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A note on completing Latin squares

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Abstract

We give a condition on the spatial distribution of filled cells in a partial Latin square P that is sufficient to ensure completability, regardless of what symbols are used in the filled cells.

For example, if P is of the order $mr + t$, where m, r are positive integers and $t \geq 0$, m is odd, and the filled cells of P are contained in the first $\frac{m+1}{2}$ $r \times r$ subsquares along the main diagonal, our condition is fulfilled, and P is completable. Another example is if P (of the same order) has non-empty cells only in the $m-1$ first $r \times r$ squares along the main diagonal and $r \geq m-2$. In this case, too, our condition holds, and P is completable.

1 Introduction

An $n \times n$ Latin square L is an $n \times n$ array filled with the symbols $1, 2, \ldots, n$ such that no symbol occurs more than once in any row or column. A *partial* $n \times n$ Latin square P (in short, a PLS) is a partially filled $n \times n$ array (using the symbols $1, 2, \ldots, n$) satisfying the condition that no symbol is used more than once in any row or column. P is said to be *completable* if there is some way of filling the empty cells of P to form a Latin square.

The purpose of this note is to generalize a result of Denley and Häggkvist on the completion of Latin squares. The theorem in question reads as follows (slightly reformulated to suit the subsequent generalizations), and can be found as Theorem 11.4.10 in [1], or in [2].

Theorem 1.1. *Let $n = 3r$ for some $r \geq 1$. Further, let P be a partial $n \times n$ Latin square with non-empty cells only in the top left $2r \times 2r$ square T. Suppose that the columns of T may be grouped together in pairs, G_i, $1 \leq i \leq r$, such that in each row there is at most one filled cell from each such pair of columns. Then P is completable if and only if there is some way of filling in the cells of T.*

In particular, we will adapt the same method of proof for the cases when $n = mr$ for any m, and also when $n = mr + t$ for any m and some $0 \leq t < r$. The inspiration for this line of research is the following conjecture.

Conjecture 1.2. *(Häggkvist, 1980) Any partial $mr \times mr$ Latin square whose filled cells lie in $(m-1)$ disjoint $r \times r$ squares can be completed.*

In the proof of the main theorem, we will need some preliminary results. The following proposition can be found as Proposition 8.2.9. in [1]. A colouring with the property described there is called a V_1-*sequential* colouring.

Proposition 1.3. *Let G be a bipartite graph with bipartition (V_1, V_2). If $d(x) \geq d(y)$ for each pair of adjacent vertices $x \in V_1$ and $y \in V_2$, then G has a colouring such that the colours of the neighbours of any $x \in V_1$ are precisely the colours $1, 2, \ldots, d(x)$.*

We shall also use a simple lemma, the proof of which is left as an easy exercise for the reader:

Lemma 1.4. *Let D be a bipartite digraph with bipartition (V_1, V_2), and $S_0 \subsetneq V_2$. If for each vertex $\sigma \in S_0$ it holds that $d^+(\sigma) \geq d^-(\sigma)$, and for each vertex $\rho \in N(S_0)$, the neighbour set of S_0, it holds that $d^+(\rho) \geq d^-(\rho)$, then for each $\sigma_0 \in S_0$ with $d^-(\sigma_0) < d^+(\sigma_0)$, there is a directed walk originating in σ_0 and ending in $V_2 \setminus S_0$.*

Finally, the following two well-known theorems will also be most useful.

Theorem 1.5. (Ryser [4]) *Let P be an $n \times n$ partial Latin square, whose upper left $r \times s$ subsquare is completely filled, and no other cells are filled. Then P is completable if and only if each symbol occurs at least $(r + s) - n$ times in P.*

Theorem 1.6. (Galvin [3]) *Let B be a bipartite multigraph, with lists L_e of permissible colours on each edge $e = (u, v)$. If $|L_e| \geq \max\{d(u), d(v)\}$ for each edge e, there exists a proper edge colouring of B using only colours from the lists.*

2 Theorem and corollaries

The main theorem may be stated as follows (Theorem 2.1), with the relevant parameters specified explicitly, but from the readability point of view, the formulation given in Theorem 2.6 might be more pleasing, where a more direct bound on the size n of the PLS is given.

Theorem 2.1. *Let P be a partial $(mr+t) \times (mr+t)$ Latin square, $0 \leq t$, whose filled cells all lie in an $\ell r \times \ell r$ subsquare T, where $\ell < m$. Further, suppose the columns of T can be grouped into r sets of ℓ columns each, G_i, where for each row ρ in T and each G_i, at most one cell in $\rho \cap G_i$ is filled (where we view both ρ and G_i as sets of cells). Finally, suppose that the parameters m, r, t and ℓ satisfy the equation $(m - \ell)r + t \geq 2\ell - m - \frac{t}{r}$. Then P is completable if and only if there is some way of filling the cells in T.*

Proof. The necessity of the final statement is trivial, and for sufficiency, we start by properly filling the cells of T in some arbitrary way, producing a new subsquare T'. We shall refer to the cells of T' that were filled from the start by T_0, and the additional cells filled by T_f. The goal will be to rearrange and exchange the symbols used in the cells of T_f in such a way that each symbol is used at least $\ell r + \ell r - (mr + t) = (2\ell - m)r - t$ times in T', and then apply Ryser's theorem, that is Theorem 1.5, to ensure the possibility of completing the Latin square.

We define a bipartite graph H which has bipartition V_1, V_2, where V_1 represents the set of ℓr rows of T and V_2 represents the set of $mr + t$ symbols. In H, we join a vertex $\rho \in V_1$ to a vertex $\sigma \in V_2$ with an edge $e_{\rho\sigma}$ whenever the symbol σ does not occur in row ρ in the original partial Latin square P. Now we colour the edge $e_{\rho\sigma}$ with colour c (a column) if the symbol σ was placed in the empty cell (ρ, c) when T_f was filled. We label this colouring C_1 and we shall refer to those edges which receive no column in this way as the *uncoloured* edges (with respect to C_1). These edges correspond to the pairs of symbols/rows where that particular symbol was not used in that particular row in T_f.

Finally we provide each vertex $\rho \in V_1$ with the list $F(\rho)$ of columns in which row ρ was filled already in P (the Filled columns), corresponding to edges that are not present in H.

We say that a symbol σ_0 is *globally deficient* if it has been used strictly less than $(2\ell - m)r - t$ times in T', and we let $\mathcal{G}$ be the total number of symbols that are globally deficient. For each globally deficient symbol, we can find a G_i where σ_0 has been used strictly less than $2\ell - m - \frac{t}{r}$ times. We say that σ_0 is *locally deficient* in G_i, and we let $\mathcal{L}$ be the *total local deficiency*, defined in the following way: For each G_i, a symbol used $L < 2\ell - m - \frac{t}{r}$ times contributes $2\ell - m - \frac{t}{r} - L$ to $\mathcal{L}$.

For a given globally deficient symbol σ_0, locally deficient in G_i, we define S_0 to be the set of symbols that are used less than or equal to $r(2\ell - m) - t$ times in T', and occur less than or equal to $2\ell - m - \frac{t}{r}$ times in G_i.

First, suppose for the sake of contradiction that T has been filled, and it is the case that $S_0 = V_2$. Then we have that the $mr + t$ symbols in S_0 have each been used at most $r(2\ell - m) - t$ times, in total $(mr + t)[(2\ell - m)r - t]$ entries. However, T has $\ell^2 r^2$ cells, and for $\ell < m$ it holds that $(mr + t)[(2\ell - m)r - t] = 2mr^2\ell + 2r\ell t - m^2r^2 - 2mrt - t^2 \leq 2mr^2\ell - m^2r^2 < \ell^2 r^2$, a contradiction. Hence $S_0 \neq V_2$.

Further, let H_0 be the subgraph of H induced by the uncoloured (with respect to C_1)edges in H incident with symbols in S_0. We define $R_0 = V_1 \cap H_0$, so that H_0 has bipartition (R_0, S_0). The degrees in H_0 satisfy $d_{H_0}(\sigma) \geq \ell r - [(2\ell - m)r - t] = (m - \ell)r + t$ (the number of rows of T in which the symbol $\sigma \in S_0$ is not used when filling T_f) for all $\sigma \in S_0$ and $d_{H_0}(\rho) \leq mr + t - \ell r = (m - \ell)r + t$ (the number of symbols in S_0 not used in row ρ) for all $\rho \in R_0$. Thus $d_{H_0}(\rho) \leq d_{H_0}(\sigma)$ for each pair $(\rho, \sigma) \in (R_0, S_0)$.

Our aim will be to redistribute the symbols in G_i so that the globally deficient symbol σ_0, locally deficient in G_i appears in one more of the columns in G_i, thus reducing $\mathcal{L}$ by one. Through this redistribution, one symbol, σ_k will be used in one less of the columns of G_i, but care will be taken so that σ_k is not locally deficient in G_i after the recolouring (but might have become globally deficient). Thus $\mathcal{L}$ is reduced by at least one.

To reach our goal, we first find a $[(m-\ell)r+t]$-factor M from S_0 into R_0. By Proposition 1.3, H_0 has an edge colouring C_2 such that exactly the colours $1, 2, \ldots, d_{H_0}(\sigma)$ are used at each $\sigma \in S_0$. We pick out the edges coloured $1, 2, \ldots, (2\ell - m - \frac{t}{r})$ in C_2, which is less than $d_{H_0}(\sigma) \geq (m-\ell)r + t$ by assumption, and call this set of edges M. Note that these edges are uncoloured in C_1.

Now, let D be the bipartite digraph with bipartition (R_0, V_2), edges M directed from the right to the left, and all edges incident with R_0 coloured with columns from G_i directed from the left to the right.

By the definition of M, it holds that the outdegree $d^+_D(\sigma) = 2\ell - m - \frac{t}{r}$ for any $\sigma \in S_0$. Also, since by the defintion of S_0 each $\sigma \in S_0$ occurs at most $(2\ell - m - \frac{t}{r})$ times in the G_i in question, we have that $d^-_D(\sigma) \leq (2\ell - m - \frac{t}{r})$, and thus $d^+_D(\sigma) \geq d^-_D(\sigma)$ for all $\sigma \in S_0$.

The indegree of a row $\rho \in R_0$ is at most $(2\ell - m - \frac{t}{r})$, since M could be coloured properly (as in C_2) with this number of colours. The outdegree at any $\rho \in V_1$ is at least $\ell - 1$ (the number of columns in $G_i \setminus F(\rho)$). For any $\ell < m$, It holds that $(2\ell - m - \frac{t}{r}) < \ell - 1$, so $d^+_D(\rho) \geq d^-_D(\rho)$ for each $\rho \in R_0$. We see also that $d^-_D(\sigma_0) < d^+_D(\sigma_0)$, since σ_0 is locally deficient in G_i.

By Lemma 1.4, we find a walk $W = (\sigma_0, \rho_1, \sigma_1, \ldots, \rho_{k-1}, \sigma_k)$ in D that starts in σ_0 and ends in $\sigma_k \in V_2 \setminus S_0$. The edges in W are thus elements of the form $e_{\rho_i\sigma_i} = (\rho_i, \sigma_i)$, which are uncoloured in C_1 (i.e. symbol σ_i was *not* used in row ρ_i), or $e_{\rho_i\sigma_{i+1}} = (\rho_i, \sigma_{i+1})$ which are coloured in C_1, where ρ_i is a row in R_0 and σ_j is a symbol in V_2.

We now recolour the edges of the walk W by colouring the edge $e_{\rho_j\sigma_j} \in W$ with the colour from the edge $e_{\rho_j\sigma_{j+1}}$ and uncolouring $e_{\rho_j\sigma_{j+1}}$ for $0 \leq j \leq k-1$, thus leaving the last edge e_{ρ_{k-1},σ_k} in W uncoloured.

Observe that after this recolouring, the symbol σ_0 will appear in precisely one more of the columns in G_i, namely column c_0. Each of the symbols $\sigma_1, \ldots, \sigma_{k-1}$ will still appear in the same number of columns in G_i.

The symbol $\sigma_k \notin S_0$ now appears in one cell less. However, since $\sigma_k \notin S_0$, by the definition of S_0, σ_k appeared at least $(2\ell - m)r - t + 1$ times in T' and thus still appears at least $(2\ell - m)r - t$ times, and has thus not become locally deficient in G_i. Therefore $\mathcal{L}$ will have been reduced by at least one.

By repeating this process for each globally deficient symbol and each set G_i where it is locally deficient, we will eventually have ensured that $\mathcal{G} = 0$, since $\mathcal{L} = 0$ implies that $\mathcal{G} = 0$. Note, however, that the algorithm can terminate even though $\mathcal{L} \neq 0$. When $\mathcal{G} = 0$, by Ryser's theorem, the modified T' can be completed to a Latin square. □

1	2				
2	1				
		1	3		
		2	1		
				1	3
				2	4

1	2	3	4	5	6
2	1	7	8	4	5
9	5	1	3	6	8
8	4	2	1	3	a
5	6	4	7	1	3
6	7	9	5	2	4

		3		5	
		7		4	
9				6	
8				3	
5		4			
6		**9**			

Figure 1: The upper left 6×6 corner of a 10×10 partial Latin square P, an arbitrary filling of its upper left 6×6 corner, and the cells of G_1 involved in the recolouring.

Example 2.2. *Let $t = 0$, $m = 5$ and $r = 2$, so that $n = 10$. Let the upper left 6×6 corner $T \subset P$ be as in Figure 1, left, and the rest of P empty. Obviously, we may take $G_1 = \{1, 3, 5\}$ and $G_2 = \{2, 4, 6\}$. The Ryser condition for P is then that each of the ten symbols be used at least $6 + 6 - 10 = 2$ times in T', (for example at least once in each of G_1 and G_2). By inspection, each of the symbols $1, \ldots, 9$ are used sufficiently many times.*

Therefore the only globally deficient symbol is a, and it is locally deficient in G_1. Also, $S_0 = \{a\}$, and $R_0 = \{1, 2, 3, 5, 6\}$, the rows in which the symbol a wasn't used. Then $H_0 \simeq K_{5,1}$, which can obviously be given a V_2-sequential colouring. Suppose we choose $M = \{e_{6,a}\}$, and find the walk $W = (e_{6,a}, e_{6,9})$, where the first edge is uncoloured, and the second edge is coloured with the column 3, that is the column in which row 6 holds the symbol 9, as indicated by bold face in Figure 1.

We now uncolour the edge $e_{6,9}$, and colour the edge $e_{6,a}$ with colour 3, which amounts to replacing the 9 with the symbol a. After this recoulouring, a is no longer deficient, neither globally nor locally, but we have created a new global deficiency, in symbol 9. Note that this possibility of creating a single new globally deficient symbol was handled in the proof of Theorem 2.1. We see that 9 is locally deficient in G_2, which was the case already before our recolouring, and a similar recolouring argument will ensure that 9 is used at least once in G_2, for example in cell $(6, 4)$. After that second recolouring, no symbols are globally deficient, and Ryser's theorem can be applied.

In general, using Galvin's theorem to ensure that completing T is possible, by first translating T into a bipartite graph, and defining lists by taking into account what symbols are already present in T, and choosing r and ℓ, we get a range of corollaries, of which we present two examples.

Note that the case $r = 1$, $\ell = m - 1$ in the next corollary, which is *not* covered byTheorem 2.1 is a special case of Evans' conjecture, that was solved by Smetaniuk:

Theorem 2.3. (Smetaniuk [5]) *Let P be an $n \times n$ partial Latin square with at most $n - 1$ cells filled. Then P is completable.*

The case $m = 2$ follows from Ryser's theorem, and the case $m = 3$, $r \geq 2$ is a corollary to Theorem 1.1, using Galvin's theorem.

Corollary 2.4. *Let P be a partial Latin $mr \times mr$ square, where m is odd, all of whose entries lie in the $\frac{m+1}{2}$ first $r \times r$ squares along the main diagonal. Then P is completable.*

Proof. By Galvin's theorem the top left $r\frac{m+1}{2} \times r\frac{m+1}{2}$ square T can be completed. The result follows from Theorem 2.1 by setting $\ell = \frac{m+1}{2}$. □

Corollary 2.5. *Let P be a partial $(mr+t) \times (mr+t)$ Latin square, where $t \geq 0$, with non-empty cells only in the $m-1$ first $r \times r$ squares along the main diagonal. Further suppose that $r \geq m-2$. Then P is completable.*

Proof. By Galvin's theorem the top left $r(m-1) \times r(m-1)$ square T can be completed. The result follows from Theorem 2.1 by setting $\ell = m-1$. The condition on r is exactly what follows from $(m-\ell)r + t \geq 2\ell - m - \frac{t}{r}$ by assuming the worst, namely that $t = 0$. □

As noted above, Theorem 2.1 may be restated as follows, by setting $n = mr + t$ and eliminating m and t.

Theorem 2.6. *Let P be a partial $n \times n$ Latin square, whose filled cells all lie in an $\ell r \times \ell r$ subsquare T, where $\frac{r+2}{r+1} r\ell \leq n$. Further, suppose the columns of T can be grouped into r sets of ℓ columns each, G_i, where for each row ρ in T and each G_i, at most one cell in $\rho \cap G_i$ is filled (where we view both ρ and G_i as sets of cells). Then P is completable if and only if there is some way of filling the cells in T.*

3 Concluding remarks

In order to approach Conjecture 1.2 from Theorem 2.1, it would seem that we need some way of grouping columns (or, by symmetry, rows) so that in each group each row is used at most once. In general, this will not be possible, and examples to this effect are easily found.

The first observation is that if no two $r \times r$ subsquares intersect any common row (or column) we can find the desired grouping G_i, $1 \leq i \leq r$. If two $r \times r$ subsquares intersect, then the column grouping condition can in general not be satisfied. In spite of this, we would like to propose the following strengthening of Conjecture 1.2. If nothing else, it at least has the merit that it might be easier to find counter-examples to it.

Conjecture 3.1. *Any partial $n \times n$ Latin square P, whose filled cells can be covered by $(m-1)$ possibly intersecting $r \times r$ subsquares, where $mr \leq n$, is completable.*

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