 2.1(ii) implies $b^+_{\mathcal{H}}(G) \leq \deg(u, G)$. So, assume henceforth $u, v \in V^-_{\mathcal{H}}(G)$ and without loss of generality let $v \notin V^-_{\mathcal{H}}(G - u)$. Since $u \in V^-_{\mathcal{H}}(G)$ it follows by Lemma 1.3(i.1) that $\gamma_{\mathcal{H}}(G - u) = \gamma_{\mathcal{H}}(G) - 1$. Since $\mathcal{H}$ is induced hereditary and additive, Observation 1.2 (c) implies $\gamma_{\mathcal{H}}(G_u) = \gamma_{\mathcal{H}}(G - u) + 1 = \gamma_{\mathcal{H}}(G)$. By Lemma 1.3(i.3), $w \in \mathbf{B}_{\mathcal{H}}(G - u)$ and by Observation 1.2(c), $w \in \mathbf{B}_{\mathcal{H}}(G_u)$. Since $v \notin V^-_{\mathcal{H}}(G - u)$, it follows that $\gamma_{\mathcal{H}}(G - \{u, v\}) = \gamma_{\mathcal{H}}((G - u) - v) = \gamma_{\mathcal{H}}(G - u) + p = \gamma_{\mathcal{H}}(G) - 1 + p$, where $p \geq 0$ is an integer. Hence by Observation 1.2(c) we have $\gamma_{\mathcal{H}}(G_{u,v}) = 2 + \gamma_{\mathcal{H}}(G - \{u, v\}) = \gamma_{\mathcal{H}}(G) + 1 + p$.

Case 1. Let $w \in \mathbf{B}_{\mathcal{H}}(G_{u,v})$. By Observation 1.2(c) it follows that w is a $\gamma_{\mathcal{H}}$-bad vertex for both $G_{u,v} - v$ and $G_{u,v} - \{u, v\}$. Now, by Lemma 1.4 applied to the graph

$G_{u,v}-v$ and the vertices u and w it follows that $\gamma_{\mathcal{H}}((G_{u,v}-v)+uw) = \gamma_{\mathcal{H}}(G_{u,v}-v)$. Hence $\gamma_{\mathcal{H}}(G_{u,v}+uw) = \gamma_{\mathcal{H}}((G_{u,v}-v)+uw)+1 = \gamma_{\mathcal{H}}(G_{u,v}-v)+1 = \gamma_{\mathcal{H}}(G_{u,v}) > \gamma_{\mathcal{H}}(G)$. This leads to $b^+_{\mathcal{H}}(G) \le \deg(u,G)+\deg(v,G)-1$.

Case 2. Let $w \in \mathbf{G}_{\mathcal{H}}(G_{u,v})$. By Observation 1.2(c) it follows that $w \in \mathbf{G}_{\mathcal{H}}(G_u-v)$. Now, by Lemma 1.4 applied to the graph $G_{u,v}$ and the vertices v and w we have $\gamma_{\mathcal{H}}(G_{u,v}+vw) = \gamma_{\mathcal{H}}(G_{u,v})-1 = \gamma_{\mathcal{H}}(G)+p$. If $p \ge 1$ then we have the result. So, let $p=0$. Thus $\gamma_{\mathcal{H}}(G_{u,v}) = \gamma_{\mathcal{H}}(G)+1$. Let M be a $\gamma_{\mathcal{H}}$-set of $G_{u,v}$ with $w \in M$. Since $\mathcal{H}$ is hereditary, $M-\{u,v\}$ is a dominating $\mathcal{H}$-set of $G_{u,v}+\{wv,wu\}$ which implies $\gamma_{\mathcal{H}}(G_{u,v}+\{wv,wu\}) \le |M-\{u,v\}| = \gamma_{\mathcal{H}}(G_{u,v})-2 < \gamma_{\mathcal{H}}(G)$. Now, if $G_{u,v}+\{uw,vw\} = G$ then we have a contradiction; if $G_{u,v}+\{uw,vw\} \ne G$ then $b^-_{\mathcal{H}}(G) \le \deg(u,G)+\deg(v,G)-1$.

(b) Since $v \in V^-_{\mathcal{H}}(G-u)$ and $u \in V^-_{\mathcal{H}}(G)$, by Lemma 1.3(i.1) we have $\gamma_{\mathcal{H}}(G-\{u,v\}) = \gamma_{\mathcal{H}}(G)-2$. Now, by Observation 1.2 (c), $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{u,v})$. Since $v \in V^-_{\mathcal{H}}(G-u)$, by Lemma 1.3(i.3) we have $w \in \mathbf{B}_{\mathcal{H}}(G-\{u,v\})$. By Observation 1.2(c) it follows that $w \in \mathbf{B}_{\mathcal{H}}(G_{u,v})$. Hence $\gamma_{\mathcal{H}}(G_{u,v,w}-w) = \gamma_{\mathcal{H}}(G_{u,v})$ (by Lemma 1.3 (ii)) and by Observation 1.2(c), $\gamma_{\mathcal{H}}(G_{u,v,w}) = \gamma_{\mathcal{H}}(G_{u,v})+1 = \gamma_{\mathcal{H}}(G)+1$. Now we have $\gamma_{\mathcal{H}}(G_1) = \gamma_{\mathcal{H}}(G_{u,v,w}-\{u,v,w\})+1 = \gamma_{\mathcal{H}}(G_{u,v,w})-2 = \gamma_{\mathcal{H}}(G)-1$. Note that $u,v \in \mathbf{B}_{\mathcal{H}}(G_1)$ by Observation 1.2 (c) and then each $\gamma_{\mathcal{H}}(G_1)$-set is a dominating $\mathcal{H}$-set of G_2. Hence $\gamma_{\mathcal{H}}(G_2) \le \gamma_{\mathcal{H}}(G_1) < \gamma_{\mathcal{H}}(G)$. Thus $b^-_{\mathcal{H}}(G) \le \deg(w,G)-2$. ∎

Example 2.7. *Let $S_{p,r}$, $2 \le p < r$ be the tree with vertex set $\{x,u,y_1,\dots,y_p,v_1,\dots,v_r\}$ and edge set $\{xu,xy_1,\dots,xy_p,uv_1,\dots,uv_r\}$. Obviously, the set $\{u,y_1,\dots,y_p\}$ is the unique $i(S_{p,r})$-set and $\{y_1,\dots,y_p\} = V^-_{\mathcal{I}}(S_{p,r})$. Since $y_2 \in V^-_{\mathcal{I}}(S_{p,r}-y_1)$ and $y_1 \in V^-_{\mathcal{I}}(S_{p,r}-y_2)$, by applying Theorem 2.6(b) to $S_{p,r}$ and the vertices y_1,x,y_2, we obtain $b^-_{\mathcal{I}}(S_{p,r}) \le \deg(x,S_{p,r})-2 = p-1$. Thus this bound is sharp for $S_{2,r}$ (observe that $i(S_{p,r}-xu) = 2 < i(S_{p,r})$). Now, applying Theorem 2.6(a) to $S_{p,r}$ and the vertices v_1 and v_2 (note that $v_1,v_2 \notin V^-_{\mathcal{I}}(S_{p,r})$) we have $b_{\mathcal{I}}(S_{p,r}) \le 1$. Hence this bound is sharp for $S_{p,r}$.*

Corollary 2.8. *(Hartnell and Rall [12] and Teschner [26]) If u and v are vertices of G such that the distance between them is 2, then $b(G) \le \deg(u,G)+\deg(v,G)-1$.*

Proof. Let w be a vertex adjacent to both u and v. Assume to the contrary that $u,v \in V^-(G)$ and without loss of generality $v \in V^-(G-u)$. But then if M is a $\gamma(G-\{u,v\})$-set then $M \cup \{w\}$ is clearly a dominating set of G with $|M \cup \{w\}| \le \gamma(G-\{u,v\})+1 = \gamma((G-u)-v)+1 = \gamma(G-u) = \gamma(G)-1$, a contradiction. The result now follows by Theorem 2.6 (a). ∎

Now, we look at the bondage number of trees.

Proposition 2.9. *(Bauer et al. [2] when $\mathcal{H} = \mathcal{G}$) Let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary. If T is a tree with at least two vertices, then $b_{\mathcal{H}}(T) \le 2$.*

Proof. Let $R : x_1,x_2,\dots,x_n$ be the longest path in T. If $n \le 3$ then T is a star and $b^+_{\mathcal{H}}(T) = 1$. So, let $n \ge 4$. Assume that $\gamma_{\mathcal{H}}(T-x_2x_3) = \gamma_{\mathcal{H}}(T)$. Let T_2 and T_3 be the components of $T-x_2x_3$ and $x_i \in V(T_i)$, $i=2,3$. Then T_2 is a star and by Observation 1.2(c), $\gamma_{\mathcal{H}}(T) = \gamma_{\mathcal{H}}(T-x_2x_3) = \gamma_{\mathcal{H}}(T_2)+\gamma_{\mathcal{H}}(T_3) = \gamma_{\mathcal{H}}(T_3)+1 < \gamma_{\mathcal{H}}(T_3)+2 = \gamma_{\mathcal{H}}(T-\{x_1x_2,x_2x_3\})$. Hence $b_{\mathcal{H}}(T) \le 2$. ∎

We conclude with results on the bondage number of planar graphs.

It is a well known fact that every planar graph has minimum degree at most 5. Hence by Proposition 2.1(iii) the following immediately follows:

Corollary 2.10. *Let G be a planar graph and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and induced-hereditary. If $V^-_{\mathcal{H}}(G) = \emptyset$ then $b^+_{\mathcal{H}}(G) \leq \delta(G) \leq 5$.*

Lemma 2.11. *(Euler's formula) Suppose that G is a connected planar graph. Then $|V(G)| - |E(G)| + |F(G)| = 2$, where $F(G)$ is the face set of any embedding of G on the plane.*

Let $e = xy$ be a non cut edge of a connected planar graph G. Following Carlson and Develin [3], we define

$$D_e = D_{xy} = \frac{1}{\deg(x, G)} + \frac{1}{\deg(y, G)} + \frac{1}{r^1_e} + \frac{1}{r^2_e} - 1,$$

where r^1_e and r^2_e are the numbers of edges comprising the faces which $e = xy$ borders (our notation is as in [7]). In view of Euler's formula, we obtain for a connected graph G without cut edges

$$\sum_{e \in E(G)} D_e = |V(G)| - |E(G)| + |F(G)| = 2. \qquad (1)$$

Theorem 2.12. *Let $\mathcal{H} \subseteq \mathcal{G}$ be an additive and induced-hereditary property. Let G be a connected planar graph.*

(i) If $\delta(G) \leq 3$ then $b^+_{\mathcal{H}}(G) \leq \delta_\delta(G) + 2$.

(ii) (Kang and Yuan [17] when $\mathcal{H} = \mathcal{G}$) $b_{\mathcal{H}}(G) \leq \Delta(G) + 2$.

Proof. (i) By Corollary 2.4, if G has any vertices of degree 3 or less, we have $b^+_{\mathcal{H}}(G) \leq \delta(G) + \delta_\delta(G) - 1 \leq \delta_\delta(G) + 2$.

(ii) We can assume $\delta(G) \geq 4$ because of (i). If G has a cut edge, then by Theorem 2.5, $b_{\mathcal{H}}(G) \leq \Delta(G)$. Hence, let in the following G be 2-edge connected. Assume to the contrary that $b_{\mathcal{H}}(G) \geq \Delta(G) + 3$. Let $e = xy$ be an arbitrary edge of G and let without loss of generality, $\deg(x, G) \leq \deg(y, G)$ and $r^1_e \leq r^2_e$. By Proposition 2.3 we have,

$$\Delta(G) + 3 \leq b_{\mathcal{H}}(G) \leq \deg(x, G) + \deg(y, G) - 1 - |N(x) \cap N(y)|. \qquad (2)$$

If $\deg(x, G) = 4$, then (2) implies $\deg(y, G) = \Delta(G)$ and $r^1_e \geq 4$; hence $D_e \leq 0$.

If $\deg(x, G) = 5$, by (2) it follows that $r^2_e \geq 4$; hence $D_e < 0$.

If $\deg(x, G) \geq 6$, then clearly $D_e \leq 0$.

Therefore $\sum_{e \in E(G)} D_e \leq 0$, which is a contradiction to (1). ■

The next conjecture provided $\mathcal{P} = \mathcal{G}$ is the main outstanding conjecture on ordinary bondage number.

Conjecture 2.13. *(Teschner [24] when $\mathcal{P} = \mathcal{G}$) Let $\mathcal{P} \subseteq \mathcal{G}$ be additive and hereditary. For any vertex-$\gamma_{\mathcal{P}}$-critical graph G, $b^+_{\mathcal{P}}(G) \leq 1.5\Delta(G)$.*

Observation 1.1(iii) gives particular support for this conjecture, namely $b_{\mathcal{P}}(C_{3k+1}) = 3 = 1.5\Delta(C_{3k+1})$. Now let $\mathcal{P} = \mathcal{G}$. Teschner [24] has shown that Conjecture 2.13 is true when $\gamma(G) \leq 3$. Observe that if $G = K_t \times K_t$ for a positive integer $t \geq 2$, then $b(G) = 1.5\Delta(G)$, as was found independently by Hartnell and Rall [11] and by Teschner [25].

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