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Vertex-magic labelings: mutations

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Abstract

In this paper, we are studying vertex-magic total labelings of simple graphs. We introduce a procedure called *mutation* which transforms one labeling into another by swapping sets of edges among vertices. The result of a mutation may be a different labeling of the same graph or a labeling of a different graph. Mutation proves to be a remarkably fruitful process — for example we are able to generate labelings for all the order 10 cubic graphs from a single initial labeling of the 5-prism. We describe all possible mutations of a labeling of the path and the cycle.

1 Introduction

A *vertex-magic total labeling* (VMTL) on a graph with v vertices and e edges is a one-to-one mapping from the vertices and edges onto the integers $1, 2, \ldots, v + e$ so that the sum of the label on a vertex and the labels of its incident edges is constant, independent of the choice of vertex. The question of determining precisely which graphs possess vertex-magic total labelings seems very difficult. So far the effort of most authors has been devoted to finding constructions of labelings for easily describable families of graphs, many of which are regular or possess some other highly restrictive feature. A serious attack on the question requires us to look for methods which do not depend of the detailed structure of the graph. The first effort along these lines is due to Dan McQuillan [5] who showed how to label any cubic graph which contains a perfect matching, without relying on its precise structure. McQuillan's approach has been extended dramatically in [1] and [2].

For a long time the authors of this paper have hoped for an approach which might use a labeling from one of these known families of graphs to create labelings for other graphs of the same order and size. In this paper we describe a process we call *mutation* which does this. It is a straightforward procedure, swapping edges between vertices in ways that are magic-preserving. Yet we will show that mutation is a remarkably fruitful operation where through repeated application a single initial labeling for one graph can generate large numbers of labelings for many different graphs. As an example, we begin with a labeling of one order 10 cubic graph and generate labelings for all the connected cubic graphs of order 10.

Readers are referred to [4] and [7] for general background and basic constructions regarding VMTLs.

2 The basic principle of mutation

A vertex magic labeling of the path P_5 is shown in Figure 1. In that labeling, the edges labelled 1 and 3 (which total to 4) are incident with vertex 7 and the edge labelled 4 is incident with vertex 5. The idea of mutation is simply to interchange the adjacencies of these edges between the 2 vertices, so that edges labelled 1 and 3 are made incident with vertex 5 and the edge labelled 4 is made incident with vertex 7. The resulting graph is shown in Figure 1 and retains the same vertex sums, hence is vertex-magic.

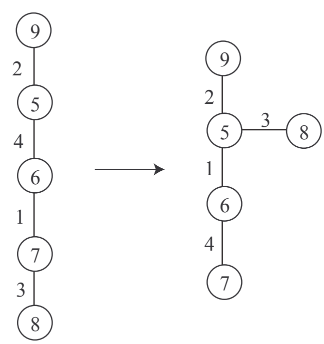

Figure 1: An example of a mutation

We bring some precision to this concept in the following theorem:

Theorem 2.1 (Mutation Theorem)**.** *Consider a graph G with a VMTL. Let u_0 and w_0 be distinct vertices such that $U = \{u_1, \ldots, u_m\}$ is the neighbourhood of u_0 and $W = \{w_1, \ldots, w_n\}$ the neighbourhood of w_0, and the following properties hold:*

(i) $\sum_{i=1}^{m} \lambda(u_0u_i) = \sum_{i=1}^{n} \lambda(w_0w_i)$

(ii) *If y is adjacent to u_0 and $y \notin U$ then $y \notin W$*

(iii) *If y is adjacent to w_0 and $y \notin W$ then $y \notin U$*

(iv) $u_0 \notin W$ *and* $w_0 \notin U$.

If we delete the edges u_0u_i and w_0w_i and add new edges w_0u_i with labels $\lambda^*(w_0u_i) = \lambda(u_0u_i)$ and edges u_0w_i with labels $\lambda^*(u_0w_i) = \lambda(w_ow_i)$, we obtain either a new VMTL of G or a VMTL of a different graph G^*.

Proof. After deleting the old edges and adding the new ones, the vertex sums for u_0 and w_0 will become:

$$\begin{aligned} k - \sum_{i=1}^{m} \lambda(u_0 u_i) + \sum_{i=1}^{n} \lambda^*(u_0 w_i) &= k - \sum_{i=1}^{m} \lambda(u_0 u_i) + \sum_{i=1}^{n} \lambda(w_0 w_i) &= k \\ k - \sum_{i=1}^{n} \lambda(w_0 w_i) + \sum_{i=1}^{m} \lambda^*(w_0 u_i) &= k - \sum_{i=1}^{n} \lambda(w_0 w_i) + \sum_{i=1}^{m} \lambda(u_0 u_i) &= k \end{aligned}$$

i.e. the vertex-sums are preserved. The second and third conditions ensure that the new graph does not contain any multiple edges, and the fourth condition ensures that the new graph contains no loops. □

We will refer to such a swapping of edges and their associated labels between two vertices as an (n, m)-*mutation*.

If $n = m$ then the new graph G^* will have the same degree sequence as the parent graph, and G^* may be:

- G itself
- a new graph isomorphic to G
- a new graph not isomorphic to G.

In particular, if G is regular then so is G^*. In some cases there are many possible pairs that can be swapped, resulting in a large number of distinct labelings of the same graph.

If $n \neq m$ then G^* may be a graph with a different degree sequence but possessing a VMTL with the same set of edge labels and same set of vertex labels, and consequently the same magic constant. Note that it is possible to start with a labeling of a disconnected graph and obtain a labeling for a connected graph and vice-versa. For example, it is easy to show that a $(2, 2)$-mutation of any labeling of $2K_n$ can only result either in a new labeling for $2K_n$ or a labeling of one particular connected graph.

The mutation process can often be repeated for successive *generations* to obtain a sequence of labelled graphs each one closely related to but different from its predecessor; hence we chose the term *mutation* by analogy with genetic mutation in biological organisms. An example of mutations of cubic graphs of order 8 is discussed in Section 5.1 in which $(2, 2)$-mutations repeated over several generations generate labelings from the cube for *all* of the cubic graphs of order 8, both connected and disconnected.

Starting from a graph with few edges obviously provides less opportunity for mutation, so we may hope to enumerate all the possible offspring of such a graph. In Section 3, we examine the (workably small) set of graphs that can be derived from paths by one or two mutations of a single initial labeling, and in Section 4 we do the same for cycles.

Finally, it is intuitively clear that mutation can be extended to the case where edges are swapped among 3 or more vertices. Describing this algebraically is complicated because of the necessity of ensuring that there are no loops or multiple edges in the resulting graph. This is touched upon briefly in Section 4.4.

3 Mutations of paths

Any graph can be represented as an edge-disjoint union of paths, cycles or both. Such a representation is generally not unique. For the purpose of describing the mutations discussed in this and the following section, we will represent a path or a cycle by a list of vertices and edges, with edges and vertices alternating and beginning and ending with a vertex. The only place at which a vertex may occur twice in a list will be at the beginning and end of that list to indicate that the path is closed (i.e. is a cycle). Note that only vertices of degree $d > 2$ in a graph will belong to more than one list.

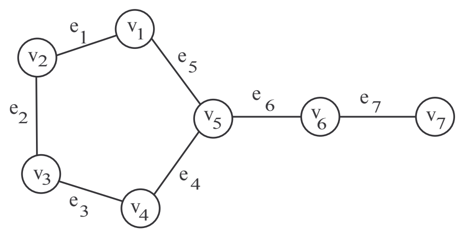

Figure 2: A list representation of a $(5, 2)$-kite

For example, we may represent the graph in Figure 2 as a set of lists by

$$(v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_1)\ (v_5, e_6, v_6, e_7, v_7)$$

where v_1 begins and ends the cycle, while v_5 links the path and the cycle. The representation is certainly not unique. For example, we could have begun and ended the cycle at any vertex, say:

$$(v_5, e_5, v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5)\ (v_5, e_6, v_6, e_6, v_7).$$

In this section we explore the labelings that result from mutating the standard caterpillar labeling of the path P_n (described in Theorem 3.2). This is the most simply structured of all VMTLs of the path, and therefore the easiest to work with. Quite possibly, some similar results could be proved if one began with some other labeling. We begin with the first generation of mutations and in the next subsection examine what can happen in the second generation.

3.1 Generalized stars

One application of the Mutation Theorem in a systematic way will yield constructions for labelings of two families of graphs which have not previously been known—the kites and generalized stars. For completeness, their definitions are included here.

Definition 1. *A* generalized star *is a tree homeomorphic to $K_{1,n}$ for some $n \geq 2$.*

Thus the generalized star has a central vertex which is the end vertex of each of n paths that are otherwise disjoint. If the n paths have respectively $a_1, a_2, \dots, a_n$ edges, we will denote this tree by $S_{a_1,a_2,\dots,a_n}$. Thus $S_{a_1,a_2,\dots,a_n}$ has $1 + \sum_{i=1}^{n} a_i$ vertices and $\sum_{i=1}^{n} a_i$ edges. Generalized stars are also known as 'star-like trees' or 'generalized comets' and by various other names in the literature. Those considered in this section are all homeomorphic to $K_{1,3}$.

Definition 2. *An* (n,t)-kite *consists of a cycle of order n one of whose vertices is the end vertex of a t-edge path which is otherwise disjoint from the cycle.*

Before proceeding with the specific constructions, we first look at the general constraint which applies to a first generation mutation of *any* VMTL of any path. We have

Theorem 3.1. *For any VMTL of a path, a first-generation mutation is always a labeling of a 3-branch generalized star or the union of a path and a kite.*

Proof. By definition, all vertices of a graph must have distinct labels under any VMTL. This implies that the edges incident to one vertex do not have the same sum as the edges incident to any other vertex, and thus that we cannot swap all of the edges incident to two vertices. Since the end vertices of the path have only one incident edge, it follows that we can only swap edges incident to internal vertices. Since this must involve a 2-for-1 swap, the degree of one vertex must increase to 3, while the other must reduce to 1. Suppose we have a path $P_m = (v_1, e_1, \dots, e_{i-1}, v_i, e_i, \dots, e_{j-1}, v_j, e_j, \dots, v_m)$ where v_i and v_j are the internal vertices of interest. Without loss of generality we will assume that the labels of the two edges incident to v_i sum to either $\lambda(e_{j-1})$ or $\lambda(e_j)$.

If they sum to $\lambda(e_{j-1})$ then after the swap we have:

$$(v_j, e_i, \dots, e_{j-1}, v_i)(v_j, e_{i-1}, \dots, e_1, v_1)(v_j, e_j, \dots, v_m)$$

i.e. a 3-branch star $S_{j-i,i-1,m-j}$.

If they sum to $\lambda(e_j)$ then after the swap we have a path $(v_i, e_j, \dots, v_m)$ and a $(j-i, i-1)$-kite $(v_j, e_i, \dots, e_{j-1}, v_j)(v_j, e_{i-1}, \dots, e_1, v_1)$.

Since these are the only two possibilities, and since we made no assumptions about the specific labeling, the result follows. □

Choosing a specific labeling of the path to mutate, we can describe some of its offspring.

Theorem 3.2. *For $n \geq 2$, the generalized star $S_{i,i+1,2n-2i-1}$ has a VMTL for all $i = 1, \dots, n-1$.*

Proof. The proof is by construction. We start with the standard caterpillar labeling of a path of order $2n+1$ defined as:

$$\lambda(e_i) = \begin{cases} \frac{1}{2}(2n+1+i) & \text{for } i \text{ odd} \\ \frac{1}{2}i & \text{for } i \text{ even} \end{cases} \tag{1}$$

where $i = 1, \ldots, 2n$. The edge labels range from 1 to n for i even and from $n+1$ to $2n$ for i odd. The vertex labels will be $\lambda(v_j) = 4n+1-j$ where j ranges from $1, \ldots, 2n$, with $\lambda(v_{2n+1}) = 4n+1$. We note that $\lambda(e_i) + \lambda(e_{i+1}) = \frac{1}{2}(2i + 2n + 2) = \lambda(e_{2i+1})$.

Starting with:
$(v_1, e_1, \ldots, e_i, v_{i+1}, e_{i+1}, \ldots, e_{2i+1}, v_{2i+2}, e_{2i+2}, \ldots, e_{2n}, v_{2n+1})$
after swapping the edges we have:
$(v_1, e_1, \ldots, e_i, v_{2i+2})$
$(v_{2i+2}, e_{i+1}, \ldots, e_{2i+1}, v_{i+1})$
$(v_{2i+2}, e_{2i+2}, \ldots, e_{2n}, v_{2n+1})$
i.e. three paths sharing the common vertex v_{2i+2}.

The first path is of length i, the second of length $i+1$, and the third path is of length $2n-2i-1$, so the result is the generalized star $S_{i,i+1,2n-2i-1}$ as required. To make sense we must have $2n-2i-1 \geq 1$, which means we require $i < n$. The result follows. □

Example. *From P_{21}, i.e. for $n = 10$, we can construct:*

$$S_{1,2,17} \quad S_{2,3,15} \quad S_{3,4,13} \quad S_{4,5,11} \quad S_{5,6,9}$$
$$S_{6,7,7} \quad S_{5,7,8} \quad S_{3,8,9} \quad S_{1,9,10}$$

Hence from a path of length $2n+1$ we can always obtain VMTLs of at least $n-1$ non-isomorphic generalized stars. (It is possible that other initial labelings of the path would yield still other stars.) This simple example demonstrates the fertility of mutation. When n is large a mutation may leave a long branch within which the edges can still be swapped, so that we can repeat this process to obtain VMTL's for a variety of trees other than generalized stars.

3.2 Second-generation mutations of paths

Starting with the construction from the previous section, we examine what further generalized stars we can obtain by mutating the first-generation stars.

The vertex v_{2i+2} is incident with three edges: e_i, e_{i+1} and e_{2i+2}. This gives us two cases to consider. Since swapping e_i and e_{i+1} with e_{2i+1} will simply give us the path we started with, we examine the other two cases:

1. $\lambda(e_i) + \lambda(e_{2i+2}) = \lambda(e_{3i+2})$ with $3i + 2 \leq 2n$
2. $\lambda(e_{i+1}) + \lambda(e_{2i+2}) = \lambda(e_{3i+3})$ with $3i + 3 \leq 2n$

In both cases, the edge involved in the swap which is not incident to v_{2i+2} is on the branch linked to v_{2i+2} by e_{2i+2}.

In case (1), we start with:
$(v_1, e_1, \ldots, e_i, v_{2i+2})$
$(v_{2i+2}, e_{i+1}, \ldots, e_{2i+1}, v_{i+1})$
$(v_{2i+2}, e_{2i+2}, \ldots, v_{3i+2}, e_{3i+2}, v_{3i+3}, \ldots, e_{2n}, v_{2n+1})$
and after the mutation we have:
$(v_{3i+3}, e_i, v_i, e_{i-1}, v_{i-1}, \ldots, e_1, v_1)$
$(v_{3i+3}, e_{2i+2}, \ldots, v_{3i+2}, e_{3i+2}, v_{2i+2}, e_{i+1}, \ldots, e_{2i+1}, v_{i+1})$

$(v_{3i+3}, \ldots, e_{2n}, v_{2n+1})$
a generalized star $S_{i,2i+2,2n-(3i+2)}$, so long as $i < \frac{2}{3}(n-1)$. Since $i \geq 1$, this mutation is only feasible if $n \geq 3$.

In case (2), we start with:
$(v_1, e_1, \ldots, e_i, v_{2i+2})$
$(v_{2i+2}, e_{i+1}, \ldots, e_{2i+1}, v_{i+1})$
$(v_{2i+2}, e_{2i+2}, \ldots, v_{3i+2}, e_{3i+2}, v_{3i+3}, e_{3i+3}, v_{3i+4}, \ldots, e_{2n}, v_{2n+1})$
and after the mutation we have:
$(v_{3i+4}, e_{i+1}, \ldots, e_{2i+1}, v_{i+1})$
$(v_{3i+4}, e_{2i+2}, \ldots, v_{3i+3}, e_{3i+3}, v_{2i+2}, e_i, v_i, e_{i-1}, v_{i-1}, \ldots, e_1, v_1)$
$(v_{3i+4}, e_{3i+4}, \ldots, e_{2n}, v_{2n+1})$
a generalized star $S_{i+1,2i+2,2n-(3i+3)}$, with $i < \frac{1}{3}(2n-3)$. Note that since $i \geq 1$, this mutation is only feasible if $n > 3$.

Returning to the example of $n = 10$ above, these two cases give us the following generalized stars:

$S_{1,4,15}$	$S_{2,6,12}$	$S_{3,8,9}*$	$S_{4,6,10}$	$S_{3,5,12}$
$S_{2,4,14}$	$S_{3,6,11}$	$S_{4,8,8}$	$S_{5,5,10}$	$S_{2,6,12}*$

where an asterisk indicates a second labeling for the generalized star that had appeared in the first generation. Together the first and second generation mutations give us VMTLs of 17 non-isomorphic generalized stars, just over half of the 33 generalized stars of order 21.

4 Mutations of cycles

4.1 Constructing kites from cycles

We first establish two results which apply regardless of the labeling.

Theorem 4.1. *A first-generation mutation of a cycle C_n is always an $(n-m, m)$-kite with $m \geq 2$*

Proof. As with the path, each vertex must have a unique label. This implies that the edges incident to one vertex do not sum to the same total as the edges incident to any other vertex and thus that we cannot swap all of the edges incident to two vertices. So any swap from this initial configuration must therefore involve swapping two edges for one.

Suppose we have the cycle $(v_1, e_1, \ldots, e_{i-1}, v_i, e_i, \ldots, v_n, e_n, v_1)$ where, without loss of generality, we let $\lambda(e_{i-1}) + \lambda(e_i) = \lambda(e_1)$. After swapping these edges we have:

$$(v_1, e_{i-1}, \ldots, e_1, v_i)(v_1, e_i, \ldots, v_n, e_n, v_1)$$

which is an $(n-(i-1), i-1)$-kite. It remains to show that $m \geq 2$. If $m = 1$ then $i = 2$ and so the two vertices involved in the mutation are adjacent. As stated in Section 2, the edge between adjacent vertices can play no active role in the mutation. Since each of the vertices has only one other incident edge this rules out any mutation. The result follows. □

Theorem 4.2. *A first-generation mutation of $C_n \cup C_m$ involving both components is either an (n,m)-kite or an (m,n)-kite*

Proof. We begin with the two cycles

$$\begin{aligned} C_n &= (v_{1,1}, e_{1,1}, \dots, v_{1,n}, e_{1,n}, v_{1,1}) \\ C_m &= (v_{2,1}, e_{2,1}, \dots, v_{2,m}, e_{2,m}, v_{2,1}). \end{aligned}$$

and suppose without loss of generality that $e_{2,1} + e_{2,m} = e_{1,1}$. Swapping the edges will result in $(v_{2,1}, e_{1,1}, v_{1,2}, \dots, v_{1,1})(v_{1,1}, e_{2,1}, \dots, v_{2,m}, e_{2,m}, v_{1,1})$ i.e. an (m,n)-kite. By the same argument, if two edges from C_n sum to an edge from C_m the result will be an (n,m)-kite. □

Which kites can be derived by a single mutation from a cycle? It depends of course on the initial labeling. We answer the question in the case of the standard labeling of the odd cycle. If we index the edges e_i of an odd cycle C_n with $i = 1, ..., n$ then edges are labeled as follows:

$$\lambda(e_i) = \begin{cases} \frac{1}{2}(i+1) & \text{for } i \text{ odd} \\ \frac{1}{2}(i+n+1) & \text{for } i \text{ even} \end{cases} \tag{2}$$

This yields:

$$\begin{aligned} \lambda(e_i) + \lambda(e_{i+1}) &= \tfrac{1}{2}(2i+n+3) &&= \lambda(e_{2i+2}) \\ \lambda(e_1) + \lambda(e_n) &= \tfrac{1}{2}(n+3) &&= \lambda(e_2) \end{aligned}$$

We need $2i + 2 \leq n - 1$, since n is odd. It follows that $i \leq \frac{1}{2}(n-3)$.

For e_i, e_{i+1}, we let $u_0 = v_{i+1}$ and $w_0 = v_{2i+3}$. We mutate the cycle from:

$$(v_1, e_1, v_2, \dots, v_i, e_i, v_{i+1}, e_{i+1}, \dots, v_{2i+2,} e_{2i+2,} v_{2i+3}, \dots, v_n, e_n, v_1)$$

to

$$\begin{aligned} &(v_1, e_1, v_2, \dots, v_i, e_i, v_{2i+3}, e_{2i+3}, \dots, v_n, e_n, v_1) \\ &(v_{2i+3}, e_{i+1}, \dots, v_{2i+2,} e_{2i+2,} v_{i+1}) \end{aligned}$$

which is an $(n-i-2, i+2)$-kite when $1 \leq i \leq \frac{1}{2}(n-3)$, in other words, the kites of order n from the $(n-3,3)$-kite down to the $(\frac{1}{2}(n-1), \frac{1}{2}(n+1))$-kite.

Alternatively, we let $u_0 = v_{i+1}$ and $w_0 = v_{2i+2}$ and mutate the cycle from:

$$(v_1, e_1, \dots, v_i, e_i, v_{i+1}, e_{i+1}, \dots, e_{2i+1}, v_{2i+2}, e_{2i+2}, v_{2i+3}, \dots, v_n, e_{n-1}, v_1)$$

to:

$$\begin{aligned} &(v_{2i+2}, e_{i+1}, v_{i+2}, e_{i+2}, \dots, e_{2i+1}, v_{2i+2}) \\ &(v_{i+2}, e_{2i+2}, v_{2i+3}, \dots, v_n, e_n, v_1, e_1, ..., v_i, e_i, v_{2i+2}) \end{aligned}$$

which is an $(i+1, n-(i+1))$-kite for $i = 2, \dots, \frac{1}{2}(n-3)$. In other words,we have all the kites of order n from the $(3, n-3)$-kite up to the $(\frac{1}{2}(n-1), \frac{1}{2}(n+1))$-kite. We cannot have $i = 1$ since this would result in v_3 and v_4 sharing two edges: e_2 and e_3.

Finally, we let $u_0 = v_1$ and $w_0 = v_3$, where $\lambda(e_n) + \lambda(e_1) = \lambda(e_2)$, and mutate the cycle from:

$$(v_1, e_1, v_2, e_2, v_3, e_3, \dots, v_n, e_n, v_1)$$

to

$$(v_3, e_3, \ldots, e_{n-1}, v_n, e_n, v_3)(v_3, e_1, v_2, e_2, v_1)$$

which is a $(n-2, 2)$-kite.

This gives us all of the order n kites from the $(3, n-3)$-kite to the $(n-2, 2)$-kite, leaving only the $(n-1, 1)$-kite without a labeling (i.e. all the kites with tails of lengths $2, \ldots, \frac{1}{2}(n+1)$ plus of course the original cycle with a tail of length 0).

This construction establishes the following theorem:

Theorem 4.3. *For odd n, the $(m, n-m)$-kites have strong VMTLs for $m = 3, \ldots, n-2$.*

4.2 Second-generation mutations of cycles

Although second-generation mutations of cycles can produce many new graphs, such as 'kites' with 2 tails, in this section we will focus on the one remaining kite for which was not obtained by the first generation mutation, the $(n-1, 1)$-kite. We will find a 2nd generation mutation of a cycle which produces a VMTL for this kite.

We return to the original labeling of a cycle of odd order, and consider the sum of $\lambda(e_i) + \lambda(e_{c-i})$, where c is an odd constant greater than i.

If i is odd then $c - i$ is even and we have:

$$\lambda(e_i) + \lambda(e_{c-i}) = \tfrac{1}{2}(i+1) + \tfrac{1}{2}(c+n+1-i) = \tfrac{1}{2}(c+n+2)$$

If i is even then $c - i$ is odd and we have:

$$\lambda(e_i) + \lambda(e_{c-i}) = \tfrac{1}{2}(c-i+1) + \tfrac{1}{2}(n+1+i) = \tfrac{1}{2}(c+n+2)$$

Hence by appropriate choices for i, j with $i \neq j$ we can obtain

$$\lambda(e_i) + \lambda(e_{c-i}) = \lambda(e_j) + \lambda(e_{c-j}) = \tfrac{1}{2}(c+n+2)$$

We proceed to use this equality to derive a labeling of the $(n-1, 1)$-kite by mutating the labeling of the $(n-3, 3)$-kite.

Starting with the cycle, for e_1, e_2, we let $u_0 = v_2$ and $w_0 = v_5$. We mutate the cycle from:

$$(v_1, e_1, v_2, e_2, v_3, \ldots, v_4, e_4, v_5, e_5, \ldots, v_n, e_n, v_1)$$

to the $(n-3, 3)$-kite:

$$(v_1, e_1, v_5, e_5, \ldots, v_n, e_n, v_1)(v_5, e_2, v_3, e_3, v_4, e_4, v_2)$$

Note that e_2 and e_5 are incident to v_5 and e_3 and e_4 are incident to v_4. Looking at the indices of these edges, we note that $2+5 = 3+4 = 7$, so we have $\lambda(e_2) + \lambda(e_5) = \lambda(e_3) + \lambda(e_4) = \frac{1}{2}(n+9)$, so we can swap these pairs of edges to give:

$$(v_1, e_n, v_n \ldots, e_5, v_4, e_2, v_3, e_3, v_5, e_1, v_1)(v_5, e_4, v_2)$$

which is the $(n-1, 1)$-kite required.

As an example, for $n = 5$ we obtain:

$$(v_1, e_5, v_4, e_2, v_3, e_3, v_5, e_1, v_1)(v_5, e_4, v_2)$$

and

$$\begin{aligned}
\lambda(v_1) + \lambda(e_5) + \lambda(e_1) &= 10 + 3 + 1 &&= 14 \\
\lambda(v_2) + \lambda(e_4) &= 9 + 5 &&= 14 \\
\lambda(v_3) + \lambda(e_2) + \lambda(e_3) &= 8 + 2 + 4 &&= 14 \\
\lambda(v_4) + \lambda(e_5) + \lambda(e_2) &= 7 + 3 + 4 &&= 14 \\
\lambda(v_5) + \lambda(e_3) + \lambda(e_1) + \lambda(e_4) &= 6 + 2 + 1 + 5 &&= 14
\end{aligned}$$

so the vertex-sums are all equal as required.

Thus, starting with a VMTL of a cycle of odd order, in 2 generations we were able to mutate the cycle to obtain VMTLs for all kites of that order.

4.3 Adjoining 2-factors to strong VMTLs

A VMTL is *strong* if it assigns the largest labels to the vertices. If we begin our mutation from a graph with a strong VMTL, we produce (sometimes large numbers of non-isomorphic) graphs with VMTLs that are also strong. This is important in the light of the construction described in [1] and exploited further in [2]. Starting from a graph with a strong VMTL, Theorem 1 of [1] shows how to add an arbitrary 2-factor to the graph to produce a graph of the same order but larger size which also has a VMTL.

For example, if we start with the labeling of C_{11} shown in Figure 3(i), we can mutate it to the $(5, 6)$-kite in Figure 3(ii) and then add a 2-factor as indicated to create a new graph with a strong VMTL as that shown in Figure 3(iii).

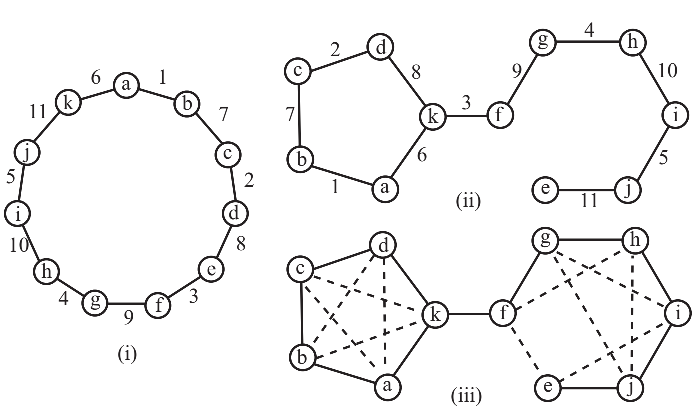

Figure 3: Mutating C_{11} to a $(5, 6)$-kite then adding a 2-factor

Because the VMTL produced by this procedure is also strong, we can repeat the construction by adjoining more 2-factors as long as there is room for the edges. While the emphasis in [1] and [2] was on regular graphs, the procedure applies equally well to non-regular graphs. Thus the mutations described in Sections 3 and 4 provide us with large numbers of starter graphs from which to build many graphs of the same order but larger size all having VMTLs.

Note that while a mutation is generally reversible, if we add a 2-factor to a mutated graph which reconnects 2 vertices that were connected prior to the mutation, then the mutation will not be reversible in the resulting graph.

4.4 Mutations of cycles involving more than 2 vertices

The Mutation Principle can be extended to include edges incident with 3 or more vertices. Consider a cycle with a VMTL such that the labels on the edges incident to three of the vertices are $\{\lambda_{1,1}, \lambda_{1,2}\}$,$\{\lambda_{2,1}, \lambda_{2,2}\}$ and $\{\lambda_{1,1}+\lambda_{1,2}-\lambda_{2,1}, \lambda_{2,1}+\lambda_{2,2}-\lambda_{1,1}\}$. Pairing the labels differently we obtain:

$$\begin{aligned}(\lambda_{2,1} + \lambda_{2,2} - \lambda_{1,1}) + \lambda_{1,1} &= (\lambda_{2,1} + \lambda_{2,2})\\(\lambda_{1,1} + \lambda_{1,2} - \lambda_{2,1}) + \lambda_{2,1} &= (\lambda_{1,1} + \lambda_{1,2})\\(\lambda_{1,2} + \lambda_{2,2}) &= (\lambda_{2,1} + \lambda_{2,2} - \lambda_{1,1}) + (\lambda_{1,1} + \lambda_{1,2} - \lambda_{2,1})\end{aligned}$$

hence we can swap the relevant edges among the three vertices in a way that preserves the vertex sums.

For example, the standard VMTL of C_9 has the sequence of edge labels $(1, 6, 2, 7, 3, 8, 4, 9, 5, 1)$. If we take the label pairs $(1, 6)$,$(7, 3)$ and $(9, 4)$, we can swap the edges to obtain equivalent weights as follows: $(3, 4)$,$(1, 9)$ and $(7, 6)$, giving us the new sequence of edge-labels $(3, 4, 8, 3)(1, 9, 5, 1)(7, 6, 2, 7)$ which is a labeling of $3C_3$.

Similarly, the standard VMTL of C_{11} has the sequence of edge-labels $(1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 1)$ and by swapping edges from $(1, 7)$, $(3, 9)$ and $(5, 11)$ to $(3, 5)$, $(1, 11)$ and $(7, 9)$, we obtain $(3, 5, 10, 4, 9, 7, 2, 8)(1, 11, 6)$, a labeling of $C_3 \cup C_8$. By generalizing these cases, we obtain labelings for the infinite families described in the following theorem:

Theorem 4.4. *$C_{2u+1} \cup C_{4u+4}$ and $3C_n$ for n odd, each possess a strong VMTL.*

Proof. For $C_{2u+1} \cup C_{4u+4}$, we start with the standard labeling for an odd cycle C_{6u+5} and mutate it by taking the edges labeled $(1, 3u+4)$,$(u+2, 4u+5)$ and $(2u+3, 5u+6)$ and reassigning them so that they are paired as $(u + 2, 2u + 3)$, $(1, 5u + 6)$ and $(3u + 4, 4u + 5)$ giving a cycle of order C_{2u+1} and a cycle of order C_{4u+4}.

For $3C_n$, we start with the standard labeling of the odd cycle C_{3n} where $n = 2u+1$ and mutate it by taking the edges labeled $(1, 3u+3)$,$(u+2, 4u+3)$ and $(2u+2, 5u+4)$ and reassigning them so that they are paired as $(u + 2, 2u + 2)$, $(1, 5u + 4)$ and $(3u + 3, 4u + 3)$. This produces the 3 cycles of length n. □

The above result on $3C_n$ is not new, since it has already been shown by another method [7] that mC_n for both m and n odd is vertex magic.

A more exotic example is given by swapping among 5 pairs of edges of C_{19}. Beginning with the standard labeling $1, 11, 2, 12, 3, 13, \ldots, 10$, we swap the equally spaced pairs $(1, 11)$, $(3, 13)$, $(5, 15)$, $(7, 17)$, $(9, 19)$ for the pairs $(3, 9)$, $(1, 15)$, $(7, 13)$, $(5, 19)$, $(11, 17)$. The result is the graph $C_8 \cup C_{11}$. This example can also be generalized to give

Theorem 4.5. *$C_{4u} \cup C_{6u-1}$ possesses a strong VMTL for all $u \geq 2$.*

Proof. Begin with the standard labeling of C_{10u-1} with $u > 1$. Mutate by reassigning the edge pairs labeled $(1, 5u+1)$, $(u+1, 6u+1)$, $(2u+1, 7u+1)$, $(3u+1, 8u+1)$, $(4u+1, 9u+1)$ which have sums respectively $5u+1, 7u+2, 9u+2, 11u+2, 13u+2$, so that they are paired as $(u+1, 4u+1)$, $(1, 7u+1)$, $(3u+1, 6u+1)$, $(2u+1, 9u+1)$, $(5u+1, 8u+1)$, which clearly have the same sums. This gives the disjoint cycles of lengths $4u$ and $6u-1$ as required. □

Applying these multiple mutations to cycles seems a very fruitful process. As a last example we mutate the standard labeling of C_{17} by reassigning the 4 pairs labeled $(1, 10), (11, 3), (13, 5), (6, 15)$ and pairing the edges as $(5, 6), (1, 13), (3, 15), (10, 11)$. The surprising result is the 3-component graph $C_3 \cup C_3 \cup C_{11}$, showing that in principal it may be possible to generate strong labelings for many of the 2-regular graphs this way. Strong VMTLs of 2-regular graphs are important because they are the starting points for the constructions described in [1] for producing labelings of odd order d-regular graphs for $d > 2$. The paper [2] treats this problem in detail.

5 Mutations of Cubic Graphs of Small Order

5.1 Cubic graphs of order 8

In this section we describe the results of a systematic computer search for strong labelings of all the cubic graphs of order 8 via $(2, 2)$-mutations of a single labeling of one graph. We choose the 3-cube to begin with, and as the seed labeling we choose the particularly pretty one illustrated at the top of Fig 4. It is a strong labeling.

For a $(2, 2)$-mutation of one graph into another, derived by reassigning the edges e_1, e_2 to vertex u and edges f_1, f_2 to the vertex v, we can indicate which edges have been reassigned by the notation:

$$\lambda(v) : (\lambda(e_1), \lambda(e_2)) \longleftrightarrow \lambda(u) : (\lambda(f_1), \lambda(f_2))$$

(Clearly, this notation can be extended to the case of more than two vertices with a variable number of edges being swapped.)

There are six non-isomorphic cubic graphs of order 8, five connected and one disconnected; and strong labelings of all of them can be obtained within three generations of $(2, 2)$-mutations. The mutations are shown in Figure 4 where the notation described above has been used. Although the mutated graphs as drawn appear quite different, it is easy to verify that only the indicated edges have been reassigned and that all of the remaining vertices have edges with the same labels after the mutation as they had before.

Table 1 shows how many different strong labelings of these cubic graphs can be obtained in relatively few generations. One thing that is notable is the wide range in the number of labelings for different graphs, which may suggest that there are properties of regular graphs that we have not yet identified which influence the *number* of possible labelings without necessarily influencing the *existence* of at least one labeling. Alternatively, it may well just be an artefact of our choice of seed graph and its labeling. Note that in Table 1, the bold column headings are *not* cycles but are references to the cubic graphs by their identification number in Read & Wilson's *An Atlas of Graphs* [6].

Gen	**C4**	**C5**	**C6**	**C7**	**C8**	2**C1**	Total
0	0	0	0	0	1	0	1
1	2	0	0	0	8	0	10
2	11	0	5	2	18	0	36
3	42	9	13	0	40	2	106
4	153	36	55	13	70	0	327
5	471	106	121	71	129	4	902
6	1262	352	348	229	191	1	2383
7	3113	860	803	646	387	7	5816
Σ	23360	8094	5399	6515	3042	34	46444

Table 1: Strong VMTLs of order 8 cubics obtained from the 3-cube

We note again that each of these labelings for each of these graphs is suitable to use as a starter for the process of adjoining 2-factors, as described in Section 4.3. This underlines how important mutation is.

The number of labelings for the graph **C4** is quite remarkable, especially when one realizes that these are only the strong labelings and that one would normally expect there to be many more labelings that are not strong. We would like to know what feature of this graph allows it to have so many more labelings than any of the other cubic graphs?

5.2 Cubic graphs of order 10

As a second illustration of how prolific the mutation process is, we will begin with a single labeling of the quasi-prism of order 10, produced by McQuillan's method as described in [1, 5]. The seed labeling will be as described in Table 2. This labeling is not strong.

By mutating this labeling through five generations, we obtain labelings for *all* connected cubic graphs of order 10. In Figure 5, each new graph is shown in the earliest generation at which one of its labelings occur.

Note that in Fig. 5, the vertex labels are given as reference points and that only the edges and their labels incident with these vertices are swapped. In order to obtain the labeling for any graph in the diagram, it is necessary to start with the

Vertex label	Labels of incident edges
11	$8, 10, 18$
12	$6, 9, 20$
13	$7, 10, 17$
14	$6, 8, 19$
15	$7, 9, 16$
21	$3, 5, 18$
22	$1, 4, 20$
23	$2, 5, 17$
24	$1, 3, 19$
25	$2, 4, 16$

Table 2: The seed labeling of the 5-prism

earliest mutation on the path which connects that graph to the seed graph **C23** and then work forward, following through the sequence of mutations in turn.

It is noteworthy that multiple labelings of some graphs (i.e. **C23**, **C11** and **C19**) were required to obtain labelings for all of the connected order 10 cubic graphs in the minimum number of generations. We expect that labelings for the two disconnected order 10 cubic graphs will emerge eventually.

We would expect to find many more labelings by mutating an order 10 graph than we found for the order 8 graphs, and that new mutations might continue to appear for more generations. We hope to implement exhaustive searches on this and other related questions in future research. The computational results of Section 5 are summarized in the following theorem:

Theorem 5.1. *All connected cubic graphs of order* 8 *and* 10 *possess VMTLs.*

6 Remarks

Mutations of strong labelings are important for several reasons. Initially, we were simply interested in finding a process to generate VMTLs for as many new graphs as possible. We have obviously been successful at this, but have really very little idea of precisely how successful. If the cubic graphs are any indication, it is a remarkably powerful tool. Is there some way we can predict theoretically that, for example, all r-regular graphs have labelings that are mutations of one properly chosen starter?

As illustrated in Section 4.3, the construction method in Theorem 2.1 of [1] begins with a *strong* VMTL of any graph and adds the edges of *any* 2-factor to produce a graph of the same order but larger size and which also has a strong VMTL. Mutation thus provides us with a large number of graphs to use as the starters for this process. The combination of these two constructions is remarkably fruitful.

It is worth remarking that after adding more edges according to Theorem 2.1, we generally increase the potential number of possible edge pairings that will enable mutation to occur. Superficially at least, this seems to provide graphs which have potential for many more mutations. We hope to explore this in future work.

We are especially interested in regular graphs because of the conjecture of the second author [3, 2] that for $r > 1$, all r-regular graphs other than $2C_3$ are vertex-magic. Given the requirement for strong labelings in the construction of [1] noted above, our hope is that there might be some way to show that mutation will generate strong labelings for all regular graphs of degree d from a single seed labeling of one graph. We have no idea how to do this yet. Further computational evidence might be helpful in this regard.

The paper [2] shows the importance of strong labelings of 2-regular graphs as the starters for building VMTLs of even-regular graphs. While we have touched on this in this paper, it would be useful if we could derive such labelings in a structured way via mutation. This suggests the following:

Problem 1. *Is there a strong VMTL for a cycle of odd order from which a fixed sequence of mutations will predictably produce* 2*-regular graphs with strong VMTLs?*

Of course, this is a special case of the much harder general question:

Problem 2. *If we start with any strong VMTL of an r-regular graph and mutate it through successive generations do we eventually obtain strong VMTLs for every r-regular graph of the same order, both connected and disconnected? If not, what are the equivalence classes of labelings that are created?*

The results in [2] applied to regular graphs of odd order, and we began there with odd order 2-regular graphs. We know much less about VMTLs for the regular graphs of even order. Cubic graphs of even order n can possess strong VMTLs only when $n = 4t$, so these are the ones suitable for use as the potential starters for building regular graphs of higher degree. This is one of the reasons we were interested in studying them in detail.

Not all VMTLs are suitable for mutation. For the cycles, for example, no mutation at all is possible if we assign either the largest integers or the odd integers to the edges of a cycle. For cycles, it seems intuitively obvious that a strong VMTL will maximize the number of possible first generation mutations, since in a strong VMTL of a cycle we are guaranteed labels of consecutive edges which sum to the edge labels $\frac{1}{2}(n+3), \ldots, n$. However it is possible that this does not produce the greatest *variety* of non-isomorphic graphs.

Problem 3. *Which labelings yield, through mutation, VMTLs of the greatest variety of non-isomorphic graphs in the smallest number of generations?*

Although mutation can be used to obtain labelings not obtainable by existing formal constructions, it has limitations. At present, mutating labelings is computer-intensive and it may be necessary to generate many labelings of intermediate graphs in order to obtain a labeling for one desired graph. We have no knowledge of how to predict what a particular labeling for a particular graph will generate. Apart from exhaustive computation, we are only able to apply mutation to graphs with a simple predictable labeling. This is the process used earlier for generalized stars and for kites, and so far is our most effective use of the method.

There is much more to be investigated. For example, we have not studied $(3, 3)$-mutations or $(2, 3)$-mutations at all. The potential is exciting.

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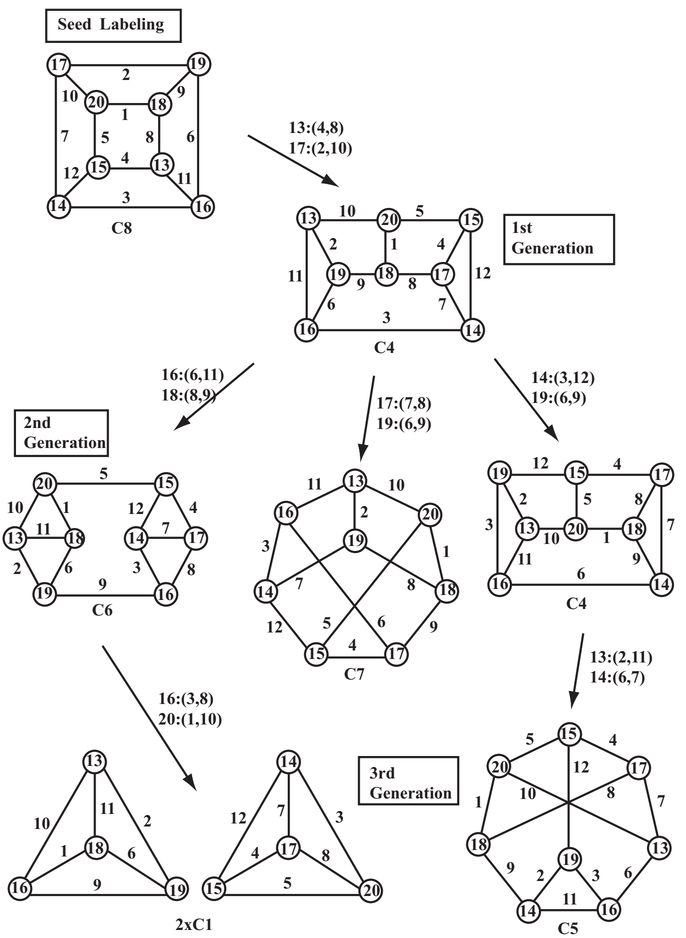

Figure 4: VMTLs of all order 8 cubic graphs via $(2, 2)$-mutations

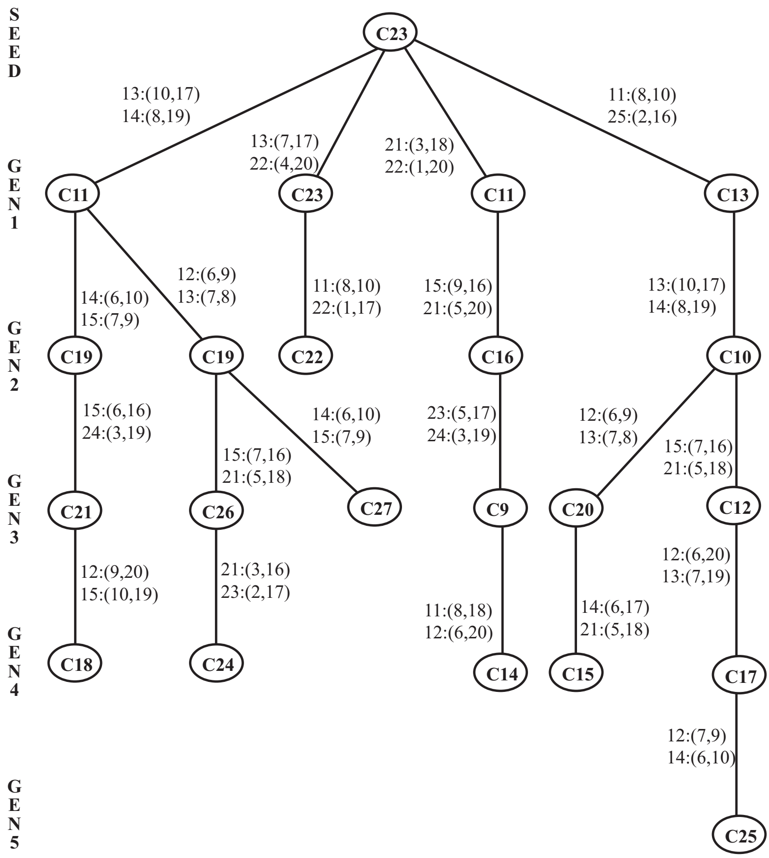

Figure 5: VMTLs of all connected order 10 cubic graphs via 5 generations of $(2, 2)$-mutations